PHASE TRANSITION ASSOCIATED WITH FORMATION OF SPATIALLY PERIODIC STRUCTURES IN A FERMI LIQUID

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The phase transition in a Fermi liquid, associated with translational symmetry breaking and the formation of periodic structures is considered. Special attention is paid to the formation of one-dimensional long-periodic structures in a three-dimensional Fermi liquid. The relation between the formation of such structures and kinetic and thermodynamic stability of the normal state of the Fermi liquid is analyzed.

The term «normal Fermi liquid» is traditionally applied to a degenerate (charged or neutral) Fermi liquid possessing main properties of a system of noninteracting fermions in the case of a quasiparticle description. Such a definition of a normal Fermi liquid presumes that the equilibrium state of the Fermi liquid is the most symmetric, i.e., the distribution function describing this state is invariant to spatial translations and rotations in the spin and momentum spaces.

In spite of differences in the behavior of charged and neutral Fermi liquids, basic concepts of the Landau–Silin theory of the normal Fermi liquid [1,2] studying low-lying excitations against the background of the equilibrium state make it possible to disregard the electric charge of quasiparticles in the description of some phenomena in charged and neutral systems of interacting fermions. Apart from the main condition of applicability of the theory of the normal Fermi liquid, i.e., the smallness of temperature T as compared to the Fermi energy ε_F ($T \ll \varepsilon_F$), the main postulate of the theory which is common for neutral and charged systems concerns the functional dependence of the energy of the system E on the fermion distribution function $f(\vec{p}, \vec{r})$: $E = E(f)^*$.

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*We are using the system of units in which Boltzmann's constant k and Plank's constant $\hbar$ are equal to unity.

In this case, the quasiparticle energy which is a functional of the distribution function is defined as

$$\varepsilon(\vec{p}, \vec{r}) = V \frac{\delta E(f)}{\delta f(\vec{p}, \vec{r})},$$

$$\varepsilon(\vec{r}, \vec{p}) = \varepsilon_p + \frac{2}{V} \sum_{\vec{p}'} \int d\vec{r}' F(\vec{r} - \vec{r}'; \vec{p}, \vec{p}') f(\vec{r}', \vec{p}'), \qquad S = 1/2, \quad (1)$$

where $F(\vec{r} - \vec{r}'; \vec{p}, \vec{p}')$ is the Landau amplitude characterizing two-particle interactions, and $\varepsilon_p \equiv F(\vec{p})$ is the fermion energy in the absence of interaction between quasiparticles. In the absence of magnetic ordering, the existence of the fermion spin $S = 1/2$ is important only for the calculation of the fermion density of states, which is reflected in the factor $2S + 1 = 2$ in the second term of formula (1). The equilibrium state of the normal fermi liquid in a spatially inhomogeneous case is described by the Fermi–Dirac distribution function

$$f(\vec{p}, \vec{r}) = \{\exp \beta(\varepsilon(\vec{p}, \vec{r}) - \mu) + 1\}^{-1} \qquad (2)$$

($\beta^{-1} = T$ is the inverse temperature and μ the chemical potential). Together with Eq. (1), this equation determines the dispersion relation for quasiparticles in the equilibrium state.

An important aspect of the theory is the determination of the stability conditions for an equilibrium state of a normal Fermi liquid. This problem was solved for the first time in the spatially homogeneous case by Pomeranchuk [3] who formulated the stability criterion for the normal state down to the temperature $T = 0$:

$$1 + \frac{\nu(\mu) F_l}{2l + 1} > 0, \qquad (3)$$

where F_l are the coefficients of the l-th harmonic in the expansion of the spatially homogeneous Landau amplitude

$$F(\vec{p}, \vec{p}') = \int d\vec{r}' F(\vec{r} - \vec{r}'; \vec{p}, \vec{p}') \qquad (4)$$

into a series in Legendre polynomials near the Fermi surface ($p \approx p' \approx p_F$).

$$F(\vec{p}, \vec{p}') = \sum_{l=0}^{\infty} F_l(p, p') P_l(\cos \theta). \qquad (4)$$

The quantity $\nu(\varepsilon)$ appearing in formula (3) is the density of energy states defined as

$$\nu(\varepsilon) = \frac{2}{(2\pi)^3} \int d^3 p \, \delta(\varepsilon - \varepsilon(p)) \qquad (5)$$

($\varepsilon(p)$ corresponds to the spatially homogeneous distribution function $f(p)$).

It should be noted that the violation of Pomeranchuk's criterion indicates that the stability of the normal state is broken at temperatures $T < T_0$ (T_0 is the transition temperature, $T \ll \varepsilon_F$).

The present work [4] is devoted to an analysis of such a phase transition, i.e., the transition involving the violation of the stability condition (3) for the zeroth harmonic ($l = 0$), for which the following relation holds:

$$\nu(\mu)F_0 \lesssim -1. \tag{6}$$

We shall prove that condition (6) characterizes a phase transition in a Fermi liquid, associated with translational symmetry breaking for the equilibrium state.

Let us seek the solution of self-consistent Eqs. (1), (2) in the form of functions periodic in x with the period $X = 2\pi/q$

$$\varepsilon(\vec{r}, \vec{p}) \equiv \varepsilon(x, p) = \varepsilon_0(p) + \widetilde{\varepsilon}(x), \tag{7}$$

where

$$\varepsilon_0(p) = \langle \varepsilon(x, p) \rangle, \qquad \langle \widetilde{\varepsilon}(x) \rangle = 0 \tag{8}$$

and the angle brackets $\langle ... \rangle$ denote the averaging over periods. Consequently, Eqs. (1), (2) can be written in the form:

$$\varepsilon_0(p) = \varepsilon_p + F_0 \langle n(\beta, \mu - \widetilde{\varepsilon}(x)) \rangle, \tag{9}$$

$$\widetilde{\varepsilon}(x) = \int dx' F(x - x')\{n(\beta, \mu - \widetilde{\varepsilon}(x')) - \langle n(\beta, \mu - \widetilde{\varepsilon}(x'')) \rangle\}, \qquad \widetilde{\varepsilon}(\vec{p}, x) \equiv \widetilde{\varepsilon}(x), \tag{10}$$

where

$$F_0 = \int_{-\infty}^{\infty} dx F(x) \tag{11}$$

and the function $n(\beta, \mu)$ is defined by the expression:

$$n(\beta, \mu) = \frac{2}{V} \sum_{\vec{p}} \frac{1}{e^{\beta(\varepsilon_0(p) - \mu)} + 1}. \tag{12}$$

Let us now go over to the solution of Eq. (10) near the phase transition point (the transition to the states with spatially periodic structure), when the quantity $\widetilde{\varepsilon}(x)$ describing the order parameter is small. Noting that the quantity $F(x - x')$ has a sharp peak at $x = x'$ ($X \gg r_0; F(x - x') \approx 0$ at $x - x' \gtrsim r_0$) and also considering that the quantity $\widetilde{\varepsilon}(x)$ varies slowly on the account of the large lattice period, we can write Eq. (10) in the form

$$\widetilde{\varepsilon}(x) = F_0\{n(\beta, \mu - \widetilde{\varepsilon}(x)) - \langle n(\beta, \mu - \widetilde{\varepsilon}(x)) \rangle\} + F_2 \frac{\partial^2 n(\beta, \mu - \widetilde{\varepsilon}(x))}{\partial x^2}, \tag{13}$$

where

$$F_2 = \frac{1}{2} \int dx' F(x - x')(x - x')^2 \qquad (14)$$

(while deriving Eq. (13), we assumed that the function $F(x - x')$ is even). Carrying out the power expansion in $\widetilde{\varepsilon}(x)$ and $(\beta - \beta_0)$ in Eq. (13) (β_0-corresponds to the transition temperature for $q = 0$) and taking into account that in accordance with (8) $\langle \widetilde{\varepsilon}(x) \rangle = 0$ we have

$$\widetilde{\varepsilon}(x) \left(1 + F_0 \frac{\partial n(\beta_0, \mu)}{\partial \mu} \right) = 0, \qquad (15)$$

$$F_0 \left\{ -(\beta - \beta_0) \frac{\partial^2 n(\beta_0, \mu)}{\partial \beta \partial \mu} \widetilde{\varepsilon}(x) + \frac{1}{2} \frac{\partial^2 n(\beta_0, \mu)}{\partial \mu^2} (\widetilde{\varepsilon}^2(x) - \langle \widetilde{\varepsilon}^2(x) \rangle) \right\} - \qquad (16)$$

$$- F_2 \frac{\partial n(\beta_0, \mu)}{\partial \mu} \frac{\partial^2 \widetilde{\varepsilon}(x)}{\partial x^2} = 0.$$

The Eq. (15) determines the transition temperature β_0. Carrying out in this equation the low-temperature expansion for the function $n(\beta_0, \mu)$ we obtain

$$T_0^2 = -\frac{6}{\pi^2} \frac{1 + F_0 \nu(\mu)}{F_0 \nu''(\mu)}. \qquad (17)$$

Since $\nu''(\mu) < 0$, it can be easily seen that the inequality $T_0^2 > 0$ holds only when the relation (6) reflecting the violation of the criterion of stability of the equilibrium state of a normal Fermi liquid is valid.

Equation (16) used for defining the quantity $\widetilde{\varepsilon}(x)$ can be written in another form more convenient for the subsequent analysis. For this purpose, we introduce the quantity $\varepsilon(x) = -\widetilde{\varepsilon}(x)$ that can be regarded as a correction to chemical potential (see (9) and (10)). Then Eq. (16) assumes the form

$$\frac{\partial^2 \varepsilon(x)}{\partial x^2} + g(\varepsilon(x)) = 0, \qquad g(\varepsilon(x)) = A\varepsilon(x) + B(\varepsilon^2(x) - \langle \varepsilon^2(x) \rangle), \qquad (18)$$

where

$$A = -\frac{F_0^2}{F_2} (\beta - \beta_0) \frac{\partial^2 n(\beta_0, \mu)}{\partial \beta_0 \partial \mu}, \qquad B = -\frac{1}{2} \frac{F_0^2}{F_2} \frac{\partial^2 n(\beta_0, \mu)}{\partial \mu^2}. \qquad (19)$$

We shall seek periodic solutions of Eq. (18) which gives

$$\varepsilon' = \pm \sqrt{2(E - U(\varepsilon))}, \qquad x = \pm \int^\varepsilon \frac{d\varepsilon}{\sqrt{2(E - U(\varepsilon))}}, \qquad (20)$$

where

$$U(\varepsilon) = \int_0^\varepsilon g(\varepsilon) d\varepsilon = \frac{1}{3} B\varepsilon^3 + \frac{1}{2} A\varepsilon^2 - Bd^2\varepsilon, \qquad d^2 = \langle \varepsilon^2(x) \rangle$$

and E is the integration constant. The cubic polynomial $E - U(\varepsilon)$ can be written in the form

$$E - U(\varepsilon) = E - \frac{1}{3}B\varepsilon^3 - \frac{1}{2}A\varepsilon^2 + Bd^2\varepsilon = -\frac{1}{3}B(\varepsilon - \varepsilon_1)(\varepsilon - \varepsilon_2)(\varepsilon - \varepsilon_3) > 0. \quad (21)$$

The points of extrema of the function $U(\varepsilon)$ are defined as

$$\varepsilon_\pm = -\frac{A}{2B} \pm \sqrt{\frac{A^2}{4B^2} + d^2}, \qquad \varepsilon_+ > 0, \ \varepsilon_- < 0.$$

Since $E - U > 0$, periodic solutions of Eq. (18) correspond to the region $\varepsilon_2 < \varepsilon < \varepsilon_1$, and since $\langle \varepsilon \rangle = 0$, we have $\varepsilon_2 < 0$, $\varepsilon_1 > 0$. Consequently we have

$$x(\varepsilon) = -\int_\varepsilon^{\varepsilon_1} \frac{d\varepsilon}{\sqrt{2(E - U(\varepsilon))}}, \qquad -\frac{X}{2} < x < 0, \quad (22)$$

$$x(\varepsilon) = \int_\varepsilon^{\varepsilon_1} \frac{d\varepsilon}{\sqrt{2(E - U(\varepsilon))}}, \qquad \frac{X}{2} > x > 0.$$

The period of the function $\varepsilon(x)$ is defined by the formula

$$X = 2\int_{\varepsilon_2}^{\varepsilon_1} \frac{d\varepsilon}{\sqrt{2(E - U(\varepsilon))}} = 2x(\varepsilon_2). \quad (23)$$

Substituting expression (21) for $E - U(\varepsilon)$ into the formula (22) for $x(\varepsilon)$ for $\frac{X}{2} > x > 0$ and transforming the corresponding integral, we obtain

$$x(\varepsilon) = \sqrt{\frac{6}{B}} \frac{1}{\sqrt{\varepsilon_1 - \varepsilon_3}} \int_0^\varphi \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \qquad k^2 = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 - \varepsilon_3},$$

$$\varphi = \arcsin\sqrt{\frac{\varepsilon_1 - \varepsilon}{\varepsilon_1 - \varepsilon_2}}.$$

Taking into account the definition of the first-order elliptical integral

$$\mathcal{F}(k, \varphi) = \int_0^\varphi \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \quad (24)$$

we can write $x(\varepsilon)$ in the form

$$x(\varepsilon) = \sqrt{\frac{6}{B}} \frac{1}{\sqrt{\varepsilon_1 - \varepsilon_3}} \mathcal{F}(k, \varphi). \quad (25)$$

In accordance with (23), in this case we have

$$X = \sqrt{\frac{6}{B}} \frac{2}{\sqrt{\varepsilon_1 - \varepsilon_3}} \mathcal{F}(k), \qquad \mathcal{F}(k) \equiv \mathcal{F}(k, \pi/2). \quad (26)$$

Let us now determine the quantities ε_1, ε_2, and ε_3. For this purpose, we note that

$$\langle \varepsilon(x) \rangle = \frac{1}{X} \int_0^{X/2} \varepsilon(x)dx + \frac{1}{X} \int_{X/2}^{X} \varepsilon(x)dx = \frac{2}{X} \int_0^{X/2} \varepsilon(x)dx,$$

or, going over to integration with respect to ε

$$\langle \varepsilon(x) \rangle = \frac{2}{X} \int_{\varepsilon_2}^{\varepsilon_1} \varepsilon \frac{d\varepsilon}{\sqrt{2(E-U)}}. \tag{27}$$

Equation (18) implies that $\langle \varepsilon \rangle = 0$. Transforming the integral appearing in (27) and taking into account (21), we obtain

$$\int_0^{\pi/2} d\varphi \frac{\varepsilon_1 - (\varepsilon_1 - \varepsilon_2)\sin^2\varphi}{\sqrt{1 - k^2\sin^2\varphi}} = 0, \qquad k^2 = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 - \varepsilon_3}.$$

Using the definition of the second-order elliptical integral

$$E(k) = \int_0^{\pi/2} d\varphi \sqrt{1 - k^2\sin^2\varphi}, \tag{28}$$

we obtain

$$E(k) + \left(k^2 \frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_2} - 1 \right) \mathcal{F}(k) = 0.$$

This expression can also be written in the form

$$\frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_2} = \frac{\mathcal{F}(k) - E(k)}{k^2 \mathcal{F}(k)}, \qquad \frac{\varepsilon_1}{\varepsilon_1 - \varepsilon_3} = \frac{\mathcal{F}(k) - E(k)}{\mathcal{F}(k)}. \tag{29}$$

These formulas indicate that the ratios $\varepsilon_1/\varepsilon_2$, $\varepsilon_1/\varepsilon_3$, $\varepsilon_2/\varepsilon_3$ can be expressed only in terms of the parameter k. Let us now find the expression for the quantity ε_1 in terms of k. For this purpose, we note that, according to (20)

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = -\frac{3}{2}\frac{A}{B} \equiv \gamma(\beta - \beta_0), \tag{30}$$

where

$$\gamma = -3 \frac{\partial^2 n(\beta_0, \mu)}{\partial\beta\partial\mu} \bigg/ \frac{\partial^2 n(\beta_0, \mu)}{\partial\mu^2}, \tag{31}$$

in accordance with (19). Using further formulas (29), we obtain the following expression for the quantity ε_1:

$$\varepsilon_1 = \frac{\gamma(\beta - \beta_0)}{3 - (1 + k^2)\dfrac{\mathcal{F}(k)}{\mathcal{F}(k) - E(k)}}. \tag{32}$$

Taking into account this relation and (29), we can easily determine the quantity $1/\sqrt{\varepsilon_1 - \varepsilon_3}$ appearing in the expression (26) for the period:

$$\frac{1}{\sqrt{\varepsilon_1 - \varepsilon_3}} = \left(\left(3\frac{\mathcal{F}(k) - E(k)}{\mathcal{F}(k)} - k^2 - 1\right)\bigg/\gamma(\beta - \beta_0)\right)^{1/2}. \qquad (33)$$

Noting that $k^2 = \dfrac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 - \varepsilon_3}$ and introducing the new variable $\varepsilon \equiv \varepsilon_1 - \varepsilon_3$, we obtain

$$\varepsilon_1 - \varepsilon_2 = \varepsilon k^2, \qquad \varepsilon_1 = \varepsilon\frac{\mathcal{F}(k) - E(k)}{\mathcal{F}(k)},$$

or

$$\varepsilon_1 = \varepsilon\left(1 - \frac{E(k)}{\mathcal{F}(k)}\right), \qquad \varepsilon_2 = \varepsilon\left(1 - k^2 - \frac{E(k)}{\mathcal{F}(k)}\right), \qquad \varepsilon_3 = -\varepsilon\frac{E(k)}{\mathcal{F}(k)}. \quad (34)$$

Since $\varepsilon > 0$ and $\gamma(\beta - \beta_0) < 0$ (see (30), (31)), the inequality $3\dfrac{\mathcal{F} - E}{\mathcal{F}} - k^2 - 1 < 0$, must hold in accordance with (33), which gives $k < k_0 \approx 0.95$.

The period X of the function $\varepsilon(x)$ is connected with the quantity q through the formula

$$X = \frac{2\pi}{q} = 2\sqrt{\frac{6}{B}}\frac{1}{\sqrt{\varepsilon}}\mathcal{F}(k). \qquad (35)$$

The variables k and ε can be taken as independent thermodynamic variables instead of β and q.

Expression for the $\varepsilon(x)$ can be expressed in the terms of Jacobi functions

$$\varepsilon(x) = \varepsilon\left(1 - \frac{E(k)}{\mathcal{F}(k)} - k^2\text{sn}^2\left(2\mathcal{F}(k)\frac{x}{X}, k\right)\right), \qquad (36)$$

where the quantity X is defined by (35). Formula (36) determines the long-periodic structure of the system under investigation at temperatures close to the transition temperature T_0.

REFERENCES

1. **Landau L.D.** — Zh. Eksp. Teor. Fiz., 1956, v.30, p.1058. [Sov. Phys. JETP, 1956, v.3, p.920]; Zh. Eksp. Teor. Fiz., 1957, v.32, p.59 [Sov. Phys. JETP, 1957, v.5, p.101].

2. **Silin V.P.** — Zh. Eksp. Teor. Fiz., 1957, v.33, p.495 [Sov. Phys. JETP, 1957, v.6, p.387]; Zh. Eksp. Teor. Fiz., 1958, v.35, p.1243 [Sov. Phys. JETP, 1958, v.8, p.870].

3. **Pomeranchuk I.Ya.** — Zh. Eksp. Teor. Fiz., 1958, v.35, p.524 [Sov. Phys. JETP, 1958, v.8, p.361].

4. **Peletminsky A.S., Peletminsky S.V., Slyusarenko Yu.V.** — Low Temperature Physics, 1999, v.25, p.303.