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NOISE INDUCED PHASE TRANSITIONS IN SPATIALLY EXTENDED SYSTEMS

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We consider systems of spatially distributed and harmonically coupled nonlinear constituents driven by a Gaussian white noise. In contrast to the single constituent one finds noise-induced nonequilibrium *phase transitions* connected with a breaking of ergodicity in the coupled infinite array depending on the control parameter and the strength of the noise and the spatial coupling. We compare the results for global coupling with those for nearest neighbor coupling on cubic lattices. The globally coupled case allows for analytical results and can be considered as a mean-field approximation for the case of nearest neighbor coupling. We discuss first a model which exhibits both continuous and discontinuous phase transitions, the latter one is connected with a hard onset of the order parameter. In a second model we consider a coupling which favours a coherent behaviour of the individual systems (ferromagnetic coupling) and also an antiferromagnetic coupling.

1. INTRODUCTION

Nonlinear systems exposed to external noise are investigated in the two past decades with growing interest both theoretically and in experiments [1].

Zero-dimensional models described by stochastic ordinary differential equations may exhibit bifurcations of the maximum of the stationary probability density for a spatially homogeneous order parameter. Models with spatially distributed nonlinear constituents subject to external noise may show noise induced nonequilibrium *phase transitions*.

In this paper we deal with systems of identical nonlinear constituents where each one is coupled in a harmonic way to any other one (global coupling) or, alternatively, to the nearest neighbors on a cubic lattice which are described by ordinary stochastic differential equations. The case of global coupling is by far easier to investigate and allows even for explicit analytical results [2–4] which may be considered as mean field approximation for the case of nearest neighbor coupling. Shiino [2] extended the concept of phase transitions to nonequilibrium phenomena in systems of globally coupled nonlinear oscillators subject to additive noise. More recently, Van den Broeck et al. [4] demonstrated the appearance of a *second* order noise induced phase transition in a model with multiplicative and additive noise which shows no transitions in the absence of noise. In [5] we constructed a model which exhibits a *first* order noise induced phase transition connected with a hard onset of the coexisting ergodic components of the system. Varying parameters of the system the order of the phase transition may be changed similar to changes from supercritical to subcritical bifurcations observed previously in zero dimensional models [6].

We consider a class of models where the dynamics of the individual constituents x_i at the lattice sites i is governed by a system of stochastic ordinary differential equations in the Stratonovich sense

$$\dot{x}_i = f(x_i) + g(x_i)\,\xi_i - \frac{D}{N} \sum_{j\in\mathcal{N}(i)} (x_i - x_j) \; , \tag{1}$$

where $\mathcal{N}(i)$ denotes the set of involved neighbors of site i and $N = \#\mathcal{N}(i)$ is equal to $L-1$ in the case of global coupling and to $2d$ in the case of nearest neighbor coupling. The parameter D is the strength of the spatial interactions. $\xi_i(t)$ is a zero mean spatially uncorrelated Gaussian white noise with autocorrelation function

$$\left\langle\, \xi_i(t)\,\xi_j(t')\,\right\rangle = \sigma^2\delta_{ij}\delta(t-t') \; , \tag{2}$$

and σ^2 is the noise strength.

The (reduced) stationary probability density $P_s(x_i)$ fulfills the Fokker–Planck equation [4]

$$0{=}\frac{\partial}{\partial x_i}\left(-f(x_i){+}\frac{D}{N}\sum_{j\in\mathcal{N}(i)}(x_i{-}\langle x_j|x_i\rangle){+}\frac{\sigma^2}{2}g(x_i)\frac{\partial}{\partial x_i}g(x_i)\right)P_s(x_i)\; , \tag{3}$$

where $\langle x_j|x_i\rangle = \int dx_j x_j P_s(x_j|x_i)$ denotes the steady state conditional average of x_j, $j\in\mathcal{N}(i)$, given x_i at site i. For the case of global coupling, fluctuations disappear in the average $1/(L-1)\sum_{j\in\mathcal{N}(i)}\langle x_j|x_i\rangle$ if $L\to\infty$, and $\langle x_j|x_i\rangle$ may be determined self-consistently.

In Section 2 we discuss the appearance of first order vs. second order phase transitions for a special model and in Section 3 we extend the method to treat spatially inhomogeneous solutions which are favoured by an antiferromagnetic coupling of the constituents.

2. FIRST VS. SECOND ORDER PHASE TRANSITIONS

Here we consider the case $D>0$ which favours a «ferromagnetic» behaviour of the constituents. Restricting to spatially homogeneous solutions for which $\langle x_j|x_i\rangle$ is independent of lattice site i, $\langle x_j|x_i\rangle$ can be replaced by the steady state mean value $\langle x\rangle$ to be determined self-consistently from

$$\langle x\rangle = \int_{-\infty}^{\infty} dx\, x P_s\left(x,\langle x\rangle\right) \equiv F\left(\langle x\rangle\right) \; , \tag{4}$$

where $P_s\,(x,\langle x\rangle)$ is the formal solution of (3) considering $\langle x\rangle$ as a parameter,

$$P_s(x,\langle x\rangle) \propto \exp\left\{\frac{2}{\sigma^2}\int_0^x dy\,\frac{f(y)-\frac{\sigma^2}{2}g(y)g'(y)-D(y-\langle x\rangle)}{g^2(y)}\right\}. \qquad (5)$$

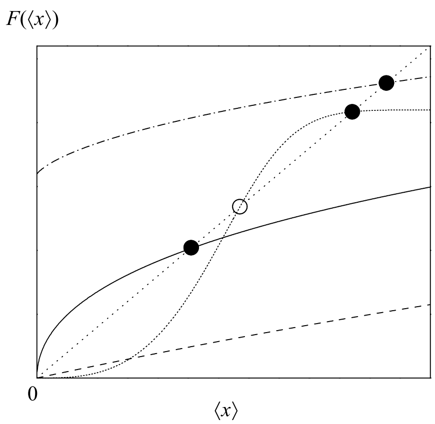

Fig. 1. Solution of the self-consistency equation (4) in typical cases

The solution of the self-consistency equation 4 is visualized in Figure 1 in different situations typically for continuous and discontinuous phase transitions, respectively. In the case of the dashed line the only solution is $\langle x\rangle = 0$. Typical for a continuous transition (solid line) are two stable solutions $\langle x\rangle = \pm x_s$ (full circle), $\langle x\rangle = 0$ is unstable. In the discontinuous case (tightly dotted line) we have a pair of unstable solutions $\langle x\rangle = \pm x_u$ (empty circle) and a pair of stable solutions $\langle x\rangle = \pm x_s$ (full circle) besides the stable solution $\langle x\rangle = 0$. In the latter case the nontrivial solutions appear with nonzero value at the critical value of the control parameter which indicates a *first* order nonequilibrium phase transition. The dash-dotted line shows a typical situation for the system of coupled Stratonovich models considered in Section 3.

A simple model [5] displaying both continuous and discontinuous phase transitions is given by

$$f(x) = ax + x^3 - x^5\,, \qquad g(x) = 1 + x^2\,. \qquad (6)$$

In the case of global coupling the stationary probability density can be obtained explicitly.

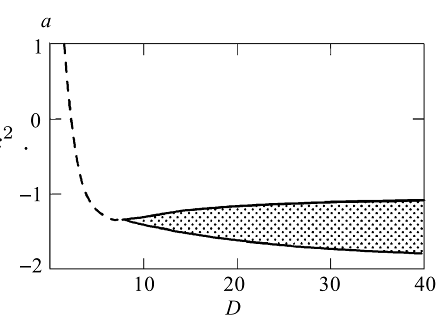

Fig. 2. Phase diagram of model (6) in the case of global coupling, cf. text

Figure 2 shows the phase diagram in the *a*-*D* plane for a given strength of the noise, $\sigma^2 = 1$. For small D we have a second order transition (dashed line). The spatial coupling favours a coherent behaviour of the constituents, acting thus opposite to the noise. With increasing coupling strength D the critical value of a is reduced and above a critical strength of D the first order transition (solid line) of the model without noise and spatial coupling is «restored». The number of ergodic components is three in the shadowed region, two in the region above and one in the region below. Hysteresis appears in the shadowed region.

3. FERROMAGNETIC VS. ANTIFERROMAGNETIC COUPLING

The analytic results for infinite systems with global coupling give a very good idea of the behaviour of finite systems or systems with nearest neighbor coupling. Figure 3 compares the order parameter obtained by simulation for a 2-dimensional square lattice of size $L = 100 \times 100$ with the results for the globally coupled model for $D = 30$. The diamonds denote the average of $x_i(t)$ over all lattice sites and over a time span of order 100 during which no jumps between the ergodic components occur. The error bars indicate the time average of the standard deviation. Figures 3,*a* and *b* show the order parameter as a function of the control parameter a ($\sigma^2 = 1$) and the noise strength σ^2 ($a = -1.5$), respectively.

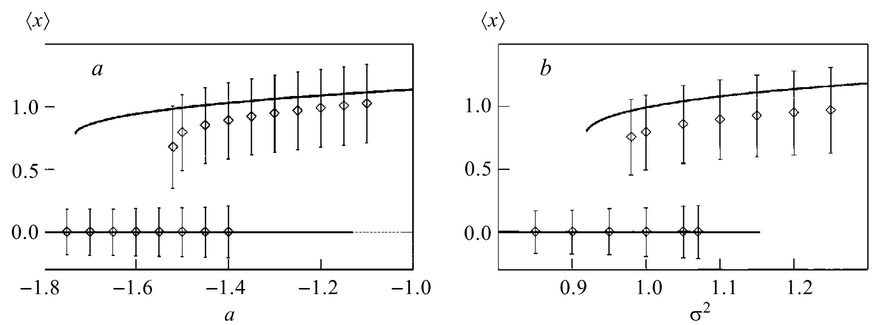

Fig. 3. Order parameter $\langle x \rangle$ as a function of a and σ^2 for a 2-dimensional square lattice (diamonds) and the globally coupled case (thick solid line)

In this Section we consider a system of coupled Stratonovich models specified by

$$f(x) = ax - x^3 \,, \qquad g(x) = x \,. \tag{7}$$

The stationary probability density in the globally coupled case is easily obtained from (5) as

$$P_s(x, \langle x \rangle_s) \propto |x|^{2(a-D)/\sigma^2 - 1} \exp\left\{-\left(x^2 + 2D\langle x \rangle_s / x\right)/\sigma^2\right\}, \tag{8}$$

provided $D\langle x \rangle_s / x \geq 0$, otherwise P_s is zero because the above expression is not normalizable then. Note that $\langle x \rangle_s$ is the *spatial* average over all involved sites.

For $D > 0$ a solution is found where all constituents have the same (statistical or temporal) average $\langle x \rangle$ as it is typical for ferromagnets; in this case we have $\langle x \rangle = \langle x \rangle_s$. Depending on the parameters a, σ, and D one finds continuous transitions from zero to nonzero values of $\langle x \rangle$ determined as solutions of Equation (4).

For $D < 0$ an 'antiferromagnetic' solution is preferred. We find two subsystems labeled by $+$ and $-$ respectively, for which the averages $\langle x \rangle$ have opposite sign. In the globally coupled case they are given by

$$\langle x_+ \rangle = \int_0^\infty dx\, x P_s(x, \langle x \rangle_s) = -\int_{-\infty}^0 dx\, x P_s(x, \langle x \rangle_s) = -\langle x_- \rangle, \tag{9}$$

where P_s is obtained from (8) by inserting $\langle x \rangle_s = (\langle x_+ \rangle + \langle x_- \rangle)/2 = 0$. For nearest neighbor coupling on a cubic lattice, the subsystems correspond just to the two Néel sublattices.

An antiferromagnetic solution exists also for $D > 0$. It is less stable than the ferromagnetic solution in the following sense. For the model with global coupling we prepared such initial conditions that a fraction λ of the constituents has, say, positive initial values; and a fraction $1 - \lambda$, negative ones. We then simulated the dynamics of the system and determined the first time for which one of the constituents changed the sign (first passage time), cf. Figure 4.

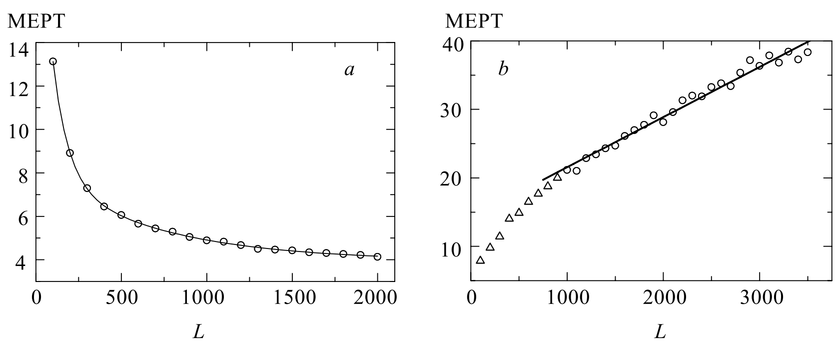

Fig. 4. Mean first passage time as a function of the system size L for globally, ferromagnetically coupled Stratonovich models. The initial conditions are characterized by $\lambda \neq 1/2$ in (a) and $\lambda = 1/2$ in (b). The lines indicate exponential and linear fits, respectively. For other parameters and discussion see text

We found that for $\lambda \neq 1/2$, the mean first passage time (MFPT) decreases exponentially and the system reaches very fast the ferromagnetic state. For $\lambda = 1/2$ the MFPT increases with the system size L and one expects that it diverges in the limit $N \to \infty$. However, the antiferromagnetic state is only metastable since an additive noise can easily lead to $\lambda \neq 1/2$ and finally to a ferromagnetic state.

Figure 4,a shows the MFPT for a system with initial conditions characterized by $\lambda = 0.4$ (parameter values $a = 1.5$, $D = 0.5$, $\sigma^2 = 0.3$) and Figure 4,b the same for $\lambda = 1/2$ (parameter values $a = 4$, $D = 2.5$, $\sigma^2 = 0.4$). The average is over 10^3 samples.

A more detailed account on the model discussed in this Section will be published elsewhere, cf. also [7].

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