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In the algebraic Interacting Vector Boson Model (IVBM) it is assumed that the nuclear dynamics can be described by means of two types of vector «quasiparticles», which are also characterized by another quantum number — a «T spin» (an analogue to the F spin). The noncompact symplectic group $\mathrm{Sp}\,(12,R)$ appears as the group of dynamical symmetry for the problem of two interacting vector bosons. The symplectic structure allows the change in the number of phonons, needed to build the collective states, that results in larger model spaces, which can accommodate the more complex structural effects as observed in the contemporary experiment.

The applications of the IVBM are extended by exploiting three new subgroup chains in the reduction of $\mathrm{Sp}\,(12,R)$ to the physical angular momentum subgroup $SO(3)$. The corresponding exactly solvable limiting cases are applied to achieve a description of complex nuclear collective spectra of even–even nuclei in the rare-earth and actinide regions up to states of very high angular momentum.

The first reduction that we exploit is one that extends the rotational limit of the number preserving version of the model; namely, $\mathrm{Sp}\,(12,R) \supset U(6) \supset U(2) \otimes SU(3)$. Another limit of the symplectic IVBM, $\mathrm{Sp}\,(12,R) \supset \mathrm{Sp}\,(2,R) \otimes SO(6)$, contains in a natural way the six-dimensional Davidson potential. In both of these cases, because collective modes can be mixed, we obtain successful descriptions of both positive and negative parity band configurations.

The structure of band-head configurations, whose importance is established in the first two limits, is also examined in the third reduction, $\mathrm{Sp}\,(12,R) \supset \mathrm{Sp}\,(4,R) \otimes SO(3)$. The distribution of energies that are obtained in this limit with respect to the number of bosons that build each of the states with fixed angular momentum, enables one to distinguish typical collective vibrational and rotational spectra. This algebraic chain also provides important links between the subgroups of the other limits.

The symplectic extension of the IVBM permits a richer classification of the states than its unitary version and is shown to be appropriate for a description of rather diverse nuclear spectra.

Алгебраическая модель взаимодействующих векторных бозонов (МВВБ) предполагает описание ядерной динамики в терминах пар векторных фононов, различающихся по квантовому числу — «T-спину» (аналог F-спина). Некомпактная симплектическая группа $\mathrm{Sp}\,(12,R)$ представляет собой группу динамической симметрии для пар взаимодействующих векторных бозонов, расширенное модельное пространство позволяет конструировать коллективные состояния при изменении числа фононов и объяснять более тонкие эффекты коллективной ядерной структуры, наблюдаемые в эксперименте.

Расширение МВВБ определяется тремя новыми цепочками редукции $\mathrm{Sp}\,(12,R)$ к физической подгруппе углового момента $SO(3)$. Точно решаемые предельные случаи с хорошей степенью точности описывают спектры четно-четных ядер из редкоземельной области и актинидов, вплоть до очень больших значений угловых моментов возбужденных состояний.

Первая рассмотренная нами цепочка редукции представляет собой расширение ротационного предела подхода, сохраняющего число бозонов $\mathrm{Sp}\,(12,R) \supset U(6) \supset U(2) \otimes SU(3)$. Вторая — $\mathrm{Sp}\,(12,R) \supset \mathrm{Sp}\,(2,R) \otimes SO(6)$ — включает в себя 6-мерный потенциал Давидсона. В обоих случаях из-за смешивания коллективных мод удается одновременно и успешно описывать полосы с положительной и отрицательной четностью.

Структура основных состояний полос, исследованных в двух первых предельных случаях, рассматривается и в случае редукции $\mathrm{Sp}\,(12,R) \supset \mathrm{Sp}\,(4,R) \otimes SO(3)$. Полученное распределение энергий как функции числа бозонов, определяющих каждое состояние, позволяет различать принадлежность этого состояния колебательному или вращательному спектру. Кроме того, последняя цепочка задает важную связь между подгруппами предыдущих редукций.

По сравнению с традиционной версией, требующей сохранения числа бозонов, предложенное симплектическое расширение МВВБ позволяет проводить более точную классификацию спектров возбужденных состояний ядер, принадлежащих различным полосам, а также учитывать взаимодействия этих полос.

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1. INTERACTING TWO-VECTOR-BOSON MODEL OF COLLECTIVE MOTIONS IN NUCLEI

1.1. Introduction. With the advent of ever improving experimental nuclear physics facilities, a large repository of data on the structure of atomic nuclei is being amassed [1]. This data reveals the complex nature of nuclear degrees of freedom, requiring different mixings of the basic rotational and vibrational collective modes. This in turn calls for better and often more sophisticated nuclear structure models, that should remain tractable and easy to apply while yielding very reasonable theoretical interpretations of the data. Algebraic models that are based on symmetry principles, such as the Interacting Boson Model (IBM) [2], have proven themselves to be useful for this purpose. Most importantly, these models have established good benchmarks for gaining a better understanding of the evolution of nuclear structure [3] and have provided an underpinning for our beginning to understand the nature of phase and shape transitions between collective modes [4]. Nevertheless, it seems that along the way some of the beauty and simplicity of the group theoretical approach to the nuclear structure has yielded to more geometrical considerations in terms of collective model variables [4, 5] and/or to more brute force large-scale shell-model investigations [6].

It is a well-known fact that nuclear collective motions can be described qualitatively by means of interacting bosons [7]. A typical example of this is the anharmonic quadrupole oscillator model. Its Hamiltonian is

$$H = \sum_{M} (-1)^M b_{2M}^{\dagger} b_{2-M} + V(b_{2M}^{\dagger}, b_{2M}),$$

where $b_{2M}^{\dagger}$ and b_{2M} are creation and annihilation operators of quadrupole phonons with an angular momentum projection M, and $V(b_{2M}^{\dagger}, b_{2M})$ is a scalar function (usually a polynomial) describing the phonon anharmonic interaction.

The analytical diagonalization of H and the exact (nonperturbational) solution of the Schrödinger equation can be significantly simplified provided that the space of the eigenstates of the system forms a basis of a single unitary irreducible representation (UIR) of a certain group G, which is a group of dynamical symmetry in the sense of Dashen and Gell-Mann [8]. The Hamiltonian H breaks the G symmetry and reduces it to the symmetry of the rotational group $O(3)$ defined by the angular momentum operators. In this way the degeneracy of the states is removed and the observable energy spectrum is generated.

A number of algebraic models describing the collective properties of nuclei have been built on this basis. The first one that is based on the symmetry group $SU(3)$ for the classification of the many nucleon states is the microscopic Elliott's model of nuclear rotations [9]. This model not only simplifies the calculations of the rotational states in the p- and sd-shell nuclei, but gives a physical insight into the structure of the nuclear wave functions and provides a way of constructing the effective internucleon interactions. Additionally the $SU(3)$ classification of many-body states allows for a geometrical analysis of the eigenstates of the nuclear system [10] and hence gives an interpretation of phenomena associated with nuclear deformation. It became a basis for the development of the algebraic approaches into the nuclear structure, many of which just explore the symmetries on a pure phenomenological level using the tensor properties of the involved physical observables.

For example, a two-vector-boson model has been constructed by Raychev [11], Raychev and Roussev [12], and Karadjov et al. [13] for the description of the collective properties of heavy even–even deformed nuclei in the framework of the broken $SU(3)$ symmetry. In the early 1980s, a boson-number-preserving version of the phenomenological algebraic Interacting Vector Boson Model (IVBM) [14], including the interactions of the bosons of different types and based on $U(6)$ spectrum generating algebra was introduced. This version of the IVBM was applied successfully [15] to a description of the low-lying collective rotational spectra of the even–even medium and heavy mass nuclei.

Another example is the phenomenological model of interacting bosons (IBM) of Arima and Iachello [16–19], which introduces s and d bosons considered as fermion pairs coupled to a total angular momentum $L = 0$ and $L = 2$, respectively. This brings in again the group $U(6)$ as a group of dynamical symmetry of collective motions in nuclei. It had been shown even earlier [20], that in the framework of a microscopic description of the collective modes, it is quite natural to introduce six collective variables related to the monopole and quadrupole degrees of freedom, as the same type of algebraic structure is generated by the five generalized coordinates of the quadrupole degree of freedom, their conjugated momenta and the commutators between them [21]. These variables also generate the group $U(6)$ proposed by Arima and Iachello [16–18] and Castanos et al. [19]. Using the method of boson representations of fermion operators, the two-quasiparticle structure of the generalized coordinates and impulses is obtained and the structure constants of the algebra are expressed as sums of products of the four amplitudes that define their microscopic structure [22]. The so-obtained collective quadrupole nuclear Hamiltonian defines the Truncated Quadrupole Phonon Model (TQM), which has successful applications in the description of nuclear collective spectra not only of spherical nuclei, but of deformed ones as well [23].

On the other hand, even during early stages of development of algebraic methods for application in nuclear structure theory [24–26], the separation of collective and intrinsic degrees of freedom was realized within the framework of the overarching symplectic geometry of the nuclear many-body problem. The Symplectic Shell Model (for example, [25, 27, 28]) which is considered as a microscopic realization of the successful Borh–Mottelson–Frankfurt [29] collective model is realized as a multy-$\hbar\Omega$ extension of the Elliott's $SU(3)$ model and is based on the noncompact $\mathrm{sp}\,(6,R)$ algebra, which has a very rich subalgebraic structure. This allows for the description of rotational dynamics in a continuous range from irrotational to rigid rotor flows. Recently, the expansion of the most important $\mathrm{sp}\,(6,R)$ symmetry adapted components of the symplectic shell model wave functions [30] in the spherical harmonic oscillator basis, utilized by the prominent — no-core shell model [31], demonstrated convincingly the importance of the symplectic symmetry in light nuclei, while reaffirming the value of the simpler $SU(3)$ model upon which it is based.

In general, the advantages of exploiting symplectic structures, in addition to gaining a direct geometrical interpretation, are quite transparent when a change in the number of phonons needed to build collective states is included in the theory as this results in larger model spaces that can accommodate the more complex structural effects as realized in nuclei with nucleon numbers that lie far from the magic numbers of closed shells.

The model discussed in this review is a natural generalization of the $SU(3)$ model developed by Raychev and co-workers [11–13]. Its main assumption is that nuclear dynamics can be described by means of two types of vector quasiparticles, which are also characterized by another quantum number, which was initially called a «pseudospin» [14]. Here we will use the more appropriate term «T spin» (an analogue to the F spin in IBM-2), as the term «pseudospin» is now known as introduced for single particle levels in [32]. These vector bosons form a «T-spin» doublet and they differ in their «T-spin» projection $\alpha = \pm 1/2$. The introduction of this additional degree of freedom leads to the extension of the $SU(3)$ dynamical symmetry to $U(6)$. It is not necessary to consider the bosons as fermion pairs coupled to $L = 1$ in order to justify this assumption; we would rather treat them as kinds of «oscillator quarks» or «Elliott quarks» as in the well-known model of Elliott [9] (see also Lipkin [33]).

In general, instead of considering the vector bosons as real nuclear excitations, there can be another approach to the problem. It is only the tensor characteristics of the operators built from the generators of some group G and belonging to its universal enveloping algebra that play an important role in the diagonalization of the Hamiltonian in the basis of the UIR of the dynamical group G, as well as in the determination of the matrix elements of the transition operators. The construction of the representations of G and the realization of its generators can be achieved by means of boson operators, which in this case will not have an immediate physical significance but will only play the role of formal auxiliary apparatus. On the other hand, however, the idea, that the bosons used in this case represent the real excitations, is more attractive, and for this reason we will retain it in our further considerations.

As suggested by Rargmann and Moshinsky in [34], two types of bosons are needed for the description of nuclear dynamics. It is shown there that the consideration of only a two-body system consisting of two different interacting particles will suffice to give a complete description of N three-dimensional oscillators with a quadrupole–quadrupole interaction.

One of the purposes of the present investigation is to show that the non-compact symplectic group $\mathrm{Sp}\,(12, R)$ is the group of dynamical symmetry for the problem of two interacting vector bosons. In this sense the two-vector-boson model is an expansion of the $\mathrm{Sp}\,(6, R)$ model of Rosensteel and Rowe [27,28,35]. It should be noted that the $\mathrm{Sp}\,(6, R)$ model was originally suggested for the description of deformed even–even nuclei by Raychev [36] and was further investigated microscopically by Asherova et al. [37], where a more general group decomposition $\mathrm{Sp}\,(6A, R) \supset \mathrm{Sp}\,(6, R) \otimes O(A)$ (A is the number of nucleons) was introduced. Although the IVBM is developed on a pure phenomenological level, the clear similarities of its group structure to the one of the $SU(3)$, $SU(6)$, and $\mathrm{Sp}\,(6, R)$ microscopic approaches gives an open possibility to explore its relations to these models, which, as a future task, will lead to establishing its relations to the microscopic foundations of the nuclear collective motion.

With the aim of extending the earlier applications of the IVBM to incorporate new experimental data on states with higher spins and to incorporate new excited bands, we explore the symplectic extension of the IVBM, for which the dynamical symmetry group is $\mathrm{Sp}\,(12, R)$. This extension is realized from, and has its physical interpretation over basis states of its maximal compact subgroup $U\,(6) \subset \mathrm{Sp}\,(12, R)$, which was identified as the rotational limit [15] of the model. This led naturally to the description [38] of not just energies, but of finer structural effects like a staggering of levels between the ground and octupole bands up to states of very high spins.

The symplectic extension of the IVBM provided a useful tool to obtain its relation [39] to a version of the IBM that contains the s and d bosons with F spin $F = 1$ and a p boson with $F = 0$, by means of a mapping procedure that uses pairs of vector bosons as constituents of the IBM ones. Even earlier the equivalence of the IBM and the microscopic TQM [27] has been established in respect to the matrix elements of their physical observables and their corresponding operators [40]. Something more, the strict mathematical proof of the unitary equivalence of the physical boson spaces of the basis states of the two models also exists [41], which is the reason for considering the two models as different realizations of the same phenomenological $SU(6)$-boson model. The established relationship between the IVBM and the IBM [39] with its equivalence to the TQM reveals another possibility to explore the microscopic foundation of the model presented in this review.

In the dynamical $\mathrm{Sp}\,(12, R)$ group there are as well additional chains of subgroups, starting with noncompact symplectic subalgebras which in larger infinite boson spaces lead to some new structural effects. One interesting example is the reduction of $\mathrm{Sp}\,(2, R) \otimes SO(6) \subset \mathrm{Sp}\,(12, R)$, which allows for the inclusion of a six-dimensional Davidson potential [42] into the theory that is known to accommodate mixing of rotational and vibrational modes and hence allows one to reproduce the spectra of transitional nuclei.

The next important chain [43] $\mathrm{Sp}\,(4, R) \otimes SO(3) \subset \mathrm{Sp}\,(12, R)$ allows the selection of states with fixed angular momentum L, given by the $SO(3)$ irreducible representations (irreps). The physics behind this dynamical symmetry outlines the importance of the structure of the band-head configurations in the development of the nuclear spectra, established in the applications of the first two reductions and yields important relations between the subgroups of the first two limits.

At the end, the dynamical symmetries mentioned above are unified in a generalized reduction scheme for the symplectic extension of the IVBM that contains the relations between the subgroups from the different chains. These clarify the physical meaning and motivation of the model and lead to rather successful applications for the description of the large amount of experimental data on the nuclear structure, while still retaining the advantages of the use of dynamical symmetries, namely the exact analytic solutions for the energy spectra of nuclei.

We start our review, by presenting the rich and beautiful structure of the maximal compact subgroup of $\mathrm{Sp}\,(12, R)$, namely the group $U(6)$, on which we base the symplectic extension of the Interacting Vector Boson Model.

1.2. Building Blocks of the Model. If $x(\alpha), y(\alpha)$, and $z(\alpha)$ are the Cartesian coordinates of a quasiparticle with projection of the «T spin» $\alpha = \pm 1/2$, the corresponding cyclic coordinates are

$$x_{\pm 1}(\alpha) = \mp \frac{x(\alpha) \pm iy(\alpha)}{\sqrt{2}}, \qquad x_0(\alpha) = z(\alpha). \tag{1}$$

The momenta associated with these coordinates are

$$q_m(\alpha) = -i\,\partial/\partial x^m(\alpha), \tag{2}$$

where

$$x^m = \sum_n g^{mn} x_n, \qquad g^{mn} = g_{mn} = (-1)^n \delta_{m,\,-n}.$$

Using (1) and (2) one can introduce creation and annihilation operators of bosons with angular momentum $l = 1$, projection m, and «T-spin» projection α as follows:

$$\begin{aligned} u_m^\dagger(\alpha) &= \big(x_m(\alpha) - iq_m(\alpha)\big)/\sqrt{2}, \\ u^m(\alpha) &= \big(x^m(\alpha) + iq^m(\alpha)\big)/\sqrt{2}. \end{aligned} \tag{3}$$

These operators satisfy the commutation relations

$$\big[u^m(\alpha), u_n^\dagger(\beta)\big] = \delta(\alpha,\beta)\,\delta_{m,\,n}$$

and Hermitian conjugation rules

$$\big[u_m^\dagger(\alpha)\big]^\dagger = u^m(\alpha), \qquad \big[u^m(\alpha)\big]^\dagger = u_m^\dagger(\alpha).$$

When convenient, we also use the notations

$$u_m^\dagger\left(\alpha = \frac{1}{2}\right) = p_m^\dagger, \qquad u_m^\dagger\left(\alpha = -\frac{1}{2}\right) = n_m^\dagger. \tag{4}$$

The operators $\mathbf{p}^\dagger$ and $\mathbf{n}^\dagger$ can be considered as three-dimensional vectors (with respect to the group $O(3)$) belonging to two independent representations $(1,0)$ of the group $SU(3)$ (the operators $\mathbf{p}$ and $\mathbf{n}$ belong to the representation $(0,1)$). The operators (3) can also be realized as double tensors with respect to the angular momentum $so_L(3)$ algebra, labelled by $l = 1$ $(-l \leqslant m \leqslant l)$, and of rank $f = 1/2$ $(-f \leqslant \tau \leqslant f)$ (with respect to the $su_T(2)$ algebra), which satisfy the commutation relations:

$$\left[u_{f\tau}^{lm}, u_{f'\tau'}^{\dagger\,l'm'}\right] = \delta_{ll'}\delta_{mm'}\delta_{ff'}\delta_{\tau\tau'}. \tag{5}$$

The vector bosons presented by (3) and (4) play the role of «quasiparticle quarks» and can be used for the construction of other quasiparticle excitations with different angular momenta. Thus, for example, the s and d bosons from the IBM can be considered as bound states of the bosons (4). In this sense the group $SU(3)$ is the group describing the fundamental collective excitations in nuclei.

Now we introduce the following operators:

$$F^L_M(\alpha,\beta) = \sum_{k,m} C^{LM}_{lk,\,lm}\, u^\dagger_k(\alpha) u^\dagger_m(\beta), \tag{6a}$$

$$G^L_M(\alpha,\beta) = \sum_{k,m} C^{LM}_{lk,\,lm}\, u_k(\alpha) u_m(\beta), \tag{6b}$$

which create or annihilate a pair of bosons with angular momentum L and projection M. We also introduce the operators

$$A^L_M(\alpha,\beta) = \sum_{k,m} C^{LM}_{lk,\,lm}\, u^\dagger_k(\alpha) u_m(\beta), \tag{6c}$$

which preserve the number of bosons. The operators (6a)–(6c) introduced so far have the following properties:

$$\begin{aligned} &F^L_M(\alpha,\beta) = (-1)^L F^L_M(\beta,\alpha), && G^L_M(\alpha,\beta) = (-1)^L G^L_M(\beta,\alpha),\\ &[F^L_M(\alpha,\beta)]^\dagger = (-1)^M G^L_{-M}(\beta,\alpha), && [G^L_M(\alpha,\beta)]^\dagger = (-1)^M F^L_{-M}(\beta,\alpha).\\ &[A^L_M(\alpha,\beta)]^\dagger = (-1)^M A^L_{-M}(\beta,\alpha), && \end{aligned} \tag{7}$$

We will show further that the operators (6a)–(6c) play the role of the generators of the $\mathrm{sp}\,(12,R)$ algebra (or, in the general case, $\mathrm{sp}\,(2d,R)$ algebra with $d=(2l+1)(2f+1)$). They can also be expressed in terms of double tensors of the angular momentum L and the T-spin T.

$$F^{LM}_{Tt} = (G^{LM}_{Tt})^\dagger = \frac{1}{\sqrt{2}} \sum_{m_1\, m_2\, \tau_1\, \tau_2} C^{LM}_{lm_1,\,lm_2} C^{Tt}_{f\tau_1,\,f\tau_2} u^{\dagger\, lm_1}_{f\tau_1} u^{\dagger\, lm_2}_{f\tau_2} = \frac{1}{\sqrt{2}} \left(u^{\dagger\, l}_f \otimes u^{\dagger\, l}_f \right)^{LM}_{Tt}, \tag{8a}$$

$$A^{LM}_{Tt} = \sum_{m_1\, m_2\, \tau_1\, \tau_2} C^{LM}_{lm_1,\,lm_2} C^{Tt}_{f\tau_1,\,f\tau_2} u^{\dagger\, lm_1}_{f\tau_1} u^{lm_2}_{f\tau_2} (-1)^{l-m_2} (-1)^{f-\tau_2} = \left(u^{\dagger\, l}_f \otimes \tilde{u}^l_f \right)^{LM}_{Tt}, \tag{8b}$$

where

$$\tilde{u}^{lm}_{f\tau} = (-1)^{l+m}(-1)^{f+\tau} u^{l\,-m}_{f\,-\tau}. \tag{9}$$

The symmetry properties of the Clebsh–Gordan coefficients $C^{LM}_{lm,\,ln}$ result in some restrictions on the possible values of the angular momentum L and isospin T of two-boson generators F^{LM}_{Tt} and G^{LM}_{Tt} (8a), namely

$$(-1)^{2l-L}(-1)^{2f-T} = 1. \tag{10}$$

2. THE ALGEBRAIC STRUCTURE GENERATED BY THE TWO VECTOR BOSONS

The algebraic structure, generated by the operators $F^L_M(\alpha,\beta)$, $G^L_M(\alpha,\beta)$ and $A^L_M(\alpha,\beta)$ (78 in number) is clarified by their commutation relations:

$$[A^{L_1}_{M_1}(\alpha,\beta),\ F^{L_2}_{M_2}(\alpha',\beta')] = \\ = (-1)^{L_2}[(2L_1+1)(2L_2+1)]^{1/2}\sum_{L,M} C^{LM}_{L_1M_1,\,L_2M_2}\left\{\begin{array}{ccc} L_1 & L_2 & L \\ 1 & 1 & 1 \end{array}\right\}\times \\ \times\left[F^L_M(\alpha',\alpha)\delta(\beta,\beta') + (-1)^{L_1}F^L_M(\beta',\alpha)\delta(\alpha',\beta)\right], \quad (11a)$$

$$[A^{L_1}_{M_1}(\alpha,\beta),\ G^{L_2}_{M_2}(\alpha',\beta')] = \\ = (-1)^{L_1}[(2L_1+1)(2L_2+1)]^{1/2}\sum_{L,M} C^{LM}_{L_1M_1,\,L_2M_2}\left\{\begin{array}{ccc} L_1 & L_2 & L \\ 1 & 1 & 1 \end{array}\right\}\times \\ \times\left[(-1)^{L_2}G^L_M(\alpha',\beta)\delta(\alpha,\beta') + G^L_M(\beta',\beta)\delta(\alpha,\alpha')\right], \quad (11b)$$

$$[F^{L_1}_{M_1}(\alpha,\beta),\ G^{L_2}_{M_2}(\alpha',\beta')] = \\ = [(2L_1+1)(2L_2+1)]^{1/2}\sum_{L,M}(-1)^L C^{LM}_{L_1M_1,\,L_2M_2}\left\{\begin{array}{ccc} L_1 & L_2 & L \\ 1 & 1 & 1 \end{array}\right\}\times \\ \times\left[(-1)^{L_2}A^L_M(\alpha,\alpha')\delta(\beta,\beta') + A^L_M(\alpha,\beta')\delta(\alpha',\beta) + \right. \\ \left. + (-1)^{L_1+L_2}A^L_M(\beta,\alpha')\delta(\alpha,\beta') + (-1)^{L_1}A^L_M(\beta,\beta')\delta(\alpha,\alpha')\right] + \\ + (-1)^{M_1+1}\delta_{L_1,\,L_2}\delta_{M_1,\,-M_2}\left[\delta(\alpha,\beta')\delta(\alpha',\beta) + (-1)^{L_2}\delta(\alpha,\alpha')\delta(\beta,\beta')\right], \quad (11c)$$

$$[A^{L_1}_{M_1}(\alpha,\beta),\ A^{L_2}_{M_2}(\alpha',\beta')] = \\ = [(2L_1+1)(2L_2+1)]^{1/2}\sum_{L,M} C^{LM}_{L_1M_1,\,L_2M_2}\left\{\begin{array}{ccc} L_1 & L_2 & L \\ 1 & 1 & 1 \end{array}\right\}\times \\ \times\left[(-1)^{L_1+L_2}A^L_M(\alpha',\beta)\delta(\alpha,\beta') - (-1)^L A^L_M(\alpha,\beta')\delta(\alpha',\beta)\right]. \quad (11d)$$

It is evident from (11a)–(11c) and the definitions (6a)–(6c) that these operators define an algebra of the noncompact symplectic group $\mathrm{Sp}\,(12,R)$ and also that the 36 operators $A^L_M(\alpha,\beta), (\alpha,\beta=\pm1/2, L=0,1,2)$ generate an algebra of the maximal compact subgroup of $\mathrm{Sp}\,(12,R)$, namely the group $U(6)$.

In terms of the double tensors (8b) the commutation relations (11d) can be expressed in a more compact form:

$$\left[A^{L_1M_1}_{T_1t_1}, A^{L_2M_2}_{T_2t_2}\right] = \hat{L}_1\hat{L}_2\hat{T}_1\hat{T}_2 \sum_{LT} \left[(-1)^{L+T} - (-1)^{L_1+L_2+T_1+T_2}\right] \times \\ \times C^{LM}_{L_1M_1,\,L_2M_2} C^{Tt}_{T_1t_1,\,T_2t_2} \left\{ \begin{array}{ccc} L_1 & L_2 & L \\ l & l & l \end{array} \right\} \left\{ \begin{array}{ccc} T_1 & T_2 & T \\ f & f & f \end{array} \right\} A^{LM}_{Tt}. \quad (12)$$

Here and later on we use the notation $\hat{L} = \sqrt{2L+1}$, and the curly brackets $\{\}$ denote the $6-j$ symbols. It is evident from (12) that, in general, operators A^{LM}_{Tt}, which do not change the number of bosons, generate a maximal compact subalgebra of $\mathrm{sp}\,(2d, R)$, namely the $u(d)$ algebra. The operator of the total number of bosons

$$N = \sum_{m\tau} u^{\dagger\, m}_{\tau} u^{m}_{\tau}$$

is connected with the multipole operator A^{00}_{00} in the following way:

$$N = \sqrt{(2l+1)(2f+1)}A^{00}_{00} = \sqrt{d}A^{00}_{00}.$$

The commutation relations for the rest of the $\mathrm{sp}\,(2d, R)$ generators are generalized in the following way:

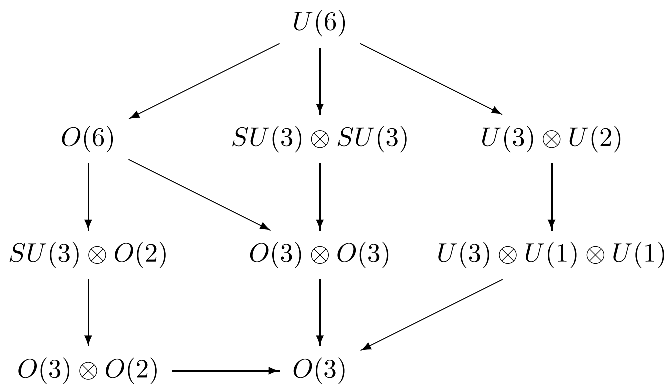
$$\left[A^{L_1M_1}_{T_1t_1}, G^{L_2M_2}_{T_2t_2}\right] = -2\hat{L}_1\hat{L}_2\hat{T}_1\hat{T}_2(-1)^{M_1+t_1} \sum_{LT} C^{LM}_{L_1M_1,\,L_2M_2} C^{Tt}_{T_1t_1,\,T_2t_2} \times \\ \times \left\{ \begin{array}{ccc} L_1 & L_2 & L \\ l & l & l \end{array} \right\} \left\{ \begin{array}{ccc} T_1 & T_2 & T \\ f & f & f \end{array} \right\} G^{LM}_{Tt}, \quad (13)$$

$$\left[G^{L_1M_1}_{T_1t_1}, F^{L_2M_2}_{T_2t_2}\right] = \delta_{L_1L_2}\delta_{M_1M_2}\delta_{T_1T_2}\delta_{t_1t_2} + 2\hat{L}_1\hat{L}_2\hat{T}_1\hat{T}_2(-1)^{M_1+t_1} \times \\ \times \sum_{LT} C^{LM}_{L_1M_1,\,L_2M_2} C^{Tt}_{T_1t_1,\,T_2t_2} \left\{ \begin{array}{ccc} L_1 & L_2 & L \\ l & l & l \end{array} \right\} \left\{ \begin{array}{ccc} T_1 & T_2 & T \\ f & f & f \end{array} \right\} A^{LM}_{Tt}, \quad (14)$$

$$\left[F^{L_1M_1}_{T_1t_1}, F^{L_2M_2}_{T_2t_2}\right] = \left[G^{L_1M_1}_{T_1t_1}, G^{L_2M_2}_{T_2t_2}\right] = 0. \quad (15)$$

These commutation relations are similar to the commutation relations of the corresponding bifermion operators [44] except for opposite signs at some terms.

2.1. $U(6)$ **Subalgebraic Structure.** From the commutation relations (11a)–(11c) it follows that the $U(6)$ algebra contains the following chains of subgroups:



(16)

Now we are going to briefly enumerate these algebras, their generators, and second-order Casimir operators:

(1) The $U(6)$ algebra is generated by the operators $A^L_M(\alpha, \beta)$ (6c) ($L = 0, 1, 2$; $\alpha, \beta = +1/2 \to (p), -1/2 \to (n)$). They satisfy the commutation relations (11c) and their second-order Casimir operator is

$$K_6 = \sum_{L,M} \sum_{\alpha,\beta} (-1)^M A^L_M(\alpha, \beta) A^L_{-M}(\beta, \alpha) = N(N+5),$$

where

$$N = -\sqrt{3}\big(A^0(p,p) + A^0(n,n)\big) \tag{17}$$

is the operator of the number of particles.

(2) The angular momentum algebra $O(3)$, defined by the angular momentum operators

$$L_M = -\sqrt{2}\big(A^1_M(p,p) + A^1_M(n,n)\big), \tag{18}$$

which satisfy the usual commutation relations

$$[L_{M_1}, L_{M_2}] = -\sqrt{2}\, C^{1M_1+M_2}_{1M_1,\,1M_2} L_{M_1+M_2}. \tag{19}$$

This group inevitably appears at the end of each chain of subgroups and has the following second-order Casimir operator:

$$\pi_3 = \sum_{M,\alpha,\beta} (-1)^M A^1_M(\alpha, \alpha) A^1_{-M}(\beta, \beta) = \frac{1}{2} L^2.$$

(3) $SU(3) \otimes U(2)$ algebra consists of two commuting sets of operators:

(a) The $U(2)$ algebra, defined by the operator of a number of particles N (17) and the «T-spin» operators $T^1_m, (m = 0, \pm 1)$ introduced through

$$
\begin{aligned}
T_1 &= \sqrt{\frac{3}{2}}\, A^0(p,n), \\
T_{-1} &= -\sqrt{\frac{3}{2}}\, A^0(n,p), \\
T_0 &= -\frac{1}{2}\sqrt{3}\left(A^0(p,p) - A^0(n,n)\right).
\end{aligned}
\tag{20}
$$

The above operators T^1_m $(m = 0, \pm 1)$ commute with N and satisfy the standard commutation relations of the type (19). Thus (20) define the subalgebra $su(2) \subset u(2)$. These operators (20) play an important role in the consideration of the nuclear system as composed by two interacting neutron and proton subsystems. The second-order Casimir operator of $U(2)$ is

$$K_2 = \frac{4}{3}T^2 + \frac{1}{3}N^2.$$

(b) The $U(3)$ algebra. It consists of the operators N (17), L_M (18) and

$$\tilde{Q}_M = \sqrt{6}\big(A^2_M(p,p) + A^2_M(n,n)\big), \tag{21}$$

which can be interpreted as operators of the truncated («Elliott») quadrupole momentum [9]. The operators L_M and $\tilde{Q}_M$ commute with N and define the subalgebra $su(3) \subset u(3)$. The standard commutation relations between these operators are

$$
\begin{aligned}
[L_{M_1}, \tilde{Q}_{M_2}] &= \sqrt{30}\, C^{1\,M_1+M_2}_{2M_1,\,2M_2} \tilde{Q}_{M_1+M_2}, \\
[\tilde{Q}_{M_1}, \tilde{Q}_{M_2}] &= 3\sqrt{10}.
\end{aligned}
$$

The second-order Casimir operator of $U(3)$ is

$$K_3 = \frac{1}{6}\tilde{Q}^2 + \frac{1}{2}L^2 + \frac{1}{3}N^2,$$

where

$$\tilde{Q}^2 = 6 \sum_{M,\alpha,\beta} (-1)^M A^2_M(\alpha,\alpha) A^2_M(\beta,\beta).$$

It should be pointed out that the groups $U(2)$ and $U(3)$ are mutually complementary in the sense that the eigenvalues of K_2 are completely determined by the eigenvalues of K_3. This is due to the relation

$$K_3 = \frac{3}{2}K_2 + N. \tag{22}$$

This means that the representations of $U(3)$ and $U(2)$ can be labeled by the same quantum numbers (for instance, the number of quasiparticles N and the «T-spin» T).

(4) $U(1) \otimes U(1)$ algebra. It is determined by the operators

$$A^0(p,p) + A^0(n,n), \qquad A^0(p,p) - A^0(n,n)$$

proportional to N (17) and T_0 (20) and its second-order Casimir operator is

$$K_1 = \frac{1}{3}N^2 + \frac{4}{3}T_0^2.$$

(5) $SU(3) \otimes SU(3)$ algebra. It consists of two sets of commuting operators $(L = 1, 2)$

$$P^L_M = \big(A^L_M(p,p) + A^L_M(n,n) + A^L_M(p,n) + A^L_M(n,p)\big)/\sqrt{2} \tag{23a}$$

and

$$S^L_M = \big(A^L_M(p,p) + A^L_M(n,n) - A^L_M(p,n) - A^L_M(n,p)\big)/\sqrt{2}. \tag{23b}$$

This group has two second-order Casimir operators

$$\begin{aligned} G_3 = \sum_{L=1,2}\sum_M (-1)^M (P^L_M P^L_{-M} + S^L_M S^L_{-M}) = \\ = \sum_{L=1,2}\sum_M (-1)^M \big(A^L_M(p,n)A^L_{-M}(p,n) + A^L_M(n,p)A^L_{-M}(n,p)\big) = \\ = K_3 - \frac{2}{3}T^2 - \frac{4}{3}T_0^2 + \frac{1}{6}N^2 + 3N \end{aligned} \tag{24}$$

and

$$\begin{aligned} G_3' = \sum_{L=1,2}\sum_M (-1)^M (P^L_M P^L_{-M} - S^L_M S^L_{-M}) = \\ = \sum_{L=1,2}\sum_M (-1)^M \big(A^L_M(p,p)A^L_{-M}(n,p) + A^L_M(p,n)A^L_{-M}(p,p) + \\ + A^L_M(n,n)A^L_{-M}(p,n) + A^L_M(n,p)A^L_{-M}(n,n)\big) - \frac{16}{3}\sqrt{3}\big(A^0(p,n) + A^0(n,p)\big). \end{aligned}$$

(6) $O(3) \otimes O(3)$ algebra. Its generators are determined by the operators (23a), (23b) with $L = 1$. This group also has two second-order Casimir

operators

$$\tilde{\pi}_3 = \sum_{L=1,2} \sum_M (-1)^M (P^1_M P^1_{-M} + S^1_M S^1_{-M}) = \\ = \sum_M (-1)^M \big(2A^1_M(p,p) A^1_{-M}(n,n) + A^1_M(p,n) A^1_{-M}(p,n) + \\ + A^1_M(n,p) A^1_{-M}(n,p)\big) = \pi_3 - T^2 + \frac{1}{4} N(N+6)$$

and

$$\tilde{\pi}'_3 = \sum_{L=1,2} \sum_M (-1)^M (P^1_M P^1_{-M} - S^1_M S^1_{-M}) = \\ = 2 \sum_M (-1)^M \big(A^1_M(p,p) A^1_{-M}(n,p) + A^1_M(p,n) A^1_{-M}(p,p) + \\ + A^1_M(n,n) A^1_{-M}(p,n) + A^1_M(n,p) A^1_{-M}(n,n)\big) - 2\sqrt{3}\big(A^0(p,n) + A^0(n,p)\big).$$

(7) $O(6)$ algebra. The set of generators is

$$\begin{array}{ccc} A^1_M(p,n) + A^1_M(n,p), & i(A^2_M(p,n) + A^2_M(n,p)), & \\ i(A^0(p,n) - A^0(n,p)), & A^1_M(p,p), & A^1_M(n,n), \end{array} \tag{25}$$

and its second-order Casimir operator is

$$\pi_6 = \sum_{L=0,1,2} \sum_M (-1)^{L+M+1} \big(A^L_M(p,n) A^L_{-M}(p,n) + A^L_M(n,p) A^L_{-M}(n,p)\big) + \\ + \frac{1}{2} N^2 - 2T_0 + 3N. \tag{26}$$

(8) $\overline{SU(3)}$ algebra. It consists of eight generators, which are part of the generators of $O(6)$, namely

$$\begin{array}{l} X^2_M = i(A^2_M(p,n) - A^2_M(n,p)), \quad M = 0, \pm 1, \pm 2, \\ Y^1_M = A^1_M(p,p) + A^1_M(n,n) = -\frac{1}{\sqrt{2}} L_M, \quad M = 0, \pm 1. \end{array} \tag{27}$$

Its second-order Casimir operator is

$$\bar{G}_3 = \sum_M (-1)^M (X_M X_{-M} + Y_M Y_{-M}). \tag{28}$$

(9) $O(2)$ algebra. It is given by the operator

$$i(A^0(p,n) - A^0(n,p))$$

and its second-order Casimir operator is

$$\pi_2 = \frac{2}{3}T^2 - \frac{2}{3}T_0^2 - A^0(p,n)A^0(p,n) - A^0(n,p)A^0(n,p).$$

2.2. Hamiltonian of the Model. The most general one- and two-body Hamiltonian, which is invariant with respect to Hermitian conjugation and with respect to the group $O(3)$, can be expressed in the form

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_{\rm int}, \tag{29a}$$

where

$$\mathscr{H}_0 = \sum_\alpha \big(\varepsilon_\alpha A^0(\alpha,\alpha) + \varepsilon_\alpha^+ F^0(\alpha,\alpha) + \varepsilon_\alpha^- G^0(\alpha,\alpha)\big), \tag{29b}$$

and $\mathscr{H}_{\rm int}$ is a linear combination of all possible scalar products of the operators (6a), (6b).

As a first approximation of the model we assumed the additional condition that the Hamiltonian should preserve the number of bosons [14, 15]. This condition restricts the region of application of the model, but makes it possible to construct in a comparatively simple way bases that can be further used for the diagonalization of the complete Hamiltonian (6a)–(6c). The truncated Hamiltonian which preserves the number of bosons is of the form

$$H = H_0 + H_{\rm int} = \\ = \sum_\alpha \varepsilon_\alpha A^0(\alpha,\alpha) + \sum_L \sum_{\alpha\beta\gamma\delta} V^L(\alpha\beta;\gamma\delta) F^L(\alpha,\beta)\, G^L(\gamma,\delta) \tag{29c}$$

and its structure is investigated below in detail.

The coefficients $V^L(\alpha\beta;\gamma\delta)$ in (29c) are real phenomenological constants related to the two-boson interaction. From (7) it follows that they have the following properties:

$$V^L(\alpha\beta;\gamma\delta) = V^L(\gamma\delta;\alpha\beta), \\ V^L(\alpha\beta;\gamma\delta) = (-1)^L V^L(\beta\alpha;\gamma\delta) = (-1)^L V^L(\alpha\beta;\delta\gamma). \tag{30}$$

In particular

$$V^L(\alpha\alpha;\gamma\delta) = V^L(\alpha\beta;\gamma\gamma) = 0, \quad \text{if} \quad L = 1.$$

It is obvious from (30) that the number of independent constants $V^L(\alpha\beta;\gamma\delta)$ is 13. This number can be reduced provided that H in (29b) is subject to certain additional restrictions. In order to define the latter it is necessary to introduce the operators of the basic physical observables of the system.

In future papers our model will be applied to even–even nuclei so that the spin of the ground state of the system is equal to zero. Its angular and quadrupole momenta are the most important physical observables in this case. Using the standard definitions and relations (1) and (2) one can express the operators of the angular and quadrupole momenta as follows:

$$L_M = -i \sum_{\alpha,k,m} C^{1\,M}_{1k,\,1m} x_m(\alpha) q_k(\alpha) = -\sqrt{2} \sum_{\alpha} A^1_M(\alpha,\alpha), \tag{31a}$$

and

$$Q_M = \sum_{\alpha,k,m} C^{2\,M}_{1k,\,1m} x_k(\alpha) x_m(\alpha) = \\ = \sqrt{6} \sum_{\alpha} \Big(\frac{1}{2} F^2_M(\alpha,\alpha) + A^2_M(\alpha,\alpha) + \frac{1}{2} G^2_M(\alpha,\alpha) \Big), \tag{31b}$$

where it is obvious that (31a) coincides with the operator (18) and the middle term in Q_M (31b) is the operator defined by (21). The «T spin» and its projection play an essential role in the classification of the states of the many-body system. Using (20) it is not difficult to show that $-F^L(p,p)/\sqrt{2}$, $F^L(p,n)$ and $F^L(n,n)/\sqrt{2}$ ($-G^L(n,n)/\sqrt{2}, G^L(n,p)$ and $G^L(p,p)/\sqrt{2}$) form a tensor of rank one with respect to the «T-spin» group defined by the operators $T_k, (k = 0, \pm 1)$. It is therefore evident that in general the Hamiltonians (29a), (29b) do not commute with the operators of the «T-spin» group. On the other hand, the difference between the p and n bosons is irrelevant in this case, and it is reasonable to consider $H_{\rm int}$ in (29b) invariant if p and n are mutually substitutable (a «T-spin» symmetry). We introduce an operator

$$R_x = \exp(i\pi T_x), \qquad T_x = (-T_1 + T_{-1})/\sqrt{2},$$

which has the property

$$R_x^{-1} u^\dagger_k(\alpha) R_x = -i u^\dagger_k(-\alpha), \qquad R_x^{-1} u_k(\alpha) R_x = i u_k(-\alpha)$$

and brings about the substitution $p \rightleftarrows n$. It is obvious that

$$R_x^{-1} F^L_M(\alpha,\beta) R_x = -F^L_M(-\alpha,-\beta),$$
$$R_x^{-1} G^L_M(\alpha,\beta) R_x = -G^L_M(-\alpha,-\beta),$$
$$R_x^{-1} A^L_M(\alpha,\beta) R_x = A^L_M(-\alpha,-\beta)$$

and that $H_{\rm int}$ is invariant with respect to the «T spin» under the condition

$$R_x^{-1} H_{\rm int} R_x = H_{\rm int}. \tag{32a}$$

The latter means that

$$V^L(\alpha\beta;\gamma\delta) = V^L(-\alpha\,-\beta;-\gamma\,-\delta) \tag{32b}$$

and the number of independent constants in (29c) is reduced from 13 to 9.

The Hamiltonians (29a), (29c) are expressed by the generators of $\mathrm{Sp}\,(12,R)$. The eigenvectors of H_{int} or H can be expanded by the basic vectors of a certain irreducible representation of the group $\mathrm{Sp}\,(12,R)$. On the other hand, (29a) and (29c) are invariant only with respect to the group $O(3)$ defined by the operators of the angular momentum (31a). Therefore, each of the Hamiltonians splits the irreducible multiplet of $\mathrm{Sp}\,(12,R)$ and generates the energy spectrum of the system, i.e., $\mathrm{Sp}\,(12,R)$ is a dynamical group for the interactions (29a), (29c). Moreover, the transition operators Q^2_M of (31b) are also expressed by the generators of $\mathrm{Sp}\,(12,R)$. This means that if one knows the representations of $\mathrm{Sp}\,(12,R)$ algebra, then not only the energy levels can be calculated but also the transition probabilities between them.

It should be noted that if $x_m(\mu)$ $(m=0,\pm1;\mu=1,2,\ldots,A)$ are the cyclic coordinates of a particle in a system of A particles, then using the formulae of Subsec. 1.2 one can construct the operators $F^L_M(\mu,\nu)$, $A^L_M(\mu,\nu)$, and $G^L_M(\mu,\nu)$ $(\mu,\nu=1,2,\ldots,A)$. These operators satisfy the commutation relations (11a)–(11c) and generate the algebra of the group $\mathrm{Sp}\,(6A,R)$ — the noncompact group of canonical transformations of a system of A particles [45]. Summing over the particle indices one obtains the generators of $\mathrm{Sp}\,(6,R)$ (21 in number) [27,28,36]:

$$F^L_M=\sum_{\mu=1}^{A}F^L_M(\mu,\mu),\qquad A^L_M=\sum_{\mu=1}^{A}A^L_M(\mu,\mu),\qquad G^L_M=\sum_{\mu=1}^{A}G^L_M(\mu,\mu).$$

The latter commute with the set of operators

$$I_{\mu\nu}=\sum_{m=0,\pm1}\left(u^\dagger_m(\mu)u^m(\nu)-u^\dagger_m(\nu)u^m(\mu)\right)$$

which generate the group $O(A)$ of rotations in A-dimensional space of the particles [37].

If the system consists of two types of particle differing, for example, in their «T-spin» projection, then the number of cyclic coordinates is doubled and the generators of the groups $\mathrm{Sp}\,(12A,R)$ and $\mathrm{Sp}\,(12,R)$ can be constructed. These generators also commute with the generators of $O(A)$.

This explains to a certain extent the basic assumption of our model — we accept that the collective properties of a system of A particles of two different types are determined by an algebraic structure generated by the group $\mathrm{Sp}\,(12,R)$.

We recall here that the group $\mathrm{Sp}\,(12,R)$ is not compact and its representations are of infinite dimension, which makes it rather difficult to diagonalize the

Hamiltonians (29a), (29c) in the basis of $\mathrm{Sp}\,(12, R)$. However, in the case of the truncated Hamiltonian (29c), this difficulty can be overcome if one uses the standard technique of recoupling noncommuting operators [46]:

$$\sum_M (-1)^M F^L_M(\alpha,\beta)\, G^L_{-M}(\gamma,\delta) = \frac{2L+1}{\sqrt{3}}\, A^0(\alpha,\delta)\delta(\gamma,\beta) + \\ + \sum_{J,\mathscr{M}} (-1)^{\mathscr{M}} (2L+1) \left\{ \begin{array}{ccc} 1 & 1 & L \\ 1 & 1 & J \end{array} \right\} A^J_{\mathscr{M}}(\alpha,\gamma) A^J_{-\mathscr{M}}(\beta,\delta).$$

In this way the truncated Hamiltonian can be written in terms of the generators of the group $U(6)$ only:

$$H'_0 = \sum_{\alpha,\beta} h_0(\alpha,\beta) A^0(\alpha,\beta),$$
$$H'_{\rm int} = \sum_{L,J,\mathscr{M}} \sum_{\alpha,\beta,\gamma,\delta} (2L+1) \left\{ \begin{array}{ccc} 1 & 1 & L \\ 1 & 1 & J \end{array} \right\} V^L(\alpha\beta;\, \gamma\delta) \times \\ \times (-1)^{\mathscr{M}} A^J_{\mathscr{M}}(\alpha,\gamma) A^J_{-\mathscr{M}}(\beta,\delta), \tag{33}$$

where

$$h_0(\alpha,\beta) = \frac{1}{\sqrt{3}} \sum_{L,\lambda} (2L+1) V^L(\alpha\lambda;\, \lambda\beta) + \frac{1}{2}\varepsilon_\alpha \delta(\alpha,\beta).$$

Now we can discuss the role of the group $\mathrm{Sp}\,(12, R)$ in the problem. First of all, the eigenvalues of the truncated Hamiltonian belong to a given UIR of $\mathrm{Sp}\,(12, R)$. Apart from that there are only two types of UIR of $\mathrm{Sp}\,(12, R)$ — even and odd, which can be constructed by means of six creation operators [23] (as is our case). On the other hand, under the decomposition $\mathrm{Sp}\,(12, R) \supset U(6)$ the UIR of $\mathrm{Sp}\,(12, R)$ splits into a countless number of symmetric UIR of $U(6)$ of the type $[N, 0, 0, 0, 0, 0]$, where $N = 0, 2, 4 \ldots$, for the even UIR of $\mathrm{Sp}\,(12, R)$, and $N = 1, 3, 5 \ldots$, for the odd UIR of $\mathrm{Sp}\,(12, R)$. This means that the matrix elements of the truncated Hamiltonian (29b) in the basis of $\mathrm{Sp}\,(12, R)$ vanish between states of different UIR of $U(6)$ given by the number N, i.e., in the basis of $\mathrm{Sp}\,(12, R)$ the matrix of H (29b) is block-diagonal. In this way the problem is reduced to the diagonalization of H in the finite subspaces of the different UIR of $U(6)$, which can always be done, at least numerically.

From all this one can get the impression that the role of $\mathrm{Sp}\,(12, R)$ in this problem has been somewhat exaggerated and that we can restrict ourselves to the dynamical $U(6)$. This, however, is not the case, because the complete spectrum of the system can be calculated only through the diagonalization of the truncated H (29b) in the subspaces of all the UIR of $U(6)$, belonging to a given UIR of $\mathrm{Sp}\,(12, R)$. On the other hand, the operators describing the transitions between the states of different UIR of $U(6)$ belong to the enveloping algebra of $\mathrm{Sp}\,(12, R)$. Finally, in the case of the complete Hamiltonian (29a), the terms which do not preserve the number of particles mix the different UIR of $U(6)$ and, in this sense, $\mathrm{Sp}\,(12, R)$ is the dynamical group for the system.

2.3. The Dynamical Symmetries of the IVBM. The diagonalization of H (29c) or (33) can easily be carried out if one is able to express the term in the sum (33) by means of the Casimir operators of the group $U(6)$ and its subgroups (16). Taking the symmetry properties (32a), (32b) into account and the recoupling relation in the case of $\alpha \neq \beta$:

$$\sum_{L,M}\sum_{\alpha\neq\beta}(-1)^M A^L_M(\alpha,\beta)A^L_{-M}(\beta,\alpha) =$$
$$= -\frac{1}{\sqrt{3}}\sum_{\alpha} A^0(\alpha,\alpha)\sum_L(2L+1) + \sum_{L,J,\mathscr{M}}\sum_{\alpha\neq\beta}(-1)^{L+J}(2L+1)\times$$
$$\times\begin{Bmatrix} 1 & 1 & L \\ 1 & 1 & J \end{Bmatrix} A^J_{\mathscr{M}}(\alpha,\alpha)A^J_{\mathscr{M}}(\beta,\beta).$$

The truncated Hamiltonian can be written explicitly in the form

$$\tilde{H}_0 = \Big(\varepsilon_p + \frac{1}{3}\sqrt{3}\,V^0(pp;\,pp) + \frac{5}{3}\sqrt{3}\,V^2(pp;\,pp)\Big)\big(A^0(p,p) + A^0(n,n)\big),$$

$$\tilde{H}_{\rm int} = \sum_{L,J,\mathscr{M}}(2L+1)\begin{Bmatrix} 1 & 1 & L \\ 1 & 1 & J \end{Bmatrix}(-1)^{\mathscr{M}}\Big[V^L(pp;pp)\big(A^J_{\mathscr{M}}(p,p)A^J_{-\mathscr{M}}(p,p)+$$
$$+ A^J_{\mathscr{M}}(n,n)A^J_{-\mathscr{M}}(n,n)\big) + 4V^L(pn;pn)A^J_{\mathscr{M}}(p,p)A^J_{-\mathscr{M}}(n,n)+$$
$$+ V^L(pp;nn)\big(A^J_{\mathscr{M}}(p,n)A^J_{-\mathscr{M}}(p,n) + A^J_{\mathscr{M}}(n,p)A^J_{-\mathscr{M}}(n,p)\big)+$$
$$+ 2V^L(pp;nn)\big(A^J_{\mathscr{M}}(p,p)A^J_{-\mathscr{M}}(n,p) + A^J_{\mathscr{M}}(p,n)A^J_{-\mathscr{M}}(p,p)+$$
$$+ A^J_{\mathscr{M}}(n,n)A^J_{-\mathscr{M}}(p,n) + A^J_{\mathscr{M}}(n,p)A^J_{-\mathscr{M}}(n,n)\big)\Big]. \quad (34)$$

From the $U(6)$ structure of the truncated Hamiltonian (the results of Sec. 2) it becomes clear that the two-boson interaction $\tilde{H}_{\rm int}$ can also be written in terms of the second-order Casimir operators of the algebras belonging to the reduction scheme (16):

$$\tilde{H}_{\rm int} = \alpha_6K_6 + \alpha_3K_3 + \alpha_2K_2 + \alpha_1K_1 + \bar{\alpha}'_3G'_3 + \beta_6\pi_6 + \bar{\beta}_3\bar{\pi}_3+$$
$$+ \bar{\beta}'_3\bar{\pi}'_3 + \beta_3\pi_3 + \beta_2\pi_2 + \gamma\big(A^0(p,n) + A^0(n,p)\big). \quad (35)$$

The Casimir operators G_3 (24) and $\bar{G}_3$ (28) have not been taken into account in (35) because of the relations

$$G_3 = \frac{2}{3}K_6 + \frac{2}{3}K_3 + K_2 - 2K_1 - 2\pi_2$$

and

$$\bar{G}_3 = -\frac{2}{3}\pi_6 + \frac{1}{3}\bar{\pi}_3 + \frac{4}{3}\pi_3.$$

Now it is easy to express the real independent constants of (34) by the coefficients of the expansion (35):

$$\begin{aligned}
V^0(pp;\,pp) &= \alpha_6 + \alpha + \frac{2}{3}\alpha_1 - \beta_6 - \bar{\beta}_3 - \beta_3,\\
V^2(pp;\,pp) &= \alpha_6 + \alpha + \frac{2}{3}\alpha_1 + \frac{1}{2}\beta_6 + \frac{1}{2}\bar{\beta}_3 + \frac{1}{2}\beta_3,\\
V^0(pn;\,pn) &= \frac{1}{2}\alpha_6 + \frac{1}{2}\alpha + \frac{1}{2}\beta_6 - \bar{\beta}_3 - \frac{1}{2}\beta_3 + \frac{1}{6}\beta_2,\\
V^1(pn;\,pn) &= -\frac{1}{2}\alpha_6 + \frac{1}{2}\alpha - \frac{1}{2}\beta_6 + \frac{1}{4}\beta_3 + \frac{1}{6}\beta_2,\\
V^2(pn;\,pn) &= \frac{1}{2}\alpha_6 + \frac{1}{2}\alpha + \frac{1}{2}\beta_6 + \frac{1}{2}\bar{\beta}_3 + \frac{1}{4}\beta_3 + \frac{1}{6}\beta_2,\\
V^0(pp;\,nn) &= -3\beta_6 - \bar{\beta}_3 - \frac{1}{3}\beta_2,\\
V^2(pp;\,nn) &= \frac{1}{2}\bar{\beta}_3 - \frac{1}{2}\beta_2,\\
V^0(pp;\,pn) &= \frac{2}{3}\bar{\alpha}'_3 - \bar{\beta}'_3,\\
V^2(pp;\,pn) &= \frac{2}{3}\bar{\alpha}'_3 + \frac{1}{2}\bar{\beta}'_3,\\
\gamma &= \frac{16}{3}\sqrt{3}\,\bar{\alpha}'_3 + 2\sqrt{3}\,\bar{\beta}'_3,
\end{aligned}$$

where

$$\alpha = \alpha_3 + \frac{2}{3}\alpha_2,$$

which is due to the fact that $U(3)$ and $U(2)$ are mutually complementary groups (22).

If one is able to find examples of nuclei for which the parameters in (35) are so special as to realize only one of the chains in the reduction scheme (16), then the truncated Hamiltonian can be diagonalized immediately in a basis of states labelled by the eigenvalues of the corresponding Casimir operators. The limiting cases realized in such a way are of interest for the determination of the region of application of the two-vector boson model, and are to be investigated in this review.

In short, the most general one- and two-body interaction has been constructed by means of two interacting vector bosons. The group of dynamical symmetry for the complete Hamiltonian is the noncompact symplectic $\mathrm{Sp}\,(12, R)$. In the basis of the group $\mathrm{Sp}\,(12, R)$ the matrix of the truncated Hamiltonian, which preserves the number of bosons, is block-diagonal with respect to the maximal compact subgroup of $\mathrm{Sp}(12, R)$, namely the group $U(6)$. In this way the diagonalization of the truncated H in the infinite space of a given UIR of $\mathrm{Sp}(12, R)$ is reduced to a diagonalization in the finite subspaces of the different UIR of $U(6)$. The $U(6)$ structure of the truncated Hamiltonian has been investigated. All possible types of two-boson interactions preserving the number of bosons have been expressed by linear combinations of second-order Casimir operators of the groups in the reduction scheme (16). As a result of the proposed representation of the truncated Hamiltonian the eigenvalue problem can be solved explicitly.

3. RELATIONSHIP OF THE INTERACTING VECTOR BOSON MODEL TO THE VERSIONS OF THE INTERACTING BOSON APPROXIMATIONS

The interacting vector boson model [14], where nuclear states are considered as a system of an even number of the p bosons with isospin $T = 1/2$, is in some aspects similar to the standard IBM [47] (for example, there is an exactly solvable $su(3)$ rotational limit and so on), but there are some differences, too. The natural question about the connections between IVBM and the standard versions of IBM arises. The answer could be obtained by means of mapping of the boson pairs with angular momentum L and isospin T $(LT = 01, 21, 10)$ on the space of the ideal S and D bosons with isospin $T = 1$ and P bosons with $T = 0$.

The boson mapping technique, which is widely applied to the problems of microscopic foundation of the IBM [48], can be applied to our problem, too. The noncompact algebra $\mathrm{sp}(12, R)$ is a dynamical symmetry algebra for IVBM. The algebraic structure of lVBM allows one to use a number of results obtained for the boson mapping of symplectic algebras [49]. The vector coherent states (VCS) and the mixed boson realization [50] are of particular importance in this case. The VCS theory is an extension of the standard theory of the generalized coherent states [51] which allows the inclusion of vector-valued functions. At first sight this approach seems a complication but in fact it brings about a major simplification because it allows the separation of the intrinsic degrees of freedom from the collective ones, thereby greatly simplifying the description of the latter. Besides, a simple $\mathbb{K}$-matrix technique has recently been elaborated, which allows one to solve the long-standing problem of transformation of the Dyson [52] representation into the Holstein–Primakoff [53] representation. Our aim is to obtain the boson realization of the $\mathrm{sp}(2d, R)$ algebra generators written in a form of double spherical tensors (8a), (8b) with respect to the $SO_L(3) \otimes SU_T(2)$ group. A similar problem has been studied in a number of papers [44, 54] where the representations of the compact $\mathrm{sp}(2d)$ algebra were mapped on the boson space. The technique suggested there for the construction of boson images of the

$\mathrm{sp}\,(2d)$ algebra generators written in a tensor form includes a solution of some additional equations. Besides, that method is applicable only to the so-called vacuum representations and cannot be used for the construction of the boson realization in a general case. In our investigation we use a different approach based on a general technique [49,50,55–57] that allows the results we are looking for to be obtained directly avoiding subsidiary calculations.

First we obtain the nonunitary Dyson mapping of the generators of the $\mathrm{sp}\,(2d,R)$ algebra. As a result, it is shown that the IVBM states with an even number of p bosons can be exactly mapped onto the states of some specific version of the IBM with S, D, and P bosons, i.e., the model that is intermediate between IBM-3 and IBM-4 and that could be conditionally named IBM-3.5. As in the case of mapping of the fermion space onto the boson space [23,58], the IVBM states are mapped only on the part of the space (physical subspace) of the IBM-3.5 model. Thus the problem of separating the physical subspace and of the identification and cancellation of the nonphysical (spurious) states arises.

It should be noted that this problem has been discussed in detail in connection with the mapping of fermion pairs (i.e., the representations of compact algebras) in [23–59]. As for the boson mapping of the boson pairs (i.e., the representations of noncompact algebras), the problem of spurious states has not been considered earlier. The reason for the appearance of spurious states when fermion pairs are mapped is usually considered to be the Pauli exclusion principle. Our considerations indicate that the spurious states can also appear in the case of the boson–boson mapping. A method of identifying and removing of spurious states from the low-lying part of the energy spectrum has been suggested by Park [58]. We will show that this method, based on pure group-theoretical analysis, can be reformulated and applied to the problem under consideration.

3.1. Mixed Dyson Boson Realization of $\mathrm{sp}\,(2d,R)$ Algebra. In order to find the relationship between the IVBM and the standard versions of IBM, the $\mathrm{sp}\,(12,R)$ algebra generated by operators A^{LM}_{Tt}, F^{LM}_{Tt}, G^{LM}_{Tt} should be mapped on the space of S, D, and P bosons with the isospins 1, 1, and 0, respectively. First we concider the general construction of Dyson mapping of the $\mathrm{sp}\,(2d,R)$ algebra and then the explicit expressions for $\mathrm{sp}\,(12,R)$ algebra are given.

In order to determine the VCS of the $\mathrm{Sp}\,(2d,R)$ group we take the subalgebra $u(d)\subset sp(2d,R)$ which contains the Cartan subalgebra and carry out the generalized Gauss decomposition

$$g = n_- + h + n_+, \tag{36}$$

where $n_-(n_+)$ is a nilpotent subalgebra of lowering G^{LM}_{Tt} (raising F^{LM}_{Tt}) operators and the subalgebra h is the $u(d)$ algebra.

According to the commutation relations (13)–(15) the operators G^{LM}_{Tt} and F^{LM}_{Tt} are tensors with respect to the $u(d)$ subalgebra. An arbitrary irreducible representation (IR) $[\sigma]$ belonging to the positive discrete series of the $\mathrm{sp}\,(2d, R)$ algebra, may be expanded in terms of the IRs $[\Lambda]$ of the $u(d)$ algebra. Of all these representations the so-called lowest-weight representation is defined by the equation

$$G^{LM}_{Tt}|\sigma\Lambda\lambda\rangle = 0. \tag{37}$$

Here $\{|\sigma\Lambda\lambda\rangle\}$ is a basis of the corresponding lowest-weight representation and $|\sigma\Lambda\Lambda\rangle$ is the highest weight vector with respect to the h subalgebra of the $\mathrm{sp}\,(2d, R)$ algebra. The VCS can be written as

$$|Z\rangle = \sum_{\lambda} \exp\left(\sum_{LMTt} Z^{*LM}_{Tt} F^{LM}_{Tt}\right)|\sigma\Lambda\lambda\rangle, \tag{38}$$

where Z^{LM}_{Tt} are complex parameters and * denotes complex conjugation.

The VCS representation of an arbitrary state $|\Psi\rangle$ is defined by

$$|\Psi(Z)\rangle = \sum_{\lambda} |\sigma\Lambda\lambda\rangle\langle\sigma\Lambda\lambda| \exp\left(\sum_{LMTt} Z^{LM}_{Tt} G^{LM}_{Tt}\right)|\Psi\rangle. \tag{39}$$

The VCS representation of an operator E

$$\begin{aligned}\Gamma(E)|\Psi(Z)\rangle &= \sum_{\lambda} |\sigma\Lambda\lambda\rangle\langle\sigma\Lambda\lambda| \exp\left(\sum_{LMTt} Z^{LM}_{Tt} G^{LM}_{Tt}\right) E|\Psi\rangle = \\ &= \sum_{\lambda} |\sigma\Lambda\lambda\rangle\langle\sigma\Lambda\lambda| \sum_{n} \frac{1}{n!}\Big[Z^{LM}_{Tt}G^{LM}_{Tt}, \ldots \Big[Z^{LM}_{Tt}G^{LM}_{Tt}, E\Big] \underbrace{\ldots}_{n}\Big] \times \\ &\qquad \times \exp\left(\sum_{LMTt} Z^{LM}_{Tt} G^{LM}_{Tt}\right)|\Psi\rangle\end{aligned} \tag{40}$$

is equivalent to some partial differential operator with respect to Z^{LM}_{Tt}.

Since vectors $|\sigma\Lambda\lambda\rangle$ belong to the lowest-weight representation we have

$$\sum_{\lambda} |\sigma\Lambda\lambda\rangle\langle\sigma\Lambda\lambda| F^{LM}_{Tt} \exp\left(\sum_{LMTt} Z^{LM}_{Tt} G^{LM}_{Tt}\right)|\Psi\rangle = 0. \tag{41}$$

The intrinsic operators $\mathcal{A}^{LM}_{Tt}$ act in a standard way on the basis vectors of the lowest-weight representation $\{|\sigma\Lambda\lambda\rangle\}$ of the $u(d)$ algebra

$$\sum_{\lambda} |\sigma\Lambda\lambda\rangle\langle\sigma\Lambda\lambda| A^{LM}_{Tt} \exp\left(\sum_{LMTt} Z^{LM}_{Tt} G^{LM}_{Tt}\right)|\Psi\rangle = \mathcal{A}^{LM}_{Tt}|\Psi(Z)\rangle. \tag{42}$$

It is assumed that the operators $\mathcal{A}_{Tt}^{LM}$ commute with Z_{Tt}^{LM} and $\partial/\partial Z_{Tt}^{LM}$.

Using the commutation relations (13)–(15) of the $\mathrm{sp}\,(2d,R)$ algebra and formulae (40)–(42), the explicit VCS representation of generators G_{Tt}^{LM}, F_{Tt}^{LM}, and A_{Tt}^{LM} can be derived. This yields immediately the Dyson boson realization of the $\mathrm{sp}\,(2d,R)$ algebra in terms of $u(d)$-boson algebra by means of the isomorphism

$$Z_{Tt}^{LM} \to b_{Tt}^{\dagger LM}, \qquad \frac{\partial}{\partial Z_{Tt}^{LM}} \to b_{Tt}^{LM}. \tag{43}$$

The latter algebra is a direct sum of the $u(d)$ algebra and the Heisenberg–Weyl algebra $hw(d(d+l)/2)$.

The ideal boson creation and annihilation operators $b_{Tt}^{\dagger LM}$ and b_{Tt}^{LM} with the angular momentum L and isospin T satisfy the standard boson commutation relations

$$\left[b_{Tt}^{LM}, b_{T't'}^{\dagger L'M'}\right] = \delta_{LL'}\delta_{MM'}\delta_{TT'}\delta_{tt'}, \tag{44}$$

$$\tilde{b}_{Tt}^{LM} = (-1)^{L+M}(-1)^{T+t}\, b_{T-t}^{L-M}. \tag{45}$$

In the particular case of $\mathrm{sp}\,(12,R)$ algebra we deal with S and D bosons of $T=1$ isospin and P bosons of $T=0$ isospin, and (44) and (45) take the following form:

$$[S_t, S_{t'}^{\dagger}] = \delta_{tt'}, \quad [P^M, P^{\dagger M'}] = \delta_{MM'}, \quad [D_t^M, D_{t'}^{\dagger M'}] = \delta_{MM'}\delta_{tt'}, \tag{46}$$

$$\tilde{S}_t = (-1)^{1+t}S_{-t}, \quad \tilde{D}_t^M = (-1)^{2+M}(-1)^{1+t}D_{-t}^{-M}, \quad \tilde{P}^M = (-1)^{1+M}P^{-M}. \tag{47}$$

The Dyson images of the G_{Tt}^{LM} generators take a simple form:

$$\rho_D(G_{Tt}^{LM}) = b_{Tt}^{LM}. \tag{48}$$

The Dyson images of the A_{Tt}^{LM} generators

$$\rho_D(A_{Tt}^{LM}) = \mathcal{A}_{Tt}^{LM} + 2(-1)^{L+T} \sum_{L_1 L_2 T_1 T_2} \hat{L}_1\hat{L}_2\hat{T}_1\hat{T}_2 \left\{ \begin{array}{ccc} L_1 & L_2 & L \\ l & l & l \end{array} \right\} \times \\ \times \left\{ \begin{array}{ccc} T_1 & T_2 & T \\ f & f & f \end{array} \right\} (b_{T_1}^{\dagger L_1} \otimes \tilde{b}_{T_2}^{L_2})_{Tt}^{LM} \tag{49}$$

contain the intrinsic operators $\mathcal{A}_{Tt}^{LM}$ of the $u(d)$ algebra which commute with the boson creation and annihilation operators $b_{Tt}^{\dagger LM}, b_{Tt}^{LM}$. Finally the Dyson images of the F^{LM}_{Tt} generators are determined by the expressions

$$\rho_D(F^{LM}_{Tt}) = b^{\dagger LM}_{Tt} + 2\sum_{L_1L_2T_1T_2} \hat{L}_1\hat{L}_2\hat{T}_1\hat{T}_2 \begin{Bmatrix} L_1 & L_2 & L \\ l & l & l \end{Bmatrix} \begin{Bmatrix} T_1 & T_2 & T \\ f & f & f \end{Bmatrix} \times$$
$$\times (-1)^{L_2+T_2}\Bigg[(b^{\dagger L_1}_{T_1} \otimes \mathcal{A}^{L_2}_{T_2})^{LM}_{Tt} + (-1)^{L_2+T_2} \sum_{L_1L_2T_1T_2} \hat{L}_1\hat{L}_2\hat{T}_1\hat{T}_2 \times$$
$$\times \begin{Bmatrix} L_3 & L_4 & L_2 \\ l & l & l \end{Bmatrix} \begin{Bmatrix} T_3 & T_4 & T_2 \\ f & f & f \end{Bmatrix} \Big((b^{\dagger L_1}_{T_1} \otimes (b^{\dagger L_3}_{T_3} \otimes \tilde{b}^{L_4}_{T_4})^{L_2}_{T_2}\Big)^{LM}_{Tt}\Bigg]. \quad (50)$$

The last formula can be written in a different form using another coupling scheme of the tensors in the last term. The corresponding transformation can be obtained by means of the standard technique of the angular momenta recoupling and results in:

$$\rho_D(F^{LM}_{Tt}) = b^{\dagger LM}_{Tt} + 2\sum_{L_1L_2T_1T_2} \hat{L}_1\hat{L}_2\hat{T}_1\hat{T}_2 \begin{Bmatrix} L_1 & L_2 & L \\ l & l & l \end{Bmatrix} \begin{Bmatrix} T_1 & T_2 & T \\ f & f & f \end{Bmatrix} \times$$
$$\times (-1)^{L_2+T_2}(b^{\dagger L_1}_{T_1} \otimes \mathcal{A}^{L_2}_{T_2})^{LM}_{Tt} + 2 \sum_{L_1L_3L_4L_5T_1T_3T_4T_5} \hat{L}_1\hat{L}_3\hat{L}_4\hat{L}_5\hat{T}_1\hat{T}_3\hat{T}_4\hat{T}_5 \times$$
$$\times \begin{Bmatrix} L_1 & L_3 & L_5 \\ l & l & L_4 \\ l & l & L \end{Bmatrix} \begin{Bmatrix} T_1 & T_3 & T_5 \\ f & f & T_4 \\ f & f & T \end{Bmatrix} \Big((b^{\dagger L_1}_{T_1} \otimes b^{\dagger L_3}_{T_3})^{L_5}_{T_5} \otimes \tilde{b}^{L_4}_{T_4}\Big)^{LM}_{Tt}. \quad (51)$$

The Dyson image of the generator F^{LM}_{Tt} can be presented in more compact form [50]. To obtain it we use the Dyson image of the second-order Casimir operator of the algebra $u(d)$, written as

$$C_2(\mathfrak{u}(d)) = \sum_{LT}(-1)^L(-1)^T\hat{L}\hat{T}\Big(\rho_D(A^L_T) \otimes \rho_D(A^L_T)\Big)^{00}_{00}. \quad (52)$$

It is convenient to separate $\rho_D(A^{LM}_{Tt})$ in two terms:

$$\rho_D(A^{LM}_{Tt}) = \mathcal{A}^{LM}_{Tt} + \rho(A^{LM}_{Tt}). \quad (53)$$

Let us denote

$$C^c_2(\mathfrak{u}(d)) = \sum_{LT}(-1)^{L+T}\hat{L}\hat{T}\Big(\rho(A^L_T) \otimes \rho(A^L_T)\Big)^{00}_{00}. \quad (54)$$

By direct calculation it can be shown that

$$\rho_D(F^{LM}_{Tt}) = -\frac{d-1}{2}\, b^{\dagger LM}_{Tt} + \frac{1}{2}\Big(C_2(\mathfrak{u}(d)) - \frac{1}{2}C^c_2(\mathfrak{u}(d))b^{\dagger LM}_{Tt}\Big). \quad (55)$$

According to the general expression (49), the Dyson image of the primary number operator N of the u bosons is

$$\rho_D(N) = \mathcal{N} + 2N_b, \tag{56}$$

where

$$N_b = \sum_{LT} \hat{L}\hat{T}\ \left(b_T^{\dagger L} \otimes \tilde{b}_T^L\right)_{00}^{00} \tag{57}$$

is the b-boson number operator.

The explicit form of the Dyson images of the multipole generators A_{Tt}^{LM} in the case of IVBM can be written explicitly as

$$\rho_D(A_{00}^{1M}) = \mathcal{A}_{00}^{1M} + \sqrt{\frac{15}{2}}(D^\dagger \otimes \tilde{D})_{00}^{1M} + \frac{1}{\sqrt{2}}(P^\dagger \otimes \tilde{P})_{00}^{1M}, \tag{58a}$$

$$\rho_D(A_{00}^{2M}) = \mathcal{A}_{00}^{2M} + \sqrt{\frac{7}{2}}(D^\dagger \otimes \tilde{D})_{00}^{2M} - \frac{1}{\sqrt{2}}(P^\dagger \otimes \tilde{P})_{00}^{2M} + \\ + \sqrt{2}\Big((S^\dagger \otimes \tilde{D})_{00}^{2M} + (D^\dagger \otimes \tilde{S})_{00}^{2M}\Big), \tag{58b}$$

$$\rho_D(A_{1t}^{00}) = \mathcal{A}_{1t}^{00} + \frac{2}{\sqrt{3}}\Big((S^\dagger \otimes \tilde{S})_{1t}^{00} + \sqrt{5}(D^\dagger \otimes \tilde{D})_{1t}^{00}\Big), \tag{58c}$$

$$\rho_D(A_{1t}^{1M}) = \mathcal{A}_{1t}^{1M} - \sqrt{\frac{2}{3}}\Big((S^\dagger \otimes \tilde{P})_{1t}^{1M} + (P^\dagger \otimes \tilde{S})_{1t}^{1M}\Big) + \sqrt{5}(D^\dagger \otimes \tilde{D})_{1t}^{1M} + \\ + \sqrt{\frac{5}{6}}\Big((D^\dagger \otimes \tilde{P})_{1t}^{1M} + (P^\dagger \otimes \tilde{D})_{1t}^{1M}\Big), \tag{58d}$$

$$\rho_D(A_{1t}^{2M}) = \mathcal{A}_{1t}^{2M} + \frac{2}{\sqrt{3}}\Big((S^\dagger \otimes \tilde{D})_{1t}^{2M} + (D^\dagger \otimes \tilde{S})_{1t}^{2M}\Big) + \sqrt{\frac{7}{3}}(D^\dagger \otimes \tilde{D})_{1t}^{2M} + \\ + \sqrt{\frac{3}{2}}\Big((S^\dagger \otimes \tilde{D})_{1t}^{2M} + (D^\dagger \otimes \tilde{S})_{1t}^{2M}\Big). \tag{58e}$$

The Dyson images $\rho_D(A_{Tt}^{LM})$ can be used to construct the image of the IVBM Hamiltonian.

3.2. Boson Mapping of the IVBM Hamiltonian. From now on we will restrict ourselves to the investigation of the most simple Dyson mapping which corresponds to the vacuum (scalar) representation of the $u(d)$ algebra, where

$$\mathcal{A}_{Tt}^{LM} = 0, \qquad \text{i.e.,} \quad A_{Tt}^{LM}|\sigma\Lambda\lambda\rangle = 0. \tag{59}$$

This particular case of boson mapping is sufficient to investigate the connections between the various versions of the interaction boson model.

It was noted in [14] that vector bosons can be treated as quarks, or elementary building blocks and the s and d bosons of the IBM can be considered as a bound states of elementary excitations generated by two vector bosons. The interaction between these secondary S, D, and P bosons is induced by the interaction between vector bosons like the nucleon–nucleon interaction which is induced by the quark–quark interaction. The comparison of the IVBM with the usual versions of the IBM is rather instructive, because in some aspects IVBM is richer than IBM. For example, two vector bosons can be coupled to create states having $L = 1$, which are absent in the IBM. In other aspects the IVBM is more restrictive, because the physical space of the IBM-3.5 corresponding to the IVBM takes up only a part of the total IBM-3.5 space.

Using the exact Dyson boson realization of the $\mathrm{sp}\,(12, R)$ algebra given by (58) we will discuss the construction of the S-, D-, and P-boson Hamiltonian based on the lVBM Hamiltonian which conserves the number of vector bosons and is invariant with respect to the rotations in the angular momentum L and isospin T spaces. This Hamiltonian can be written only in terms of the generators of the $u(3)$ algebra

$$H = \epsilon_1 N + \epsilon_2 N^2 + \epsilon_3 \mathbf{L}^2 + \epsilon_4 \mathbf{Q}^2, \tag{60}$$

where Q_M are the components of Elliott quadrupole operator (21) and L_M are the angular momentum components (18). The Dyson image of the angular momentum operator (58a) contains, together with the standard D-boson term, an additional contribution corresponding to the P bosons, as well as the Dyson image of the quadrupole operator (58b).

Thus the Dyson image of the Hamiltonian (60) is of the form

$$\begin{aligned}
\rho_D(H) = 2\epsilon_1(N_S + N_P + N_D) +& \\
+ 4\epsilon_2(N_S^2 + N_P^2 + N_D^2 + 2N_S N_P + 2N_S N_D + 2N_D N_P) +& \\
+ \epsilon_3 \Big[30(D^\dagger \otimes \tilde{D})^1_0 \cdot (D^\dagger \otimes \tilde{D})^1_0 + 4\sqrt{15}(D^\dagger \otimes \tilde{D})^1_0 \cdot (P^\dagger \otimes \tilde{P})^1_0 +& \\
+ 2(P^\dagger \otimes \tilde{P})^1_0 \cdot (P^\dagger \otimes \tilde{P})^1_0\Big] + \epsilon_4 \Big[42(D^\dagger \otimes \tilde{D})^2_0 \cdot (D^\dagger \otimes \tilde{D})^2_0 +& \\
+ 24((S^\dagger \otimes \tilde{D})^2_0 + (D^\dagger \otimes \tilde{S})^2_0) \cdot ((S^\dagger \otimes \tilde{D})^2_0 + (D^\dagger \otimes \tilde{S})^2_0) +& \\
+ 24\sqrt{7}(D^\dagger \otimes \tilde{D})^2_0 \cdot ((S^\dagger \otimes \tilde{D})^2_0 + (D^\dagger \otimes \tilde{S})^2_0) +& \\
+ 6(P^\dagger \otimes \tilde{P})^2_0 \cdot (P^\dagger \otimes \tilde{P})^2_0 - 6\sqrt{7}(D^\dagger \otimes \tilde{D})^2_0 \cdot (P^\dagger \otimes \tilde{P})^2_0 -& \\
- 12(P^\dagger \otimes \tilde{P})^2_0 \cdot ((S^\dagger \otimes \tilde{D})^2_0 + (D^\dagger \otimes \tilde{S})^2_0)\Big],&
\end{aligned} \tag{61}$$

where $T^k \cdot R^k = \sum_q (-1)^q T^{kq} R^{k\,-q}$ is the scalar product of two tensors.

This expression contains almost all terms that can be constructed from S-, D-, and P-interacting bosons. In the general case, the one- and two-boson interaction Hamiltonian that conserves the total number of S, D, and P bosons and is invariant with respect to the rotations in the angular momentum and isospin spaces contains 23 independent constants. In the rotational limit [15] of the IVBM discussed in this section, all these constants can be reduced to the four constants of the Hamiltonian (60).

3.3. Spurious States. It is well known [23,58] that the Dyson boson mapping of a fermion space on the ideal boson space gives rise to spurious states. It was shown by Park [58] that this problem can be solved by pure algebraic analysis. We show in [39] that the mapping of the boson pairs on the space of ideal bosons leads to a similar problem and that Park's approach can be applied to the case of boson–boson mapping.

In the case under consideration, there are only two types of IRs of the $\mathrm{sp}\,(2d,R)$ algebra $[N,0,\ldots,0]$ — even with $N=0,2,4,\ldots$ and odd with $N=1,3,5,\ldots$, which can be constructed by means of $d=(2l+1)(2f+1)$ creation operators $u^{\dagger lm}_{f\tau}$ [60]. The wave function, corresponding to the fixed boson number N, belongs to the symmetric IR $[N]$ of the $u(d)$ algebra and is constructed from the vacuum state using the creation boson operators $u^{\dagger lm}_{f\tau}$:

$$\prod_{i=1}^{N} u^{\dagger lm_i}_{f\tau_i}|0\rangle, \tag{62}$$

where $|0\rangle$ is the u-boson vacuum.

The two-boson operators F^{LM}_{Tt} of noncompact symplectic algebras $\mathrm{sp}\,(2d,R)$ transform under the algebra $u(d)$ as symmetric tensors of the rank [2]. After the Dyson mapping the so-obtained boson images $\rho_D(F^{LM}_{Tt})$ will also have the same tensor property with respect to $u(d)$ algebras. A wave function in the ideal boson space is constructed using the operators $b^{\dagger LM}_{Tt}$:

$$\prod_{i=1}^{N_b} b^{\dagger L_iM_i}_{T_it_i}|0), \tag{63}$$

where $|0)$ is b-boson vacuum.

The biboson combinations of the new boson operators

$$b^{\dagger LM}_{Tt}\, b^{L'M'}_{T't'} \tag{64}$$

can be considered as the generators of the algebra $u(d(d+1)/2)$ and the wave function $|n)$, corresponding to a fixed boson number N_b, belongs to the symmetric IR $[N_b]$ of this algebra. The symmetric representation $[N_b]$ of the algebra $u(d(d+1)/2)$ is reducible with respect to the $u(d)$ algebra. It can be expanded in terms of the IRs of the $u(d)$ algebra [61] in the following way:

$$[N_b]_{U(d(d+1)/2)} = \sum_g [g]_{U(d)}, \tag{65}$$

where

$$[g] = [g_1, g_2, \ldots, g_d], \qquad g_1 \geqslant g_2 \geqslant \ldots \geqslant g_d, \qquad \sum_{k=1}^{d} g_k = 2N_b = N$$

and g_k are even positive integer numbers.

However, the true wave function belongs only to the symmetric IR $[N]$ of the $u(d)$ algebra. As a result only part of the ideal boson space, which contains the states with the correct transformation property with respect to the $u(d)$ algebra, will correspond to the true wave function. These states form the so-called physical subspace. In the problem under consideration, the symmetric IR $[N]$ appears at each value of N_b in the decomposition (65). Thus the true component always exists in the wave function.

The dimension of the boson space corresponding to the symmetric IR $[N_b]$ of $u(d(d+1)/2)$ algebra is

$$\dim [N_b] = C^{N_b}_{N_b + \frac{d(d+1)}{2} - 1}, \tag{66}$$

where C^n_m is a binomial coefficient. On the other hand, the dimension of the physical subspace corresponding to the symmetric IR $[2N_b]$ of $u(d)$ algebra is

$$\dim [2N_b] = C^{2N_b}_{2N_b + d - 1}. \tag{67}$$

The greater the boson number N_b is, the smaller part of ideal boson space is occupied by the physical subspace.

As an explicit example we consider the case of the IVBM. The decomposition of a symmetric IR $[N_b]$ of $u(21)$ algebra formed by the bilinear combinations of the ideal boson operators S_t, $S_t^\dagger$, P^m, $P^{\dagger m}$, D_t^m, $D_t^{\dagger m}$ in tems of the IRs of the $u(6)$ algebra is

$U(21)$	$U(6)$
$[1]$	$[2]$
$[2]$	$[4]; [22]$
$[3]$	$[6]; [42], [222]$
$[4]$	$[8]; [62], [44], [422], [2222]$
$\vdots$	$\vdots$
$[N_b]$	$[2N_b]; \ldots$

In analogy with [58], one can introduce the classification operator $\hat{S}$

$$\hat{S} = \hat{N}_H^2 - \hat{N}_{HN}^2, \tag{68}$$

where

$$\hat{N}_H^2 = \sum_{m_1 m_2 \tau_1 \tau_2} u_{\tau_1}^{\dagger m_1} u_{\tau_1}^{m_1} u_{\tau_2}^{\dagger m_2} u_{\tau_2}^{m_2}, \tag{69}$$

$$\hat{N}_{HN}^2 = \sum_{m\tau} u_\tau^{\dagger m} u_\tau^m + \sum_{m_1 m_2 \tau_1 \tau_2} u_{\tau_1}^{\dagger m_1} u_{\tau_2}^{\dagger m_2} u_{\tau_1}^{m_1} u_{\tau_2}^{m_2}. \tag{70}$$

Strictly speaking the operator S is the vanishing operator, i.e., $\hat{S}$, because N_{HN}^2 is simply the N_H^2 written in the normal ordered form. Its Dyson image is

$$S = 4N_b N_b - 2N_b - 2 \sum_{LMTt} \rho(F_{Tt}^{LM}) b_{Tt}^{LM}. \tag{71}$$

It is a Majorana-type operator whose eigenvalues depend only on the $U(d)$ symmetry. The operator S is scalar with respect to the $u(d)$ algebra and contains one- and two-body boson operators. Hence, it could also be written as a linear combination of one- and two-body Casimir operators of the $u(d)$ algebra

$$S = N(N + d - 1) - C_2(\mathfrak{u}(d)). \tag{72}$$

The eigenvalues of the Casimir operator $C_2(u(d))$ [62] for the symmetric representation $[N]$ are $C_2(\mathfrak{u}(d))_{[N]} = N(N + d - 1)$. Hence, the eigenvalues of the operator S are equal to zero for the symmetric IRs of $u(d)$ algebra, while they are positive for all other IRs which occur. As a result by including the S operator with a suitable amplitude factor into the Hamiltonian all spurious states can be shifted to the higher part of the energy spectrum without affecting the physical eigenvalues and eigenfunctions. Other ways of separation of the spurious states are given in [59, 63].

3.4. Hermitization of Dyson Mapping. The main advantage of Dyson mapping is that the generators of the $\mathrm{sp}\,(2d, R)$ algebra are expressed as finite boson expansions and include only two-boson terms in the case of $\rho(A_{Tt}^{LM})$ operators, or one-boson and three-boson terms in the case of $\rho(F_{Tt}^{LM})$ and $\rho(G_{Tt}^{LM})$ operators. The price for this advantage is its non-Hermiticity which is evident from (48) and (50):

$$(\rho(G_{Tt}^{LM}))^\dagger \neq \rho(F_{Tt}^{LM}). \tag{73}$$

However, this disadvantage can be avoided by performing similarity transformation [44, 50, 54, 64] that converts a Dyson boson realization into a Holstein–Primakoff-type boson realization. This transformation makes the $\rho_D(G_{Tt}^{LM})$ and $\rho_D(F_{Tt}^{LM})$ operators Hermitian, without changing the multipole operators

$\rho_D(A^{LM}_{Tt})$, i.e.,

$$K\rho(F^{LM}_{Tt})K^{-1} = \rho_{\text{HP}}(F^{LM}_{Tt}), \tag{74a}$$

$$K\rho(G^{LM}_{Tt})K^{-1} = \rho_{\text{HP}}(G^{LM}_{Tt}), \tag{74b}$$

$$K\rho(A^{LM}_{Tt})K^{-1} = \rho_{\text{HP}}(A^{LM}_{Tt}) = \rho(A^{LM}_{Tt}), \tag{74c}$$

$$(\rho_{\text{HP}}(G^{LM}_{Tt}))^\dagger = \rho_{\text{HP}}(F^{LM}_{Tt}). \tag{74d}$$

If we impose the condition $K^\dagger = K =$ real, we can define an operator V:

$$V = K^\dagger K = K^2. \tag{75}$$

Note that $V^\dagger \neq V^{-1}$. From (74a)–(74d) and (75) we can write

$$V^{-1}(\rho(F^{LM}_{Tt}))^\dagger V = \rho(G^{LM}_{Tt}), \tag{76a}$$

$$V^{-1}(\rho(G^{LM}_{Tt}))^\dagger V = \rho(F^{LM}_{Tt}), \tag{76b}$$

$$V^{-1}\rho(A^{LM}_{Tt})V = (\rho(A^{LM}_{Tt}))^\dagger = \rho(A^{LM}_{Tt}). \tag{76c}$$

It follows from (76c) that V commutes with the operators $\rho(A^{LM}_{Tt})$ (satisfying $((\rho(A^{LM}_{Tt}))^\dagger = \rho(A^{LM}_{Tt})))$. This, in turn, implies that V, and therefore S, are functions of the Casimir invariants of the $u(d)$ algebra.

Within the physical subspace only the Casimir operators $C_2(u(d))$ (52) may be replaced by functions of the number operator. This replacement leads to an essential simplification of (55):

$$\rho(F^{LM}_{Tt}) = b^{\dagger LM}_{Tt}(1 + 2N_b). \tag{77}$$

It is important to emphasize that the F^{LM}_{Tt} and G^{LM}_{Tt} operators, when applied to the wave function corresponding to the symmetric IR $[N]$, transform it to the wave function corresponding to the symmetric IRs $[N \pm 2]$ again. This allows the Hermitian boson realization to be obtained in a simple analytic form. Substituting (77) and (48) in (76b) we get

$$V^{-1}b^{\dagger LM}_{Tt}V = b^{\dagger LM}_{Tt}(1 + 2N_b). \tag{78}$$

Let us choose a basis in the IR $[2N_b]$ of the $u(d)$ algebra. For our aims it is sufficient to mark this basis only by the boson number operator eigenvalue. The operator V is diagonal in this basis: $V|[2N_b]\rangle = V([2N_b])|[2N_b]\rangle$. The operators $b^{\dagger LM}_{Tt}$ lead only to a change of the boson number in the wave function, as $b^{\dagger LM}_{Tt}|[2N_b]\rangle \sim |[2N_b + 2]\rangle$. Hence we have from (78)

$$V^{-1}([2N_b + 2])V([2N_b]) = (1 + 2N_b), \tag{79}$$

$$K^{-1}([2N_b + 2])K([2N_b]) = \sqrt{(1 + 2N_b)}. \tag{80}$$

Formula (80) can be used as a recurrent relation to find the matrix elements of the K operator. Thus we have in an operator form

$$\rho_{\mathrm{HP}}(G_{Tt}^{LM}) = \sqrt{(1+2N_b)}\, b_{Tt}^{LM}, \tag{81a}$$

$$\rho_{\mathrm{HP}}(F_{Tt}^{LM}) = b_{Tt}^{\dagger LM} \sqrt{(1+2N_b)}. \tag{81b}$$

These formulae are similar to the well-known boson realization of $\mathrm{sp}\,(2,R)$ algebra. As a matter of fact this analogy is deeper. Actually the origin of the non-Hermiticity of the Dyson mapping lies in the fact that it maps a general nonorthogonal basis of a Lie algeba IR on an orthonormal basis in the ideal boson space [50]. Dyson mapping is therefore, by definition, nonunitary (non-Hermitian). However, in the case of odd and even IR of the $\mathrm{sp}\,(2d,R)$ algebra, which are under consideration only, the vectors forming the basis in the primary u-boson space are orthogonal to each other but this basis is not normalized. Hence, the Hermitization procedure is reduced to the calculation of the necessary normalization factor in order to restore the norm of the basis.

In summary, the general features of the mapping of the pairs of bosons were investigated. The general results obtained for the exact boson mapping of the $\mathrm{sp}\,(2d,R)$ algebra on the $u(d)$-boson algebra are then applied to the special case of algebra $\mathrm{sp}\,(12,R)$ which is the dynamical symmetry algebra of the IVBM. The Dyson boson image of the IVBM Hamiltonian is written and its connection with a version of standard IBM is discussed.

From physical point of view, a similar situation occurs in the isospin-invariant version of the fermion dynamical symmetry model (FDSM) [65, 66] applied to the sd-shell nuclei. In this version of the FDSM, the nucleons occupying the single-particle levels $2s(1/2)$, $d(3/2)$, and $d(5/2)$ are considered as the particles with the pseudoorbital angular momentum $k = 1$, the pseudospin $i = 3/2$, and the isospin $t = 1/2$. In the k-invariant version of the FDSM it is assumed that the pseudospins of the particles are paired and only the states constructed from the nucleon pairs in the states with $K = 0, 2$; $I = 0$, $J = K$; T are taken into account. In correspondence with the Pauli principle such pairs have the following combinations of the total angular momentum and isospin:

$$(J, T) = (0, 1), (2, 1), (1, 0).$$

In order to describe these states in the framework of the interacting boson model it is necessary to introduce S, D bosons with the isospin $T = 1$ and P boson with $T = 0$. This set of bosons coincides identically with the bosons taking part in the IVBM mapping discussed in the present paper. It is important to note also that in the microscopic FDSM picture these bosons are constructed from two p objects ($k = 1$) as well as S, P, D bosons in the IVBM.

The similarity of the k-active version of the FDSM and the IVBM allows us to hope that the structure of the sd-shell nuclei (or some other medium mass nuclei with $k = 1$) can be interpreted in terms of the IVBM in the same manner, as it was done in [67] for the nuclei from the beginning of the sd shell in the framework of the IBM-4. The investigation in this direction will be the subject of our further investigation.

4. THE SYMPLECTIC EXTENSION OF THE IVBM

We consider $\mathrm{Sp}\,(12, R)$ to be the group of the dynamical symmetry of the model [14], because the most general one- and two-body Hamiltonian can be expressed in terms of its generators (29a). Using the commutation relations (11c) between the $F^L_M(\alpha, \beta)$ and $G^L_M(\alpha, \beta)$, the full range of number of bosons preserving Hamiltonian (33), (34) can be expressed in terms of the operators $A^L_M(\alpha, \beta)$.

Being a noncompact group, the representations of $\mathrm{Sp}\,(12, R)$ are of infinite dimension, which makes it difficult to diagonalize the most general Hamiltonian. The operators $A^L_M(\alpha, \beta)$ generate the maximal compact subgroup of $\mathrm{Sp}\,(12, R)$, namely the group $U(6) \subset \mathrm{Sp}\,(12, R)$. The $U(6)$ subspaces contained in the even and odd boson representations of $\mathrm{Sp}\,(12, R)$ are of finite dimension, which simplifies the problem of diagonalization. Therefore the *complete* spectrum of the system can be calculated only through the diagonalization of the Hamiltonian in the subspaces of *all* the UIR of $U(6)$, belonging to a given UIR of $\mathrm{Sp}\,(12, R)$.

In this section we will present an application of this approach, where we use the «unitary» limit [15] of the model, in the sense that the basis states and the Hamiltonian are defined by means of the dynamical symmetry along the chain (see the right-hand side reduction chain of (16)):

$$\mathrm{Sp}\,(12, R) \supset U(6) \supset SU(3) \otimes U(2) \supset SO(3) \otimes U(1), \tag{82}$$

$$[N] \quad (\lambda, \mu) \quad (N, T)\ K \quad L \qquad T_0, \tag{83}$$

where the labels below the subgroups are the quantum numbers (83) corresponding to their irreducible representations. Their values are obtained by means of standard reduction rules and are given in [15]. As already shown, in this limit the operators of the physical observables are the angular momentum operator L_M (18) and the truncated («Elliott's») quadrupole operator Q_M (21) which define the algebra of $SU(3)$. The «T-spin» operators (20) and the number of bosons N (17) define the algebra of $U(2)$. Since the reduction from $U(6)$ to $SO(3)$ is carried out by the mutually complementary groups $SU(3)$ and $U(2)$, their quantum numbers are related in the following way:

$$T = \frac{\lambda}{2}, \qquad N = 2\mu + \lambda. \tag{84}$$

Making use of the latter we can write the basis as

$$\mid [N]_6; (\lambda, \mu); K, L; T_0\rangle = \mid (N, T); K, L; T_0\rangle. \tag{85}$$

The ground state of the system is $\mid 0\rangle = \mid (N=0, T=0); K=0, L=0; T_0=0\rangle$ the vacuum state for the $\mathrm{Sp}\,(12,R)$ group.

4.1. The Basis in the Symplectic Extension of IVBM. The $\mathrm{Sp}\,(12,R)$ classification scheme for the $SU(3)$ boson representations obtained by applying the reduction rules [15] for the irreps in the chain (82) for even value of the number of bosons N is shown in Table 1. Each row (fixed N) of the table corresponds to a given irreducible representation of the $U(6)$ algebra. Then the possible values for the «T spin», given in the column next to the respective value of N, are $T = N/2, N/2-1, \ldots, 0$. Thus when N and T are fixed, $2T+1$ equivalent representations of the group $SU(3)$ arise. Each of them is distinguished by the eigenvalues of the operator $T_0 : -T, -T+1, \ldots, T$, defining the columns of Table 1. The same $SU(3)$ representations (λ,μ) arise for the positive and negative eigenvalues of T_0.

Table 1. **Classification of the basis states in the even H_+ space of $\mathrm{Sp}\,(12,R)$ in the $U(6)$ limit of the IVBM**

N	T	$T_0\backslash\ \ldots\ \pm 4$	± 3	± 2	± 1	0
0	0				$\swarrow F^{[2]_6}_{[2]_3[2]_2}$	$(0,0)$
2	1			$F^{[2]_6}_{[1,1]_3[0]_2} \downarrow$	$\Longrightarrow$ $(2,0)$	$(2,0)$
	0				$A^{[1-1]_6}_{[2,1]_3[0]_2}$ —	$(0,1)$
4	2			$(4,0)$	$(4,0)$	$(4,0)$
	1			—	$A^{[1-1]_6}_{[2,1]_3[2]_2} \Downarrow$ $(2,1)$	$(2,1)$
	0			—	—	$(0,2)$
6	3		$(6,0)$	$(6,0)$	$(6,0)$	$(6,0)$
	2	$A^{[1-1]_6}_{[0]_3[2]_2}$ $\rightarrow$	—	$(4,1)$	$(4,1)$	$(4,1)$
	1		—	—	$(2,2)$	$(2,2)$
	0		—	—	—	$(0,3)$
8	4	$(8,0)$	$(8,0)$	$(8,0)$	$(8,0)$	$(8,0)$
	3	—	$(6,1)$	$(6,1)$	$(6,1)$	$(6,1)$
	2	—	—	$(4,2)$	$(4,2)$	$(4,2)$
	1	—	—	—	$(2,3)$	$(2,3)$
	0	—	—	—	—	$(0,4)$
⋮		⋮	⋮	⋮	⋮	⋮

Hence, in the framework of the discussed boson representation of the $\mathrm{Sp}\,(12,R)$ algebra all possible irreducible representations of the group $SU(3)$ are determined uniquely through all possible sets of the eigenvalues of the Hermitian operators N and T^2. The equivalent use of the (λ,μ) labels facilitates the final reduction to the $SO(3)$ representations, which define the angular momentum L and its projection M, as we make use of the standard reduction rules for the $SU(3) \supset SO(3)$ chain [9]:

$$\begin{aligned} K &= \min(\lambda,\mu), \min(\lambda,\mu)-2,\ldots,0(1),\\ L &= \max(\lambda,\mu), \max(\lambda,\mu)-2,\ldots,0(1); \quad K=0,\\ L &= K, K+1,\ldots,K+\max(\lambda,\mu); \quad K\neq 0. \end{aligned} \tag{86}$$

The multiplicity index K appearing in this reduction is related to the projection of L on the body fixed frame and is used with the parity (π) to label the different bands (K^{π}) in the energy spectra of the nuclei. We define the parity of the states as $\pi = (-1)^T$. This allows us to describe both positive and negative parity bands.

4.2. The Energy Spectrum. The Hamiltonian, corresponding to the considered unitary limit of IVBM, is expressed in terms of the first and the second order invariant operators of the different subgroups in the chain (82):

$$H = a'N + \alpha_6 K_6 + \alpha_3 K_3 + \alpha_1 K_1 + \beta_3 \pi_3, \tag{87}$$

where K_n are the quadratic invariant operators of the $U(n)$ groups; π_3 is the second order Casimir operator of the $SO(3)$ group. As a result of the connections (84), the Casimir operator K_3 with eigenvalue $(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu)$ is expressed in terms of the operators N and T:

$$K_3 = 2Q^2 + \frac{3}{4}L^2 = \frac{1}{2}N^2 + N + T^2.$$

Making use of the above relation, the Hamiltonian (87) takes the form

$$H = aN + bN^2 + \alpha_3 T^2 + \beta_3 \pi_3 + \alpha_1 T_0^2 \tag{88}$$

and is obviously diagonal in the basis (85) labelled by the quantum numbers (83) of the subgroups of the chain (82). Its eigenvalues are the energies of the basis states of the boson representations of $\mathrm{Sp}\,(12,R)$:

$$E((N,T);KLM;T_0) = aN + bN^2 + \alpha_3 T(T+1) + \beta_3 L(L+1) + \alpha_1 T_0^2. \tag{89}$$

The energy of the ground state of the system is obviously 0.

4.3. Application of IVBM for the Description of the Ground and Octupole Bands Energies. The existence of nuclei with stable deformed shapes was realized early in the history of nuclear physics. The observation of large quadrupole moments led to the suggestion that some nuclei might have spheroidal shapes, which was confirmed by the observation of rotational band structures and measurements of their properties. For most deformed nuclei, a description as an axial- and reflection-symmetric spheroid is adequate to reproduce the band's spectroscopy. Because such a shape is symmetric under space inversion, all members of the rotational band have the same parity. However, with the first observation of negative parity states near the ground state, the possibility that some nuclei might have an asymmetric shape under reflection arose.

On the other hand, whenever symmetry breaking appears new behavior of the many-body system is expected. Reflection symmetry breaking is associated with a static octupole deformation which is expected to determine new collective features for the nuclear system.

Extensive investigations into the structure of nuclei with low-lying negative parity states have led to the conclusion that, while reflection asymmetric shapes can play a role in the band structure, they are not as stable as the familiar quadrupole deformations. The rotational spectra of some even–even nuclei in the rare-earth and light actinide regions exhibit, next to the ground band, a negative parity band with $K^{\pi} = 0^{-}$, which consists of the states with $I^{\pi} = 1^{-}, 3^{-}, 5^{-}, \ldots$ These two bands are displaced from each other, which means that fluctuations back to space symmetric shapes must also be significant. Experimentally, the presence of «octupole» bands for some isotopes from the light actinide and rare-earth region [68] is firmly established.

There is a large variety of models that try to describe this behaviour of the low-lying states of deformed nuclei [69]. Particularly successful are the algebraic ones, that are based on symmetry principles. The introduction of an additional octupole degrees of freedom is a common feature of the most of those models.

The prescription for describing negative parity states by the addition of an f boson to the usual s and d bosons of the IBM was first mentioned by Iachello and Arima [2]. It was suggested [70] that the inclusion of a p boson to the s, d, and f bosons may play an important role in the description of these collective states.

The coherent state method (CSM) was applied by Alonso et al. to the spdf $SU(3)$ Hamiltonian with quadrupole and octupole interactions [71]. A. A. Raduta and D. Ionescu [72, 73] have used a generalization of the CSM. They suggested that both ground and octupole bands may be considered as being projected from a single deformed intrinsic state that exhibits both quadrupole and octupole deformations.

The introduction of an octupole degrees of freedom in the presence of comparatively large number of free parameters in all of these models allows for the reproduction of the experimental data of the energies of the negative parity states, at least in the low-spin region.

As mentioned earlier, the Interacting Vector Boson Model (IVBM), introduced [14] in the beginning of the 1980s, was a generalization of the phenomenological broken $SU(3)$-symmetry model [12], which provided a good description of the low-lying ground and γ bands of the well-deformed even–even nuclei. Its advantages were incorporated into the rotational limit of the IVBM [15], resulting in a good description of all the positive parity bands of nuclei in the rare-earth and the actinide regions. Moreover, the $U(6)$ extension of the model contains such sequences of $SU(3)$ multiplets, some of which prove to be convenient for the description of the low-lying negative parity bands [74].

With the recent advance of the experimental technique the investigated bands are extended to very high angular momenta [68]. This motivated a new approach within the framework of the model aimed at a description of the first positive and negative bands, up to very high spins. In this new application, we make use of the symplectic extension of the IVBM [38, 75]. This allows these bands to be considered as yrast bands in the sense, that we take into account the states with a given L, which minimize the energy values with respect to N. N is the eigenvalue of the total number of bosons that build the basis states of the model. As a first test of the symplectic extension of the IVBM [74], we modify its earlier application for the description of the first excited even and odd parity bands in order to reach much higher angular momentum states in both band types. We will apply the model to even–even deformed nuclei, which exhibit a low-lying negative parity band next to the ground band traditionally considered to be an octupole band [68]. In order to do this we first have to identify these experimentally observed bands with the sequences of basis states for the even representation of $\mathrm{Sp}\,(12, R)$ given in Table 1. We choose the $SU(3)$ multiplets $(0, \mu)$ for a description of the ground band, whereas for the octupole band the $SU(3)$ multiplets $(2, \mu - 1)$ are used. In terms of (N, T) this choice corresponds to $(N = 2\mu, T = 0)$ for the positive $(K^\pi = 0^+)$ and $(N = 2\mu + 2, T = 1)$ for the negative $(K^\pi = 0^-)$ parity band, respectively. Hence in the framework of the symplectic extension of boson representations of number preserving $U(6)$ symmetry we are able to consider all even eigenvalues of the number of vector bosons N with the corresponding set of T values.

First, we define the energies of each state with given L as yrast energy with respect to N in the two considered bands. This approach is based on the fact that the energies (89) increase with increasing of N. Hence their minimum values are obtained at $N = 2L$ for the ground band, and $N = 2L + 2$ for the octupole band, respectively. So for the description of the ground band our choice corresponds to the sequence of states with different numbers of bosons, $N = 0, 4, 8, \ldots$ and «T spin» $T = 0$ in the column labelled $T_0 = 0$ of Table 1. Respectively for a description of the negative parity band, we choose the set of states with quantum numbers $N = 8, 12, \ldots$ and $T = 1$ from the same column $T_0 = 0$. In the so-defined $SU(3)$ representations for each N the maximal values of L appear for the first time (see Table 1).

Because for the basis states corresponding to the experimental ones of the ground and octupole bands, $T_0 = 0$, the last term in the energy formula (89) vanishes. The phenomenological model parameters a, b, α_3, and β_3 are evaluated by a fit to the experimental data. The comparison between the experimental spectra and our calculations using the values of the model parameters given in the figure caption for the ground and octupole bands of the nuclei ^{226}Th and ^{168}Yb is

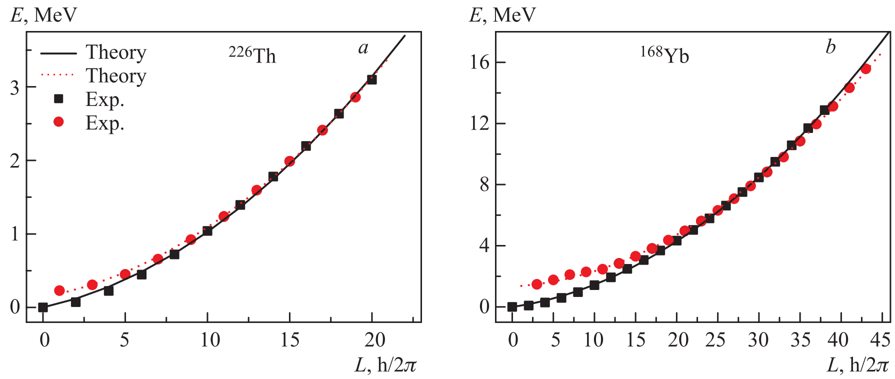


Fig. 1. Comparison between the calculated (89) (Theory) and experimental (Exp.) energies of the ground and octupole bands for the nuclei ^{226}Th (a) and ^{168}Yb (b). The latter are calculated with the parameters $a = 0.0194$, $b = -0.0009$, $\alpha_3 = 0.0522$, and $\beta_3 = 0.0094$ for the ^{226}Th and $a = 0.0235$, $b = -0.0056$, $\alpha_3 = 0.6512$, and $\beta_3 = 0.0295$ for the ^{168}Yb

illustrated in Fig. 1. All experimental data are taken from [1, 68]. The agreement between the theoretical values obtained with only four model parameters and the experimental data is rather good.

Applying the yrast conditions relating N and L, the energies (89) for the two considered bands can be rewritten as

$$E(L) = \beta L(L+1) + (\gamma + 8bT)L + \xi T(T+1) + \gamma T. \tag{90}$$

The new free parameters β, γ, and ξ are related to the previous ones as follows:

$$\beta = 4b + \beta_3, \quad \gamma = 2a - 4b, \quad \xi = 2a + 4b + 2\alpha_3. \tag{91}$$

The values of the parameters (91) determine the behavior of the energies of the two bands and their position with respect to each other. In some cases, the two bands are almost parallel. The shift between them depends on the parameter ξ. When they are very close, they interact through the L-dependent interaction with a strength $\gamma + 8bT$.

As a result of our theoretical assumptions, we obtained a simple formula for the energy levels. From (90) we can see that the energies of the eigenstates of the first positive and negative bands consist of rotational $L(L+1)$ and vibrational L modes. The rotational interaction is with equal strength β in both of the bands.

Some fine structure effects, as back-banding and staggering behavior are observed in the collective rotational spectra of deformed even–even nuclei in this mass region. Odd–even staggering patterns between ground and octupole bands

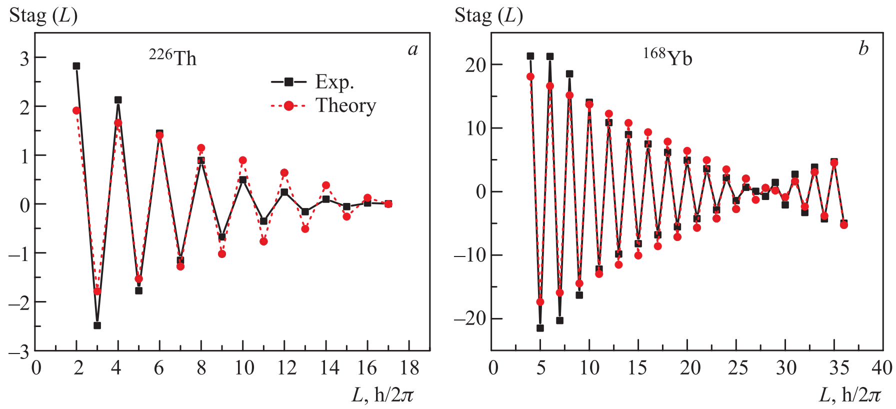


Fig. 2. Comparison between calculated (Theory) and experimental (Exp.) staggering functions Stag (L) (92) for the energies of the ground and octupole bands with $\Delta L = 1$ for the nuclei ^{226}Th (a) and ^{168}Yb (b)

have been investigated recently [76]. In order to test further our model we applied on the energies the staggering function defined as

$$\mathrm{Stag}\,(L) = 6\Delta E(L) - 4\Delta E(L-1) - 4\Delta E(L+1) + \Delta E(L+2) + \Delta E(L-2), \quad (92)$$

where $\Delta E(L) = E(L) - E(L-1)$. This function is a finite difference of the fourth order in respect to $\Delta E(L)$ or of the fifth order in respect to energy $E(L)$. The calculated and experimental staggering patterns are illustrated in Fig. 2. One can see a good agreement with experiment, as well as the reproduction of the «beat» patterns of the staggering behavior. They occur in the region where the interaction between the two considered bands is most strong or they cross (see Fig. 1). The correct reproduction of the experimental staggering patterns is due to the interaction term depending on L in (90) between the positive and negative parity bands, which is a result of the introduced notion of yrast energies in the framework of the symplectic extension of the IVBM.

4.4. Application to Other Excited Bands. In order to demonstrate further applicability of this dynamical symmetry, here in what follows we use the theory to describe some additional positive and negative parity bands. The next most important bands for determining the collective properties of heavy nuclei are the positive parity β $(K^{\pi} = 0^{+})$ [77] and γ $(K^{\pi} = 2^{+})$ collective bands and low-lying negative parity bands with $K^{\pi} = 1^{-}, 3^{-}$ (see, for example, [78]).

These excited bands have to be mapped on the basis states as well. In general, the appropriate [25] subset of $SU(3)$ states are the so-called «stretched» states. Their domination is determined by the important role of the quadrupole–quadrupole interactions in the collective excitations. Thus, the most important

$SU(3)$ states will be those with maximal weight, i.e., those which have maximal eigenvalues of the second order $SU(3)$ Casimir operator. For the considered chain we have two types of stretched states.

The first one, defined by the $SU(3)$ irreps $(\lambda, \mu) = (\lambda_0, \mu_0 + i)$ is a generalization of the eigenstates attributed to the ground and octupole bands, which were selected above by means of the algebraic notion of yrast bands. Obviously in this case $\lambda_0 = 2T$ is fixed, which fixes the parity of the bands. We require that $\lambda_0 \leqslant \mu_0$ and this relates it to the value of K, that labels the bands. For the $K = 0$ bands we have $i = 0, 2, 4\ldots$ and N changing in steps of 4 and for the $K \neq 0$ — the values of $i = 0, 1, 2\ldots$ and $\Delta N = 2$. Hence, it is obvious that the changing value of $\mu = \mu_0 + i$ is associated with the corresponding value of the angular momentum $L = \mu$ of the states belonging to the considered band. As a result, the yrast condition that relates the number of bosons $N = \lambda + 2\mu$ with the angular momenta is $N = \lambda_0 + 2\mu_0 + 2i = 2T + 2L$. As a result of this generalization we get the same expression for the energies as in (90). The constant α_3 in the Hamiltonian enters here through the parameter ξ as an additive constant energy for each band.

For the second type of stretched states $(\lambda, \mu) = (\lambda_0 + 2i, \mu_0)$, obviously $T = (\lambda_0 + 2i)/2$ is changing and in order to preserve the parity of the bands we need to have $i = 0, 2, 4\ldots$ and so $\Delta N = 4$. So, for $\lambda_0/2$ being even, we obtain the positive parity states and for the odd values of $\lambda_0/2$ — the negative ones. Now $\mu_0 \leqslant \lambda_0$ is fixed and related to the value of K, that labels the bands. In this case the yrast condition is a result of the definition $\lambda = 2L = 2T$ and is given by the relations $N = 2L + 2\mu_0 = 2L + 2N_0 - 4\lambda_0$. Hence, the values of the energies given by the equation

$$E(L) = \beta L(L+1) + (\gamma + 8b\mu_0)L + 4\xi\mu_0(\mu_0 + 1) + \gamma\mu_0, \tag{93}$$

are very similar to the ones (90) in the previous case, but T is changed to μ_0 and now α_3 enters in the inertial parameter $\beta = 4b + \beta_3 + \alpha_3$, while γ is the same as in (91).

For the excited β bands, we can use the above sequences of $SU(3)$ irreps $(\lambda, \mu) \equiv (2T, L)$ or $(2L, \mu_0)$ with fixed λ or fixed μ and L changing. These bands are raised in energy in respect to the ground state band (GSB) by the number of bosons $N_0 = \lambda_0 + 2\mu_0$ that go into building the band-head configuration.

If we consider T_0 as fixed in both cases, the stretched $SU(3)$ irreps belong to a column in Table 1, and if they change, one moves along a diagonal. In the second case T_0 plays a more important role because the value of T is changing with the development of the bands and we have more choices in changing, respectively, the values of $T_0 = \pm T$, $\pm(\mathrm{T}-1), \ldots, 0$, and hence we can make use of the additional parameter α_1 in (89) when fitting the values of the energies. For the excited β and γ bands, we use the sequences of stretched $SU(3)$ irreps determined by the «diagonals» $(\lambda_0 + 2i, \mu_0)$ with $\mu_0 = 0$ or 2.

Table 2. **Values of the parameters of the model Hamiltonian (88) obtained in the fitting to the experimental spectra of the considered nuclei**

Nucleus	Bands	s	T_0	N_0	χ^2	Parameters
^{172}Yb	Ground	8	0	0	0.00049	$a = 0.01063$ $b = 0.00145$
	$\beta_1(K^\pi = 0^+)$	7	0	48	0.00213	$\alpha_3 = -0.00485$
$R_{4/2} = 3.33$	$\gamma_1(K^\pi = 2^+)$	4	0	48	0.00062	
	$\gamma_2(K^\pi = 2^+)$	5	1	52	0.02155	$\beta_3 = 0.00517$
	$K^\pi = 3^+$	12	2	44	0.00188	$\alpha_1 = -0.16628$
^{162}Dy	Ground	9	0	0	0.00150	$a = 0.01574$ $b = 0.00131$
	$\beta_1(K^\pi = 0^+)$	8	2	10	0.00084	$\alpha_1 = 0.30902$
$R_{4/2} = 3.31$	$\gamma_1(K^\pi = 2^+)$	14	0	24	0.00212	$\alpha'_3 = -0.00328$
	$K^\pi = 1^-$	5	0	62	0.00160	$\alpha_3 = -0.00476$
	$K^\pi = 2^-$	13	0	40	0.00562	$\beta_3 = 0.00473$
^{152}Sm	Ground	9	0	0	0.00225	$a = 0.03083$ $b = 0.00118$
	$\beta(K^\pi = 0^+)$	8	0	16	0.00250	$\alpha_3 = -0.00311$
$R_{4/2}$ =3.025	$\gamma(K^\pi = 2^+)$	8	0	24	0.0064	$\alpha_1 = -0.00184$
	$K^\pi = 5^-$	6	$L + 8$	34	0.00432	$\beta_3 = 0.00442$

It is well known that β bands have rather irregular behavior even in neighboring nuclei of the region under consideration [77]. The reason for this was established and correctly reproduced by applying a proton–neutron version of the algebraic shell model with pseudo-$SU(3)$ symmetry [79]. The microscopic interpretation of the relative position of collective bands, as well as that of the levels within these bands, follows from an evaluation of the primary $SU(3)$ content of the collective states [80], and the established connection between the microscopic quantum numbers (λ, μ) and the collective shape variables (β,γ) [81]. In our phenomenological approach, because of the symplectic extension of the number preserving model [15], we have a great flexibility in identifying bands in any of the ways described above. As a result of the $N \longleftrightarrow L$ connections we always get a mixing of the rotational and vibrational collective modes, represented by the $L(L + 1)$ and L terms, respectively.

We illustrate these advantages for the nuclei ^{172}Yb, ^{162}Dy, and ^{152}Sm. The values of the Hamiltonian (88) parameters a, b, β_3, α_3, and α_1, obtained in a fitting procedure for all the states of the considered bands (second column) are given in Table 2. In the third column, s gives the number of experimentally observed states in each band and next to it the values of T_0, are also given. For the values of N_0, given in the fifth column, we get the ones, that give the best χ^2 values for the corresponding band. In the present application all the bands except the ground band are considered as developing in the $SU(3)$ multiplets

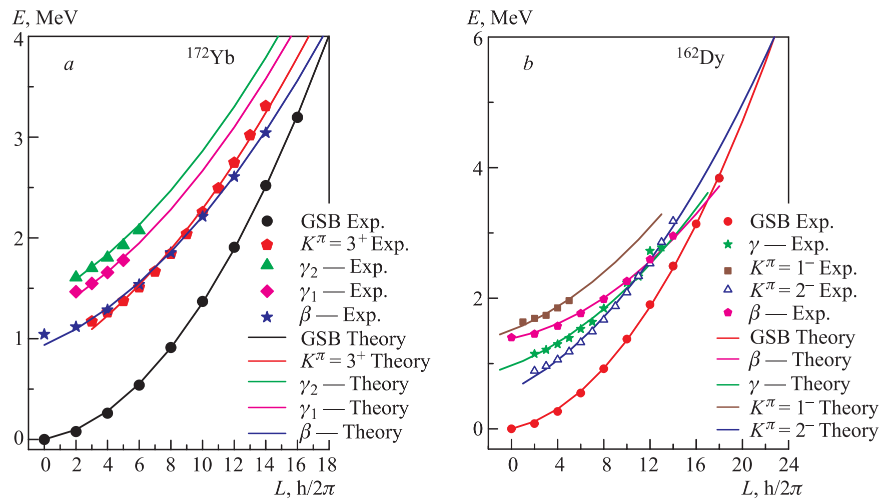


Fig. 3. Comparison of the $U(6)$-limit theoretical and experimental energies of the states from several collective positive and negative parity bands of the rotational ^{172}Yb (a) and ^{162}Dy (b) nuclei

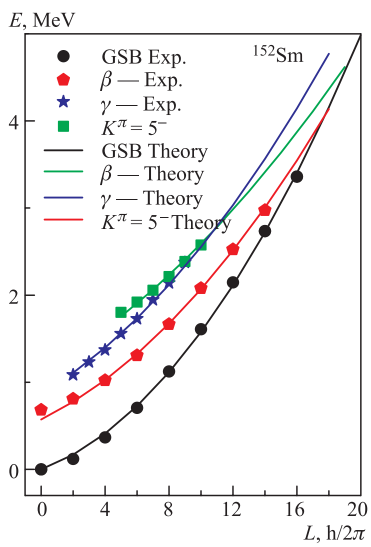


Fig. 4. The same as in Fig. 3 for the ^{152}Sm

of the second type with changing λ and fixed μ. The comparison of the theoretical results with the experimental data is illustrated in Figs. 3 and 4. It is clear from them that we can get rather good results in reproducing the different positions of the β and γ bands in respect to each other, the γ bands are above the β ones in the ^{152}Sm and ^{172}Yb nuclei and almost parallel. In the spectrum of ^{162}Dy they exchange positions and cross at relatively low spin $L \approx 12$. This requires the introduction of an additional parameter α_3' (given in Table 2) for the description of the γ band. The position and the development of the negative parity bands are also reproduced quite well. In general, the band crossings that are obtained in the description of the collective bands in the framework of the model, rather well reproduce the experimental situation (see Figs. 3 and 4) and can be used further for studying finer effects in the structure of nuclear spectra.

We have considered, except for spectra of the two well-deformed nuclei ^{172}Yb and ^{162}Dy, given in Fig. 3 with $R_{4/2} = 3.3$, the ^{152}Sm, which is an example [82] of a nucleus at the critical point symmetry $X(5)$ [4] and whose collective bands are rather well described as well (see Fig. 4). The energies of the gsb are reproduced very well by means of the same correspondence of the basis states $(0, \mu)$ to the yrast states of the ground band and the yrast condition $N = 2\mu = 2L$, which gives a mixing of the rotational and vibrational modes (see (90)), which depends on the parameters a, b, and β_3, given in Table 5. The next important feature of the nuclei at the $X(5)$ critical point symmetry is defined by the position of the β band [82], which in our model is very well adjusted due to the possibility of choosing the value of the number of bosons N_0, that build the band-head configuration. Then the development of the energies in the band along the $(2T, 0)$ — $SU(3)$ multiplets (T-even) involves the unfixed yet parameter α_3 (93) and adjusts them additionally. In this way we prove that the symplectic extension of the rotational $U(6)$ limit of the IVBM is appropriate for the description of nuclei with more complex features than the rotational ones. Nevertheless, a more detailed and systematic investigation of the values of N_0 and the Hamiltonian parameters for sequences of nuclei in the regions, where some type of phase/shape transitions is expected, is necessary and comes into our future intents.

A good illustration of the model's qualities is presented by its recent use, in a slightly modified form [83], for the description of the new experimental data on high spin states in the ^{160}Dy [84], that has very similar properties to the example given here with ^{162}Dy (see the right side of Fig. 3). In [83], three positive bands: the ground band, $K^\pi = 0^+$ — S band, $K^\pi = 2^+$ — γ band, as well as two negative $K^\pi = 1^-$ and $K^\pi = 2^-$ bands are described rather accurately up to very high spins, by means of corresponding different choices of $SU(3)$ multiplets with fixed λ or μ for each band. In addition possible band crossing, between collective bands with different $SU(3)$ irreps associated, was considered, which considerably improved the reproduction of the collective behavior in particular of the higher lying states. The investigation of the odd–even staggering between states from different bands is also used to obtain the correct ordering of the high spin states in the bands. The success of this application was based on the concept of the correct energy distribution of the states with fixed angular momentum, and, in particular, the 0^+ band-head configurations.

We have applied the Interacting Vector Boson Model for the description of the ground and octupole bands in some even–even rare-earth and actinide nuclei up to very high spins. In spite of the simplicity of the model, without introducing additional degrees of freedom we are able to describe both positive and negative parity bands. This is due to the specific definition of the states parity depending on the quantum number T. The successful reproduction of the experimental energies and of their odd–even staggering was achieved as a result of their consideration as yrast energies in respect to the number of phonon excitation N that build the collective states. The introduction of this notion was possible, as we extended the IVBM to its symplectic dynamical symmetry $\mathrm{Sp}\,(12, R)$, which allows the change of the number of bosons that are the building blocks of the model Hamiltonian. Nevertheless, the Hamiltonian remains with only few phenomenological parameters and is still exactly solvable. In the considered unitary limit of the model in addition to the rotational degree of freedom in the ground and excited bands a purely vibrational mode is appearing, which introduces also some interaction between them. This is the reason for the reproduction also of the fine effect of the structure of these bands. The obtained physically meaningful results are also simple and easy for use and they permit the application of the model to larger class of nuclei than the purely rotational ones.

The symplectic extension of the IVBM permits a richer classification of the states than its unitary version and gives the possibility for a further consideration of other collective bands.

5. TRANSITION PROBABILITIES IN THE $U(6)$ LIMIT OF THE SYMPLECTIC INTERACTING VECTOR BOSON MODEL

Up till now we have used the $U(6)$ dynamical symmetry defined by the reduction chain (82) to obtain exact solutions for the eigenvalues and eigenfunctions of the model Hamiltonian, which is constructed from the invariant operators of the subgroups in the chain. But it is well known that the comparison of the experimental data with the calculated transition probabilities is one of the best tests of the validity of the employed algebraic model. With the aim of such applications of the symplectic extension of the considered dynamical symmetry of the IVBM, we develop in this section a practical mathematical approach for explicit evaluation of the matrix elements of the transitional operators in this limit [85].

In the algebraic approach, the evaluation of the matrix elements of transition operators between the eigenstates, makes use of the fact that both — the basis states and the operators, can be defined as tensors in respect to the considered dynamical symmetry. Then the calculation of matrix elements is simplified by the application of a respective generalization of the Wigner–Eckart theorem, which requires the calculation of the isoscalar factors and reduced matrix elements. The transition probabilities between the collective states attributed to the basis states of the Hamiltonian are by definition the $SO(3)$ reduced matrix elements of transition operators T^{E2} between the $|i\rangle$-initial and $|f\rangle$-final collective states

$$B(E2; L_i \to L_f) = \frac{1}{2L_i + 1} |\langle f \parallel T^{E2} \parallel i\rangle|^2. \qquad (94)$$

As a first step we test the theory on the transitions between the states belonging to the ground bands in the even–even nuclei from the rare earths and the actinides [85], where the energies and the staggering between the states are rather well reproduced in our model approach [38]. This proves the correct mapping of the basis states to the experimentally observed ones and their band systematics, which is very important for the theoretical reproduction of the behavior of the physical observables in the framework of the considered model.

5.1. Tensorial Properties of the Generators of the $\mathrm{Sp}(12, R)$ Group and the Basis States of IVBM. As already introduced, the basic building blocks of the IVBM [14] are the creation and annihilation operators of the vector bosons $u_m^{\dagger}(\alpha)$ and $u_m(\alpha)$ $(m = 0, \pm 1; \alpha = \pm 1/2)$, which can be considered as components of a 6-dimensional vector, which transform according to the fundamental $U(6)$ irreducible representations $[1,0,0,0,0,0]_6 \equiv [1]_6$ and $[0,0,0,0,0,-1]_6 \equiv [1]_6^*$, respectively. These irreducible representations become reducible along the chain of subgroups (82) defining the dynamical symmetry [15]. This means that along with the quantum number characterizing the representations of $U(6)$, the operators are also characterized by the quantum numbers of the subgroups of chain (82).

The only possible representation of the direct product of $U(3) \otimes U(2)$ belonging to the representation $[1]_6$ of $U(6)$ is $[1]_3 \cdot [1]_2$, i.e., $[1]_6 = [1]_3 \cdot [1]_2$. According to the reduction rules for the decomposition $U(3) \supset O(3)$ the representation $[1]_3$ of $U(3)$ contains the representation $(1)_3$ of the group $O(3)$ giving the angular momentum of the bosons $l = 1$ with a projection $m = 0, \pm 1$. The representation $[1]_2$ of $U(2)$ defines the «T spin» of the bosons $T = 1/2$, whose projection is given by the corresponding representation of $U(1)$, i.e., $\alpha = \pm 1/2$. In this way, the creation and annihilation operators $u_m^{\dagger}(\alpha)$ and $u_m(\alpha)$ are defined as irreducible tensors along the chain (82) and the used phase convention and commutation relations are the following [86]:

$$\begin{gathered}\left(u^{[1]_6}_{[1]_3[1]_2 m\alpha}\right)^{\dagger} = u^{[1]_6^* \quad m\alpha}_{[1]_3^*[1]_2^*} = (-1)^{m+\frac{1}{2}-\alpha} u^{[1]_6^*}_{[1]_3^*[1]_2^* -m-\alpha}, \\ \left[u^{[1]_6^* \quad m\alpha}_{[1]_3^*[1]_2^*}, u^{[1]_6}_{[1]_3[1]_2 n\beta}\right] = \delta_{m,n}\delta_{\alpha,\beta}.\end{gathered} \tag{95}$$

We do not consider here the microscopic structure of the so-introduced vector bosons. In the IVBM they serve as a convenient mathematical tool and in the present work only their tensor properties are important, as they generate the transition operators and the basis states.

Initially, the generators of the symplectic group $\mathrm{Sp}(12, R)$ (8a), (8b) were written as double tensors [39] with respect to the $O(3) \supset O(2)$ and $U(2) \supset U(1)$ reductions. Further they can be defined as irreducible tensor operators according to the whole chain (82) of subgroups and expressed in terms of (8a) and (8b) in the folowing way:

$$A^{[\chi]_6\ \ LM}_{[\lambda]_3[2T]_2\ TT_0} = C^{[1]_6\ \ \ \ \ \ [1]^*_6\ \ \ \ \ \ [\chi]_6}_{[1]_3[1]_2[1]^*_3[1]^*_2\ \ [\lambda]_3[2T]_2} C^{[1]_3[1]^*_3[\lambda]_3}_{(1)_3(1)_3(L)_3} A^{LM}_{TT_0}, \tag{96}$$

$$F^{[\chi]_6\ \ LM}_{[\lambda]_3[2T]_2\ TT_0} = C^{[1]_6\ \ \ \ \ \ [1]_6\ \ \ \ \ \ [\chi]_6}_{[1]_3[1]_2[1]_3[1]_2\ \ [\lambda]_3[2T]_2} C^{[1]_3[1]_3[\lambda]_3}_{(1)_3(1)_3(L)_3} F^{LM}_{TT_0}, \tag{97}$$

$$G^{[\chi]_6\ \ LM}_{[\lambda]_3[2T]_2\ TT_0} = C^{[1]^*_6\ \ \ \ \ \ [1]^*_6\ \ \ \ \ \ [\chi]_6}_{[1]^*_3[1]^*_2[1]^*_3[1]^*_2\ \ [\lambda]_3[2T]_2} C^{[1]^*_3[1]^*_3[\lambda]_3}_{(1)_3(1)_3(L)_3} G^{LM}_{TT_0}, \tag{98}$$

where, according to the lemma of Racah [87], the Clebsch–Gordan coefficients along the chain are factorized by means of the isoscalar factors (IF), defined for each step of decomposition (82). It should be pointed out [86] that the $U(6)$- and $U(3)$-IFs, entering in (96), (97), and (98), are equal to ± 1.

The tensors (96), transform according to the direct product $[\chi]_6$ of the corresponding conjugated $U(6)$ representations $[1]_6$ and $[1]^*_6$ [86], namely

$$[1]_6 \times [1]^*_6 = [1,-1]_6 + [0]_6, \tag{99}$$

where $[1,-1]_6 = [2,1,1,1,1,0]_6$ and $[0]_6 = [1,1,1,1,1,1]_6$ is the scalar $U(6)$ representation. Further we multiply the two conjugated fundamental representations of $U(3) \otimes U(2)$

$$\begin{aligned}[1]_3[1]_2 \times [1]^*_3[1]^*_2 &= ([1]_3 \times [1]^*_3)([1]_2 \times [1]^*_2) = \\ &= ([210]_3 \oplus [1,1,1]_3) \times ([2,0]_2 \oplus [1,1]_2) = \\ &= [210]_3[2]_2 \oplus [210]_3[0]_2 \oplus [0]_3[2]_2 \oplus [0]_3[0]_2. \end{aligned} \tag{100}$$

Obviously, the first three $U(3) \otimes U(2)$ irreducible representations in the resulting decomposition (100) belong to the $[1,-1]_6$ of $U(6)$; and the last one, to $[0]_6$. In terms of Elliott's notations [9] (λ, μ), where $\lambda = n_1 - n_2$, $\mu = n_2 - n_3$, we have $[210]_3 = (1,1)$ and $[0]_3 = (0,0)$. The corresponding values of L from the $SU(3) \supset O(3)$ reduction rules are $L = 1,2$ in the $(1,1)$ irrep and $L = 0$ in the $(0,0)$. The values of T are 1 and 0 for the $U(2)$ irreps $[2]_2$ and $[0]_2$, respectively. Hence, the $U(2)$ irreps in the direct product distinguish the equivalent $U(3)$ irreps that appear in this reduction and there is not degeneracy. The tensors $A^{[1-1]_6\ LM}_{[210]_3[0]_2\ 00}$ with $T = 0$ correspond to the $SU(3)$ generators, to the total number of bosons N with $L = 0$ (17), to the angular momentum operator L_M with $L = 1$ (18) and to the truncated («Elliott») quadrupole momentum [9] $\tilde{Q}_M$ with $L = 2$ (21).

The tensors $A^{[1-1]_6\ 00}_{[0]_3[2]_2\ 1t}$ correspond to the $SU(2)$ generators, which are the components of the «T-spin» operator T^1_m (20). And finally the tensors $A^{[1-1]_6\ LM}_{[210]_3[2]_2\ 1t}$ with $L = 1,2$ and $M = -L, -L+1, \ldots, L$ extend the $U(3) \otimes U(2)$ algebra to the $U(6)$ one. By analogy, the tensors (97) and (98) transform according to [86]

$$[1]_6 \times [1]_6 = [2]_6 + [1,1]_6 \tag{101}$$

and

$$[1]^*_6 \times [1]^*_6 = [-2]_6 + [-1,-1]_6,$$

respectively. But, since the basis states of the IVBM are fully symmetric, we consider only the fully symmetric $U(6)$ representations $[2]_6$ and their conjugated $[-2]_6$, since for the operators (97) and (98) we have $(F^{[\chi]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0})^\dagger = (-1)^{\lambda+\mu+L-M+T-T_0}\ G^{[\chi]^*_6}_{[\lambda]^*_3[2T]^*_2}{}^{L-M}_{T-T_0}$, where $[\lambda]_3 = (\lambda,\mu)$. Hence, we present the next decompositions only for the F tensors (101). According to the decomposition rules for the fully symmetric $U(6)$ irreps [86] we have

$$[2]_6 = [2]_3[2]_2 + [1,1]_3[0]_2 = (2,0)[2]_2 + (0,1)[0]_2, \qquad (102)$$

which further contain in $(2,0)$ $L = 0,2$ with $T = 1$ and in $(0,1)$ $L = 1$ with $T = 0$. Hence, in addition to the $SU(3)$ raising generators $F^{[2]_6}_{(2,0)[2]_2}$ we have the operator $F^{[2]_6}_{(0,1)[0]_2}$, which is a new one compared to the generators of the $\mathrm{Sp}\,(6,R)$ model of Rosensteel and Rowe [28].

The above operators and their conjugated ones $G^{[\chi]^*_6}_{[\lambda]^*_3[2T]^*_2}{}^{LM}_{TT_0}$ change the number of bosons by two and realize the symplectic extension of the $U(6)$ algebra. In this way we have listed all the irreducible tensor operators in respect to the reduction chain (82), that correspond to the infinitesimal operators of the $\mathrm{Sp}\,(12,R)$ algebra.

In order to clarify the role of the tensor operators introduced in this section as transition operators and to simplify the calculation of their matrix elements, the basis for the Hilbert space must be symmetry adapted to the algebraic structure along the considered subgroup chain (82). It is evident that the basis states of the IVBM in the $\mathcal{H}_+$ (N-even) subspace of the boson representations of $\mathrm{Sp}\,(12,R)$ can be obtained by a consecutive application of the raising operators $F^{[2]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0}$ on the boson vacuum $\mid 0\rangle$ (ground state), annihilated by the tensor operators $G^{[\chi]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0} \mid 0\rangle = 0$ and $A^{[\chi]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0} \mid 0\rangle = 0$.

Thus, in general a basis for the considered dynamical symmetry of the IVBM can be constructed by applying the multiple symmetric coupling of the raising tensors $F^{[2]_6}_{[\lambda_i]_3[2T_i]_2}{}^{L_iM_i}_{T_iT_{0i}}$ with itself

$$[F \times \cdots \times F]^{[\chi]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0}.$$

Note that only fully symmetric tensor products $[\chi]_6 \equiv [N]_6$ are nonzero, since the raising operator commutes with itself. The possible $U(3)$ couplings are enumerated by the set

$$[\lambda]_3 = \{[n_1, n_2, 0] \equiv (\lambda = n_1 - n_2, \mu = n_2); n_1 \geqslant n_2 \geqslant 0\}.$$

The number of copies of the operator F in the symmetric product tensor $[N]_6$ is $N/2$, where $N = \lambda + 2\mu$ [38].

Now it is clear which of the tensor operators act as transition operators between the basis states ordered in the classification scheme presented in Table 1. The operators $F^{[2]_6}_{[\lambda]_3[2T]_2}{}^{LM}_{TT_0}$ with $T_0 = 0$ give the transitions between two neighboring cells ($\downarrow$) from one column, while the ones with $T_0 = \pm 1$ change the column as well ($\swarrow$). The tensors $A^{[1-1]_6}_{[2,1]_3[0]_2}$, which correspond to the $SU(3)$ generators, do not change the $SU(3)$ representations (λ, μ), but can change the angular momentum L inside it ($\Longrightarrow$). The $SU(2)$ generating tensors $A^{[1-1]_6}_{[0]_3[2]_2}$ change the projection $T_0(\rightarrow)$ of the «T spin» T and in this way distinguish the equivalent $SU(3)$ irreps belonging to the different columns of the same row of Table 1. Inside a given cell, the transition between the different $SU(3)$ irreps ($\Downarrow$) is realized by the operators $A^{[1-1]_6}_{[2,1]_3[2]_2}$, that represent the $U(6)$ generators. The action of the tensor operators on the $SU(3)$ vectors inside a given cell or between the cells of Table 1 is also schematically presented in it with corresponding arrows, given above in parentheses.

5.2. $B(E2)$ **Transition Probabilities for the Ground Band.** In the symplectic extension of the IVBM the complete spectrum of the system is obtained in all the even subspaces with fixed N-even of the UIR $[N]_6$ of $U(6)$, belonging to a given even UIR of $\mathrm{Sp}\,(12, R)$.

Here we give as an example the evaluation of the $E2$ transition probabilities of the ground state band (GSB) [38], whose states were identified with the $SU(3)$ multiplets $(0, \mu)$ or what is the same $(N = 2\mu, T = 0)$. In terms of the yrast energy with respect to N for the ground band their minimum values are obtained at $N = 2L$. Using the tensorial properties of the $\mathrm{Sp}\,(12, R)$ generators it is easy to define the $E2$ transition operator between the states of the considered band:

$$T^{E2} = e\left[A^{[1-1]_6}_{[210]_3[0]_2}{}^{20}_{00} + \theta([F\times F]^{[4]_6}_{(0,2)[0]_2}{}^{20}_{00} + [G\times G]^{[-4]_6}_{(2,0)[0]_2}{}^{20}_{00})\right], \quad (103)$$

where the first tensor operator is a part of the $SU(3)$ generators, and actually changes only the angular momentum with $\Delta L = 2$. The tensor product

$$[F\times F]^{[4]_6}_{(0,2)[0]_2}{}^{20}_{00} = \sum C^{[2]_6\ \ [2]_6\ \ [4]_6}_{(2,0)[2]_2\,(2,0)[2]_2\,(0,2)[0]_2} C^{(2,0)\ (2,0)\ \ (0,2)}_{(2)_3\ \ (2)_3\ \ (2)_3}\times$$
$$\times C^{20}_{20\ 20}C^{10}_{11\ 1-1}\,F^{[2]_6}_{(2,0)[2]_2}{}^{20}_{11}F^{[2]_6}_{(2,0)[2]_2}{}^{20}_{1-1} \quad (104)$$

of the raising $\mathrm{Sp}\,(12, R)$ generators changes the number of bosons by $\Delta N = 4$ and $\Delta L = 2$.

Thus, for calculating their matrix elements, we have the advantage of using the Wigner–Eckart theorem in two steps. For the $SU(3) \rightarrow SO(3)$ and $SU(2) \rightarrow$

$U(1)$ reduction we need the standard $SU(2)$ Clebsch–Gordan coefficient (CGC)

$$\langle [N']\,(\lambda',\mu'); K'L'M'; T'T_0'|T^{[\chi]_6\ \ lm}_{[\sigma]_3[2t]_2\ tt_0}|[N]\,(\lambda,\mu); KLM; TT_0\,\rangle = \\ = \langle [N'](\lambda',\mu'); K'L'||T^{[\chi]_6\ \ lm}_{[\sigma]_3[2t]_2\ tt_0}||[N](\lambda,\mu); KL\rangle C^{L'M'}_{LMlm} C^{T'T_0'}_{TT_0tt_0}. \quad (105)$$

For the calculation of the double-barred reduced matrix elements in (105) we use the next step:

$$\langle [N']\,(\lambda',\mu'); K'L'||T^{[\chi]_6\ \ lm}_{[\sigma]_3[2t]_2\ tt_0}||[N]\,(\lambda,\mu); KL\rangle = \\ = \langle [N']|||T^{[\chi]_6}_{[\sigma]_3[2t]_2}|||[N]\rangle C^{[N]_6\quad\ [\chi]_6\quad\ [N']_6}_{(\lambda,\mu)[2T]_2\ [\sigma]_3[2t]_2\ (\lambda',\mu')[2T']_2} C^{(\lambda,\mu)\ [\lambda]_3\ (\lambda',\mu')}_{KL\quad k(l)_3\ K'L'}, \quad (106)$$

where $C^{[N]_6\quad\ [\chi]_6\quad\ [N']_6}_{(\lambda,\mu)[2T]_2\ [\sigma]_3[2t]_2\ (\lambda',\mu')[2T']_2}$ and $C^{(\lambda,\mu)\ [\lambda]_3\ (\lambda',\mu')}_{KL\quad k(l)_3\ K'L'}$ are $U(6)$ and $SU(3)$ IFs. Obviously, the practical value of the application of the generalized Wigner–Eckart theorem for the calculation of the matrix elements of the $\mathrm{Sp}\,(12,R)$ generators and the construction of the symplectic basis depends on the knowledge of the isoscalar factors for the reductions $U(6)\supset U(3)\otimes U(2)$ and $U(3)\supset O(3)$, respectively. For the evaluation of the matrix elements (105) of the $\mathrm{Sp}\,(12,R)$ operators in respect to the chain (82) the reduced triple-barred $U(6)$ matrix elements are also required (106).

However, the $SU(3)$ generators are scalars with respect to the isospin group $U(2)$, so they act only on the $SU(3)$ part of the wave function and the Wigner–Eckart theorem is applied in respect to the $SU(3)$ subgroup [88]

$$\langle [N],(\lambda',\mu'); K'L'M'; T'T_0'|\ A^{[1,-1]_6\quad lm}_{(1,1)[0]_2\ \ 00}\ |[N],(\lambda,\mu); KLM; TT_0\rangle = \\ = \delta_{TT'}\delta_{T_0T_0'}\delta_{\lambda\lambda'}\delta_{\mu\mu'} \sum_{\rho=1,2} C^{(\lambda,\mu)\ \ (1,1)\ \ \rho(\lambda',\mu')}_{K(L)\ \ k(l)\ \ K'(L')} C^{L'M'}_{LM\ \ lm}\times \\ \times \langle [N],(\lambda',\mu')|||\ A^{[1,-1]_6}_{(1,1)[0]_2}\ |||[N],(\lambda,\mu)\rangle.$$

The sum over ρ runs over terms containing products of IFs of $SU(3)$ and $U(6)$, respectively. The reduced triple-barred matrix elements are well known and are given for $\rho = 1$ by [28]

$$\langle [N],(\lambda,\mu)|||A^{[1-1]_6}_{(1,1)_3[0]_2}\quad|||[N],(\lambda,\mu)\rangle_1 = \begin{cases} g_{\lambda\mu}, & \mu = 0, \\ -g_{\lambda\mu}, & \mu \neq 0, \end{cases} \quad (107)$$

where

$$g_{\lambda\mu} = 2\left(\frac{\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu}{3}\right)^{1/2} \quad (108)$$

and the phase convention is chosen to agree with that of Draayer and Akiyama [89]. For $\rho = 2$ we have $\langle [N], (\lambda, \mu) ||| A^{[1-1]_6}_{[210]_3 [0]_2} ||| [N], (\lambda, \mu) \rangle_2 = 0$. Thus, for the matrix elements of $A^{[1-1]_6}_{[210]_3 [0]_2}{}^{20}_{00}$ between the states attributed to the GSB we obtain

$$\begin{aligned} &\langle [N], (0, \mu); 0L-20; 00 | A^{[1-1]_6}_{(1,1)\ [0]_2}{}^{20}_{00} | [N], (0, \mu); 0L0; 00 \rangle = \\ &= C^{(0,\mu)}_{L-2}{}^{(1,1)}_{2}{}^{(0,\mu)}_{L} C^{L,0}_{L-2,0\ \ 2,0} \langle [N], (0, \mu) ||| A^{[1-1]_6}_{(1,1)\ [0]_2} ||| [N], (0, \mu) \rangle = \\ &= 2 \left[\frac{(\mu - L + 2)(\mu + L + 1)(L - 1)L}{2(2L - 1)(2L + 1)} \right]^{1/2} C^{L,0}_{L-2,0\ \ 2,0}. \end{aligned} \tag{109}$$

The value of the reduced $SU(3)$ Clebsch–Gordan coefficient (IF) is taken from [90]. Actually, we are interested in the $SO(3)$ reduced matrix elements which enter in (94). Thus taking into account the yrast conditions $\mu = L$ we obtain

$$\langle [N], (0, \mu); 0L-2; 00 || A^{[1-1]_6}_{(1,1)\ [0]_2} || [N], (0, \mu); 0L; 00 \rangle = 2 \left[\frac{(L-1)L}{(2L-1)} \right]^{1/2}. \tag{110}$$

For the calculation of the matrix element

$$\begin{aligned} &\langle [N+4], (0, \mu+2); 0L+20; 00 | [F \times F]^{[4]_6}_{(0,2)[0]_2}{}^{20}_{00} | [N], (0, \mu); 0L0; 00 \rangle = \\ &= C^{[N]_6}_{(0,\mu)[0]_2}{}^{[4]_6}_{(0,2)[0]_2}{}^{[N+4]_6}_{(0,\mu+2)[0]_2} C^{(0,\mu)}_{L}{}^{(0,2)}_{2}{}^{(0,\mu+2)}_{L+2} C^{L+2,0}_{L,0\ \ 2,0} \times \\ &\times \langle [N+4], (0, \mu+2) ||| [F \times F]^{[4]_6}_{(0,2)[0]_2} ||| [N], (0, \mu) \rangle \end{aligned} \tag{111}$$

we use the standard recoupling technique for two coupled $U(6)$ tensors [91]:

$$\begin{aligned} &\langle [N'] ||| [T^{[\alpha]_6} \times T^{[\beta]_6}]^{\sigma [\gamma]_6} ||| [N] \rangle = \\ &= \sum_{c, \rho_1, \rho_2} U([N]_6; [\beta]_6; [N']_6; [\alpha]_6 | [N_c]_6 \rho_2 \rho_1; [\gamma]_6\ \sigma) \times \\ &\times \langle [N'] ||| T^{[\alpha]_6} ||| [N_c] \rangle \langle [N_c] ||| T^{[\beta]_6} ||| [N] \rangle, \end{aligned} \tag{112}$$

where $U(\ldots)$ are the $U(6)$ Racah coefficients in a unitary form [92]. For the reduced triple-bared matrix element in our case, which is multiplicity-free and hence there is no sum, we have

$$\begin{aligned} &\langle [N+4] ||| [F \times F]^{[4]_6}_{(0,2)[0]_2} ||| [N] \rangle = \\ &= U([N]_6; [2]_6; [N+4]_6; [2]_6 | [N+2]_6; [4]_6) \times \\ &\times \langle [N+4] ||| F^{(2,0)} ||| [N+2] \rangle \langle [N+2] ||| F^{(2,0)} ||| [N] \rangle, \end{aligned}$$

where the corresponding Racah coefficient for maximal coupling representations is equal to unity ([91]; see also formula $A9$ of [92]). Applying again formula (112) with respect to coupled tensor $F^{[2]_6}$ and using the fact that in the case of vector bosons which span the fundamental irrep $[1]$ of $u(n)$ algebra the $u(n)$-reduced matrix element of raising generators has the well-known form [93]

$$\langle [N+1]||| \; u^{\dagger}_m(\alpha) \; |||[N]\rangle = \sqrt{N+1}, \tag{113}$$

we obtain

$$\begin{aligned}\langle [N+2]|||F^{[2]_6}|||[N]\rangle = \\ &= U([N]_6;[1]_6;[N+2]_6;[1]_6|[N+1]_6;[2]_6)\times \\ &\times \langle [N+2]||| \; p^{\dagger \; [1]_6} \; |||[N+1]\rangle\langle [N+1]||| \; p^{\dagger \; [1]_6} \; |||[N]\rangle = \\ &= \sqrt{(N+1)(N+2)}\end{aligned}$$

and in analogy

$$\langle [N+4]|||F^{[2]_6}|||[N+2]\rangle = \sqrt{(N+3)(N+4)}.$$

Introducing in (111) the above results and the value of the coefficient $C^{(0,\mu)}_{L}\,{}^{(2,0)}_{2}\,{}^{(0,\mu+2)}_{L+2}$ from [90] (the corresponding fully stretched [91] $U(6)$ IF for maximal coupling representations is equal to 1), we finally derive for the $SO(3)$ reduced matrix element

$$\begin{aligned}\langle [N+4],&(0,\mu+2);0L+2;00||[F\times F]^{[4]_6}_{(0,2)[0]_2}\,{}^{20}_{00}||[N],(0,\mu);0L;00\rangle = \\ &= \left[\frac{(\mu+L+3)(\mu+L+5)(L+1)(L+2)}{(\mu+1)(\mu+2)(2L+3)(2L+5)}\right]^{1/2}\times \\ &\times\sqrt{(N+1)(N+2)(N+3)(N+4)}, \\ &= \sqrt{(2L+1)(2L+2)(2L+3)(2L+4)},\end{aligned} \tag{114}$$

where $N = 2\mu+\lambda$ and for the last row the yrast condition $\mu = L$ is taken into account. For the calculation of the matrix element of $[G\times G]^{[-4]_6}_{(2,0)[0]_2}\,{}^{20}_{00}$ we use the conjugation property

$$\begin{aligned}&\langle [N-4],(0,\mu-2);0L-20;00|[G\times G]^{[-4]_6}_{(2,0)[0]_2}\,{}^{20}_{00}|[N],(0,\mu);0L0;00\rangle = \\ &= (\langle [N],(0,\mu);0L0;00|[F\times F]^{[4]_6}_{(0,2)[0]_2}\,{}^{20}_{00}|[N-4],(0,\mu-2);0L-20;00\rangle)^* = \\ &= C^{[N-4]_6}_{(0,\mu-2)[0]_2}\,{}^{[4]_6}_{(0,2)[0]_2}\,{}^{[N]_6}_{(0,\mu)[0]_2} C^{(0,\mu-2)}_{L-2}\,{}^{(0,2)}_{2}\,{}^{(0,\mu)}_{L} C^{L,0}_{L-2,0\;\;20}\times \\ &\times\sqrt{(N-3)(N-2)(N-1)N} = \\ &= C^{L,0}_{L-2,0\;\;20}\sqrt{(2L-3)(2L-2)(2L-1)2L}.\end{aligned} \tag{115}$$

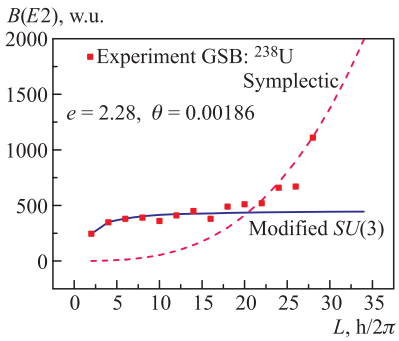


Fig. 5. The behavior of the number conserving and symplectic terms of the matrix elements of the transition operator T^{E2} (103)

With the help of the above analytic expressions ((110), (114), and (115)) for the matrix elements of the tensor operators forming the $E2$ transition operator we can calculate the transition probabilities (94) between the states in the ground band as attributed to the $SU(3)$ symmetry-adapted basis states of the model (85). It is obvious that the second term in T^{E2} (103) comes from the symplectic extension of the model. The behavior of each term of the transition operator is plotted as a function of the angular momentum L in Fig. 5 where for comparison typical experimental data for the GSB are also shown. It can be seen that because of the yrast conditions ($\mu = L$), the well-known parabolic behavior corresponding to Elliott's quadrupole operator is modified and looks like a rigid rotor curve. In this case, the rigid rotor predictions are asymptotically determined by the ordinary $SO(3)$ Clebsch–Gordan coefficient. Such type of curve is obtained in the limit of large-dimensional irreducible representations $2\lambda + \mu \to \infty$ when $su(3)$ algebra contracts to the rigid rotor algebra $\mathrm{rot}(3) = [R^5]so(3)$ [94]. It is obvious that the experimental points are well reproduced by the modified $SU(3)$ term up to $L \approx 20$, while for the description of the states with $L > 20$ the symplectic term is appropriate.

The various values of the parameters θ and e can reproduce the two main types of $B(E2)$ behavior — the enhancement or the reduction of the $B(E2)$ values. The strongly enhanced values, which are an indication for increased collectivity in the high angular momentum domain, are easily obtained for positive values of the parameter θ. For negative values of the parameter θ we obtain behavior similar to that of the standard $SU(3)$ one and it can be used to reproduce the well-known cutoff effect. Such saturation effect is also a characteristic feature of the IBM based calculations in its $SU(3)$ limit. Although the coefficient in front of symplectic term is about four orders of magnitude smaller than the $SU(3)$ contribution to the transition operator, its role in reproducing the correct behavior (with or without cutoff) of the transition probabilities between the states of the GSB band is very important.

5.3. Application to Real Nuclei. In order to prove the correct predictions following from our theoretical results we apply the theory to real nuclei for which there is available experimental data for the transition probabilities [95,96] between the states of the ground bands up to very high angular momenta. The application actually consists of fitting the two parameters of the transition operator T^{E2} (103) to the experiment for each of the considered bands.

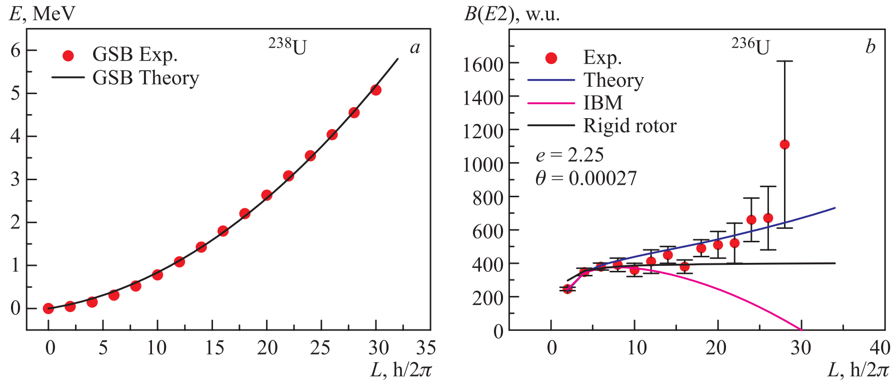


Fig. 6. Comparison of theoretical and experimental values for the energies (a) and $B(E2)$ transition probabilities for the ^{236}U (b)

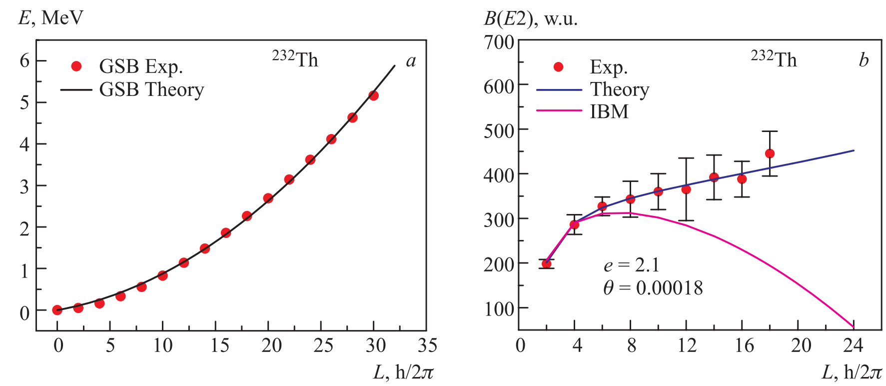


Fig. 7. Comparison of theoretical and experimental values for the energies (a) and $B(E2)$ transition probabilities of the ^{232}Th (b)

As examples we present the intraband $B(E2)$ transitions in the GSB for the nuclei ^{232}Th and ^{236}U, for which the experimental data exist up to $L = 28$ [96]. The energies of the GSB plot a and $B(E2)$ values for transitions between their states plot b are compared with the theoretical results of the IVBM in Figs. 6 and 7. The results for the $B(E2)$ values of the $SU(3)$ limit of the IBM and the rigid rotor are also shown. One can see that the IBM works well for the transitions between the first excited states ($L = 2{-}10$). The rigid rotor describes well the experimental states in the middle spin region ($L = 4{-}16$), while for the high spins the $B(E2)$ values must be enhanced due to the observed collectivity excess. Thus, at high spins in the yrast band the calculations of the IBM and the rigid rotor model cannot reproduce the fine structure of the $B(E2)$ data. As was mentioned in the preceding section, such an enhancement can be obtained for slightly positive values of the parameter θ in the transition operator T^{E2}. From the presented examples we see how sensitive is the theory to the term coming from the symplectic extension and in particular from the sign of the parameter θ entering in the transition operator (103).

In the present new application of the unitary limit of the symplectic extension of the IVBM, the model was tested on the more complicated and complex problem of reproducing the $B(E2)$ transition probabilities between the states of the ground band up to very high spins. In developing the theory, the advantages of the algebraic approach were used first for the proper assignment of the basis states to the experimentally observed states of the collective bands. Here the construction of the $E2$ transition operator as linear combination of tensor operators representing the generators of the subgroups of the respective chain is a basic result that allows the application of a specific version of the Wigner–Eckart theorem and consecutively leads to analytic results for their matrix elements in $U(6)$ symmetry-adapted basis, that give the transition probabilities.

Analyzing the terms taking part in the construction of the $E2$ transition operator, the important role of the symplectic extension of the model is revealed. The experimental data for the presented examples is reproduced rather well, although the results are very sensitive to the values of the parameters, hence more experiment on transition probabilities is needed.

The presented approach is rather general and universal and can be used for the calculation of transitions in other collective bands, in particular in the similarly constructed negative parity bands and the excited β bands, which are of great interest lately in the nuclear structure.

6. REDUCTION THROUGH THE NONCOMPACT $\mathrm{Sp}\,(2, R)$

6.1. Algebraic Construction Containing the Six-Dimensional Davidson Potential. The need for a description of nuclei in which rotational-vibrational interactions are taken into account has led to a search for algebraically solvable potentials and a meaningful set of basis states that make the transitional nature of these systems more transparent. An algebraically solvable theory that can describe systems with rotational-vibrational interactions and which has known algebraic solutions when applied to diatomic molecules, is the one containing the Davidson potential [97]. In an algebraic approach for either the nuclear many-body problem or the Bohr–Mottelson collective model, the addition of the Davidson potential to the Hamiltonian requires the consideration of a dynamical subgroup chain that starts with the direct product $\mathrm{Sp}\,(2, R) \otimes SO(n) \subset \mathrm{Sp}\,(2n, R)$, with $n = 3$ and 5, respectively [98].

Indeed if the local isomorphism of the $\mathrm{sp}\,(2,R) \approx su(1,1)$ algebras is taken into account, its relation to the spectrum generating algebra of the many body nuclear system with the Davidson interaction becomes explicit. This provides the motivation for considering this reduction in seeking a description of a more complex modes that includes rotational-vibrational interactions. Within the framework of the phenomenological interacting vector boson model [14], the more general case of a six-dimensional Davidson potential naturally appears [43]. The new reduction chain of its dynamical symmetry group $\mathrm{Sp}\,(12,R)$ [26, 99, 100]:

$$\mathrm{Sp}\,(12,R) \supset \mathrm{Sp}\,(2,R) \otimes SO(6) \tag{116}$$

extends the applicability of the theory to include mixed-modes collective interactions.

As can be deduced from the considerations given in [98], this construction obviously survives the addition of Davidson potential. The infinitesimal generators of the $\mathrm{Sp}\,(2,R)$ algebra

$$\begin{aligned} F &= \sum_{k,m,\alpha} C^{00}_{1k1m} u^+_k(\alpha) u^+_m(\alpha) = 2S^+, \\ G &= \sum_{k,m,\alpha} C^{00}_{1k1m} u_k(\alpha) u_m(\alpha) = 2S^-, \\ A &= \sum_{k,m,\alpha} C^{00}_{1k1m} u^+_k(\alpha) u_m(\alpha) = \frac{1}{\sqrt{3}} N = 2S^0 - 1 \end{aligned} \tag{117}$$

are obtained from the $\mathrm{Sp}\,(12,R)$ generators (6a), (6b), and (6c) by means of contraction with respect to both the spatial $m = 0, \pm 1$ and the «T-spin» $\alpha = p = 1/2$, $\alpha = n = -1/2$ indices. It is straightforward to show that the operators $S^\tau, \tau = 0, \pm$ commute in a standard way for the $SU(1,1)$ algebra generators [101]

$$[S^0, S^\pm] = S^\pm,\ [S^+, S^-] = -2S^0,$$

so the $\mathrm{sp}\,(2,R)$ and the $su(1,1)$ algebras are locally isomorphic with a Casimir operator written as $C_2(SU(1,1)) = S^0(S^0 - 1) - S^+ S^-$.

By construction, the operators (117) are scalars with respect to six-dimensional rotations and they commute with the components of the six-dimensional momentum operators (2.1) [14],

$$\Lambda^L_M(\alpha,\beta) = A^L_M(\alpha,\beta) - (-1)^L A^L_M(\beta,\alpha), \tag{118}$$

which obey the property

$$\Lambda^L_M(\alpha,\beta) = (-1)^L \Lambda^L_M(\beta,\alpha)$$

and generate the $SO(6) \supset U(6)$ algebra. In this way, the direct product of the two groups (116) is realized. The second-order invariant for the $SO(6)$ group is defined (see (26)) as

$$\Lambda^2 = \sum_{L,\alpha,\beta} (-1)^M \Lambda^L_M(\alpha,\beta)\Lambda^L_{-M}(\beta,\alpha), \tag{119}$$

and in this case is related to the second-order invariant of the $\mathrm{Sp}\,(2,R)$, as in the direct product (116) the two groups are complementary [99], which means that the irreps of the group $SO(6)$ determine those of $\mathrm{Sp}\,(2,R) \approx SU(1,1)$ and vice versa.

In order to define the basis of the system with (116) as a dynamical symmetry that allows one to include the six-dimensional Davidson potential, we consider the reduction of the $SO(6)$ algebra to the $SO(3)$ algebra of the angular momentum through the following chain (see the left-hand side of the diagram (16)) [14], [102]:

$$\begin{array}{ccccccc} SO(6) & \supset & \overline{SU(3)} & \otimes & O(2) & \supset & SO(3) \\ \omega & & (\bar{\lambda},\bar{\mu}) & & \nu & & L \end{array}, \tag{120}$$

which could be defined as the γ-unstable limit of the IVBM. The single infinitesimal operator of $O(2)$ is proportional to the scalar operator $\Lambda^0(\alpha,\beta)$ from the $SO(6)$ generators (118),

$$M_{\alpha\beta} = -\sqrt{3}\Lambda^0(\alpha,\beta) = -\sqrt{3}[A^0(p,n) - A^0(n,p)]. \tag{121}$$

The generators of $\overline{SU(3)}$ [14] are given by (27) and for this realization of the algebra the quadrupole moment X^2_M is the proton–neutron quadrupole interaction, which makes the difference with the subgroup $SU(3) \subset U(6)$ considered in the previous limit. The second-order Casimir invariants of the two groups in the direct product in (120) are given by (28) and (26).

In the introduced above $SO(6) \subset U(6)$ (16) reduction, the obtained symmetric representations $[N]_6$ of $U(6)$ decompose into fully symmetric $(\omega,0,0)_6 \equiv (\omega)_6$ irreps of $SO(6)$ according to the rule

$$[N]_6 = \bigoplus_{\omega=N,N-2,\ldots,0(1)} (\omega,0,0)_6 = \bigoplus_{i=0}^{\langle N/2\rangle} (N-2i)_6, \tag{122}$$

where $\langle N/2\rangle = N/2$ if N is even and $(N-1)/2$ if N is odd. Furthermore, the following relation between the quadratic Casimir operators $\overline{G}_3 = \sum\limits_M (-1)^M (X_M X_{-M} + Y_M Y_{-M})$ (28) of $\overline{SU(3)}$, π_2 of $O(2)$ and π_6 (119) of $SO(6)$ holds [103]:

$$\pi_6 = 2\overline{G}_3 - \frac{1}{3}\pi_6, \tag{123}$$

which means that the reduction from $SO(6)$ to the rotational group $SO(3)$ is carried out through the complementary groups $O(2)$ and $\overline{SU(3)}$ [99]. As a consequence, the $[f_1, f_2, 0]_3$ labels of the $\overline{SU(3)}$ irreps are determined by $(\omega)_6$ of $SO(6)$ and by the integer label $(\nu)_2$ of the associated irrep of $O(2)$, i.e.,

$$(\omega)_6 = \bigoplus [f_1, f_2, 0]_3 \otimes (\nu)_2. \tag{124}$$

Using the relation (123) of the Casimir operators, for their respective eigenvalues in terms of Elliott's notation [104] $(\bar{\lambda}, \bar{\mu})$ one obtains

$$(\omega)_6 = \bigoplus_{\nu=\pm\omega,\pm\omega-2,\ldots,0(\pm 1)} \left(\bar{\lambda} = \frac{\omega+\nu}{2}, \bar{\mu} = \frac{\omega-\nu}{2}\right) \otimes (\nu)_2. \tag{125}$$

Finally, the convenience of this reduction can be further enhanced through the use of the standard rules for the reduction of the $SU(3) \supset SO(3)$ chain. Hence, the basis, labelled by the quantum numbers classified by the group-subgroup chain (120), can be written as

$$|N\omega; (\bar{\lambda}, \bar{\mu})\nu; K, L\rangle, \tag{126}$$

where the reduction rules for obtaining specific values for each state were given above. By means of these labels, the basis states can be classified in each of the two irreducible even H_+ with $N = 0, 2, 4, \ldots,$ and odd H_- with $N = 1, 3, 5, \ldots$ representations of $\mathrm{Sp}\,(12, R)$. We illustrate this in Table 3 for the even H_+ irreducible representation, where N with the set of ω contained in it (122) label the rows and the values of the quantum number ν label the columns. The $\overline{SU(3)}$ quantum numbers $(\bar{\lambda}, \bar{\mu})$ define the cells of Table 3 as they are obtained with the help of ω and ν (125).

At this point some of the similarities between the reductions through the $U(6)$ (82) and $O(6)$ (120) become apparent. The values of the quantum number ω are clearly related with the values of the «T-spin» $T = \omega/2$, so we can introduce a parity operator defined as $(-1)^{\omega/2}$ like in the $U(6)$-limit (82). Respectively, the values of ν have similar relation to the values of $T_0 = \nu/2$. The important difference in this case is that there is no degeneracy in the values of the $\overline{SU(3)}$ irreps $(\bar{\lambda}, \bar{\mu})$, that belong to a given row defined by ω, but in the columns (fixed value of ν) the $\overline{SU(3)}$ irreps repeat each other except the ones corresponding to the maximal value of $\omega = N$ (the first row for each N) added for each $\nu = \bar{\lambda} - \bar{\mu}$. Also in this case $\bar{\lambda}$ and $\bar{\mu}$ are both always even or odd.

The Hamiltonian with the considered dynamical symmetry (130) is expressed in terms of the first and second order Casimir operators of the different subgroups in its corresponding chain (120):

$$H = aN + bN^2 + \beta_6\Lambda^2 + \beta_2 M^2 + \beta_3 L^2, \tag{127}$$

Table 3. **Classification scheme of the basis states in the even $\mathbf{H}_+$ space of** $\mathrm{Sp}\,(12, R)$ **according to the decompositions given by (116) and (120)**

N	ω	$\nu/\;\;6$	4	2	0	-2	-4	-6
0	0				$(0,0)$			
2	2 0			$(2,0)$	$(1,1)$ $(0,0)$	$(0,2)$		
4	4 2 0		$(4,0)$	$(3,1)$ $(2,0)$	$(2,2)$ $(1,1)$ $(0,0)$	$(1,3)$ $(0,2)$	$(0,4)$	
6	6 4 2 0	$(6,0)$	$(5,1)$ $(4,0)$	$(4,2)$ $(3,1)$ $(2,0)$	$(3,3)$ $(2,2)$ $(1,1)$ $(0,0)$	$(2,4)$ $(1,3)$ $(0,2)$	$(1,5)$ $(0,4)$	$(0,6)$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

and it is obviously diagonal in the basis (126) labelled by the quantum numbers of their representations. The second order invariant of $\overline{SU(3)}$ is dropped in (127), because of its linear dependence on the Casimir operators of the $SO(6)$ and $O(2)$ (123). Then the eigenvalues of the Hamiltonian (127) that yield the spectrum of a system interacting with six-dimensional Davidson potential are:

$$E(N, \omega, \nu, L) = aN + bN^2 + \beta_6\omega(\omega + 4) + \beta_2\nu^2 + \beta_3 L(L + 1). \qquad (128)$$

This expression is very similar to the one obtained in the $U(6)$ (82) case (89), with a difference coming only from the $SO(6)$ second order invariant Λ^2, with the parameter β_6. Other differences in the applications to real nuclei follow from the possible choices in mapping the experimentally observed collective states to the basis of this dynamical symmetry.

6.2. Application to Real Nuclei. In the applications of this new dynamical symmetry of the IVBM to real nuclear systems that we present here, we exploit again the «algebraic» definition of yrast states as introduced earlier and in [38]. By means of the later, the states of the ground band, which are the yrast states of the nucleus, correspond to the basis states with $N = \omega = \mp\nu = 2L = 0, 4, 8, \ldots$ ($\Delta N = 4$). At $-\nu = 2L$ we obtain the left-to-right diagonal $(0, 2L)$ of Table 3 with $\overline{\lambda} = 0$ and changing $\overline{\mu} = L = 0, 2, 4, \ldots$ At $\nu = 2L$ the respective diagonal of Table 3 is $(2L, 0)$. Hence, $T = \omega/2$ is always even and we consider the states belonging to the $K = 0$ band. Then the ground state band's energies are obtained by the expression

$$E_g(L) = \alpha L + \beta L(L + 1), \qquad (129)$$

where $\alpha = 2a + 8\beta_6 + \beta_3$ and $\beta = 4b + 4\beta_6 + 4\beta_2 + \beta_3$. From (129) it is obvious that in the GSB we will have a certain degree of mixing of the vibrational and

Table 4. **Values of the parameters of the model Hamiltonian (127) obtained in the fitting to the experimental spectra of the considered nuclei**

Nucleus	N_0	Bands	χ^2	Parameters	ν
^{162}Dy	0	GSB	0.0018	$a = 0.01909$ $b = 0.00019$	$2L$
$R_4 = 3.31$	34	γ	0.0004	$\beta_6 = 0.00007$	$2L-4$
	46	β	0.0055	$\beta_2 = 0.00202$	$2L$
	54	$K^\pi = 1^-$	0.0013	$\beta_3 = 0.00074$	$2L-2$
	42	$K^\pi = 2^-$	0.0436	$\beta_2^{'} = 0.00021$	$2L+2$
^{154}Gd	0	GSB	0.0101	$a = 0.04731$ $b = 0.00041$	$2L$
$R_4 = 3.02$	18	γ	0.0049	$\beta_6 = 0.00043$	$2L-4$
	12	β	0.0331	$\beta_2 = 0.00041$	$2L+12$
	30	$K^\pi = 7^-$	0.0175	$\beta_3 = 0.00171$	$2L-12$
	32	$K^\pi = 8^-$	0.0320	$\beta_2^{'} = -0.00031$	$2L-14$
	26	$K^\pi = 4^+$	0.0043		$2L-8$
	18	$K^\pi = 2^-$	0.0028		$2L+6$
^{156}Er	0	GSB	0.0099	$a = 0.10625$ $b = 0.00018$	$2L$
$R_4 = 2.31$	8	γ	0.0060	$\beta_6 = 0.00031$	$2L-4$
	6	β	0.0204	$\beta_2 = 0.00018$	$2L$
	16	$K^\pi = 6^-$	0.0180	$\beta_3 = 0.00101$	$2L-12$
				$\beta_2^{'} = -0.000006$	

rotational modes depending on the values of the parameters α and β. The latter depends on all the Hamiltonian (127) parameters, except a. There is no an additive constant to the energies, like in the $U(6)$ case. The difference is that we cannot describe octupole bands ($K^\pi = 0^-$) here, because $\overline{\lambda}$ and $\overline{\mu}$ are both always even or odd. In this case it is obvious that we can relate the quantum number of the «T-spin» T to the quantum number of the $O(6)$ irreps $\omega/2$, which can be even or odd. Hence, we can describe both positive and negative parity bands.

The states of the excited bands are mapped on the theoretical ones, generally in two ways described in terms of stretched states $(\bar{\lambda}, \bar{\mu}) = (\bar{\lambda}_0 + k, \bar{\mu}_0)$, where $\bar{\mu}_0$ is fixed, or $(\bar{\lambda}_0, \bar{\mu}_0 + k)$, where $\bar{\lambda}_0$ is fixed, which in turn can belong to certain «column» (ν fixed) or «diagonal» (ν changing) of the basis states presented in Table 3.

Along the left or right «diagonals» of Table 3 we have a sitiuation similar to the case described for the ground state band, where $\bar{\lambda}$ is changing and $\bar{\mu}$ is fixed or vice versa. In this case we have different sequences of $(\bar{\lambda}, \bar{\mu})$ multiplets, defining the bands along the diagonals with $\pm\nu = \bar{\lambda} - \bar{\mu} = \pm(2L - k)$.

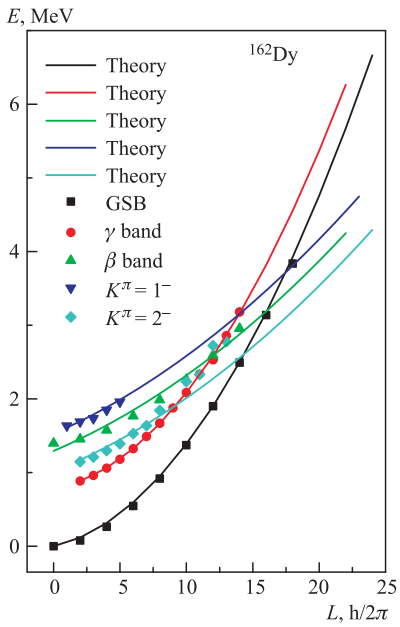


Fig. 8. Comparison of the theoretical and experimental energies for ground and excited bands for ^{162}Dy

As discussed above, the correct placement of the bands in the spectrum strongly depends on their band-head configurations and, in particular, on the number of bosons N_0, from which they are built [43]. So we define the number of bosons for each state in the band with a given L in the following way: $N = N_0 + \nu + 2L$ and the respective $\omega = \nu + 2L$.

The variety of possible choices for the correspondence of the excited bands to sequences of states in the symplectic space and the mixing of the rotational and vibrational degrees of freedom (129), like in the $U(6)$ limit, allows us to reproduce correctly the behavior of the excited bands with respect to one another, which can change a lot even in neighboring nuclei [80].

The five free parameters of the Hamiltonian (127) are determined by fitting the theoretical predictions for the energies of the ground and few excited bands to the experimental data [1], using a χ^2 procedure. The bands are developed according to the rules described above for their corresponding K^π values. We choose for the value of N_0 corresponding to the band-head states the one that gives the lowest χ^2, after obtaining the parameters mainly from the ground state band. That is why, we usually choose in the applications nuclei with long spin sequences in their ground bands.

In [43] and [105], we presented examples of a rather precise description of only positive parity collective bands (ground, β and γ) in well-deformed nuclei, corresponding to the $SU(3)$ limit of the IBM, γ-soft ones, belonging to the $SO(6)$ limit of the IBM and of the ^{152}Sm nucleus with $(R_{4/2} = 3.1)$, which is transitional between the γ-soft $(R_{4/2} = 2.5)$ and rotational nuclei with $(R_{4/2} = 3.33)$ and is a typical example for being at the critical point of phase/shape transition [82] with $X(5)$ symmetry [4].

Here we illustrate the possible description of negative parity bands with $K^\pi \neq 0 = 1^-, 2^-, 3^- \ldots$ in addition to the basic ground, β and γ bands, like in the $U(6)$-reduction chain and use this application to compare the results of both limits. Hence, in the presented below applications we fit the Hamiltonian parameters for this limit again for the rotational ^{162}Dy nucleus, we describe another $X(5)$ symmetry nucleus ^{154}Gd [106] and make an attempt to deal with

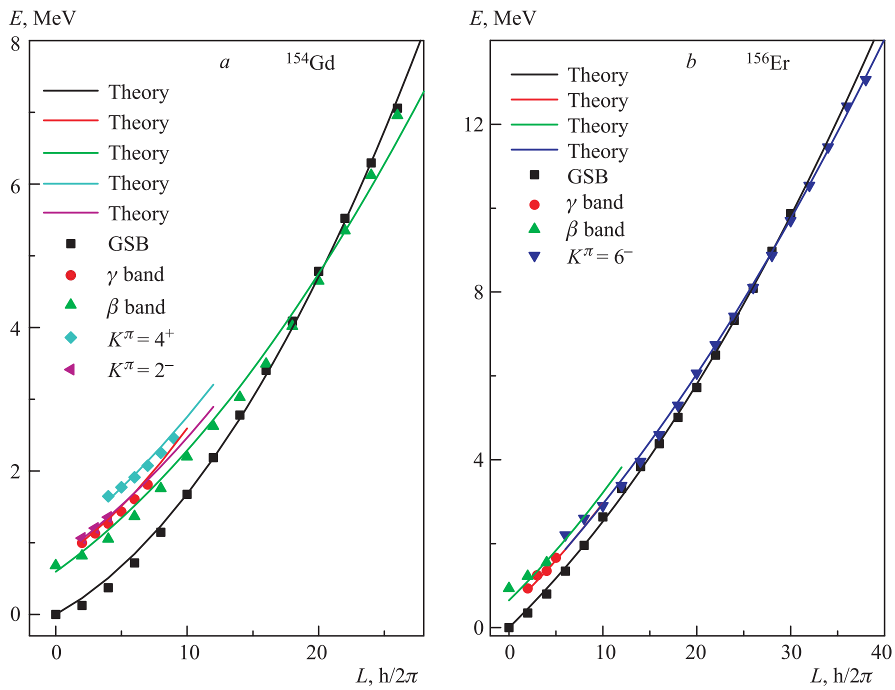


Fig. 9. The same as in Fig. 8 for ^{154}Gd (a) and ^{156}Er (b)

a spectrum that shows a behavior close to the other critical point with $E(5)$ symmetry [107].

The parameters that were obtained, the number s of experimental states, χ^2, N_0, and ν are all given in Table 4 for the three different nuclei, where the ^{156}Er is also considered.

As shown in Figs. 8 and 9, the experimental data is reproduced remarkably well, especially for the β and γ bands, as well as for the newly described negative parity bands. In this case most of the parameters of the Hamiltonian are obtained through a fit to the energies of the states from the ground state band. The energies of the other excited bands are reproduced mainly by the correct choice of the quantum number N_0 and, in general, the correct identification of the experimental states with the basis states. This reveals again the importance of N_0, which vary quite a lot, but much less than the parameters of the Hamiltonian.

Comparing the parameters of the Hamiltonians obtained for the ^{162}Dy in the $U(6)$ and this limit it is clear that they are rather close to each other which confirms the observed correspondence between the terms of the two Hamiltonians (88) and (127).

From the presented results, it could be seen that any of the above two limits of the symplectic extension of the IVBM is equally appropriate for the description of a rather broad range of nuclei, and most importantly, nuclei that display different degree of mixing of the rotational and vibrational degrees of freedom.

7. THE REDUCTION THROUGH THE NONCOMPACT $\mathrm{Sp}\,(4,R)$

7.1. The Algebraic Realization. Finally we will introduce the group — theoretical background [43] of the application of the symplectic IVBM, for the description of the energy distributions of collective excited states with fixed angular momenta. This new reduction further elucidates the importance of considering the structure of the band-head's configurations in respect to the number of bosons N that build them, established through the applications of the first two chains. It also plays the role of a connecting branch in the general reduction scheme of $\mathrm{Sp}\,(12,R)$, clarifying the relations between the subgroups of the above two chains.

In terms of the introduced boson representations of $\mathrm{Sp}\,(12,R)$, the third chain of subgroups starts with the reduction [26, 43, 99]:

$$\mathrm{Sp}\,(12,R) \supset \mathrm{Sp}\,(4,R) \otimes SO(3). \tag{130}$$

The infinitesimal operators of the $\mathrm{Sp}\,(4,R)$ algebra are the $L=0$ part

$$\begin{aligned} F^0(\alpha,\beta) &= \sum_{k,m} C^{00}_{1k1m} u^+_k(\alpha) u^+_m(\beta), \\ G^0(\alpha,\beta) &= \sum_{k,m} C^{00}_{1k1m} u_k(\alpha) u_m(\beta), \\ A^0(\alpha,\beta) &= \sum_{k,m} C^{00}_{1k1m} u^+_k(\alpha) u_m(\beta) \end{aligned} \tag{131}$$

of the $\mathrm{Sp}\,(12,R)$ generators $F^L_M(\alpha,\beta), G^L_M(\alpha,\beta)$ (6a), (6b) and $A^L_M(\alpha,\beta)$ (6c) $\alpha,\beta = 1/2(p)$, or $= -1/2(n)$. Hence by construction, all these operators are scalars in respect to the three-dimensional rotations. Obviously the generators (131) commute with the components of the angular momentum L^1_M (18), that generate the $SO(3)$ algebra, i.e., we have a direct product of the two groups (130). As a result the $\mathrm{Sp}\,(4,R)$ irreps can be labelled by the quantum numbers of their corresponding $SO(3)$ ones, namely the angular momentum L.

The maximal compact subalgebra $u(2)$ of $\mathrm{sp}\,(4,R)$ is generated by the Weyl generators $A^0(\alpha,\beta)$ of (131), and is the same one that corresponds to the «T-spin» algebra of $U(2)$ (20) in the direct product with the $SU(3)$ in (82). The operator N generates $u(1)$ and plays the role of the first-order invariant of

$u(2) \supset su_T(2) \otimes u_N(1)$. Hence, the following correspondence between the chains of subalgebras of $\mathrm{sp}\,(12,R)$ — through $u(6)$ and through $\mathrm{sp}\,(4,R)$, exists [26]:

$$\begin{array}{lllll} \mathrm{sp}\,(12,R) & \supset & \mathrm{sp}\,(4,R) & \otimes & so(3) \\ \cup & & \cup & & \cap \\ u(6) & \supset & u(2) & \otimes & su(3) \end{array} \tag{132}$$

Each of the $\mathrm{sp}\,(4,R)$ irreps that are contained in the $\mathrm{sp}\,(12,R)$ boson representations is of infinite dimension and consists of countless number of $u(2)$ irreps. A basis for the $\mathrm{sp}\,(4,R)$ representations is generated by a consecutive application of the symmetrically coupled products of the operators $F^0(\alpha,\beta)$ (131) to the lowest weight state (lws) with angular momentum L that labels the considered $\mathrm{Sp}\,(4,R)$ irrep [26]. Each starting $u(2)$ configuration is characterized by a totally symmetric representation $[L]_2$ formed by $L = N_{\min}$ vector bosons. The procedure to obtain the rest of the $su(2)$ irreps that are contained in a given L irrep of $\mathrm{sp}\,(4,R)$ is given in detail in [43].

We first decompose all the even numbers $n = 0, 2, 4, 6, \ldots$, where $n/2$ gives the degree of the $F^0(\alpha,\beta)$ that are applied to the lws, into a direct sum of $u(2)$ irreps $[n_1, n_2]$ with both n_1,n_2 even and $n = n_1 + n_2$. After symmetrization of the irreps $[n_1, n_2] \to [n_1 - n_2, 0] \equiv [n_1 - n_2]_2$ we get the decomposition:

$$[n]_2 \to [n]_2 \oplus [n-4]_2 \oplus \ldots \oplus [2]_2 \text{ or } [0]_2. \tag{133}$$

Next, all the outer products of the representations $[L]_2$ with the reduced $[n]_2$ (133) are calculated and restricted to the two-dimensional Young diagrams of integers. We obtain as a result the $su(2)$ representations $[k]_2$, which correspond to $N = N_{\min} + n$, $n = 0, 2, 4, 6, \ldots$ and $T = k/2 = N/2, (N/2) - 1, \ldots, 0$ (for even L). This technique is illustrated for the cases $L = 0$ and $L = 2$ with Tables 5 and 6. The columns are defined by the quantum number T and the rows by the eigenvalues of $N = N_{\min} + n$ for $n = 0, 2, 4, 6, \ldots$ Table 5 for the $L = 0$ states actually coincides with the decomposition of the even numbers n (133) as in it the outer product is with the $[0]_2$ irrep. By means of the correspondence (132) between the two considered chains of subgroups of $\mathrm{Sp}\,(12,R)$ and the relations (84) between the $U(2)$ and $SU(3)$ quantum numbers, we were able also to present the respective ($\lambda = k$, $\mu = (N-k)/2$) irreps in the cells of Tables $5, 6$. For a given value of N these could be compared to the classification scheme of the $SU(3)$ irreps contained in the even $U(6)$ irreps of $\mathrm{Sp}\,(12,R)$ given in Table 1. Note that the missing $su(2)$ irreps on the above tables do not contain in their corresponding (λ,μ) representations the states with the considered value of L according to the $SU(3) \supset SO(3)$ reduction rules (86) [15]. In the obtained decomposition of the $\mathrm{sp}\,(4,R)$ representations L into $[k]_2$ — $su(2)$ ones, except for the $L = 0$ case in Table 5, there is a multiplicity, denoted as $\rho\times$ in the appearance of some of the irreps, that shows how many times the

Table 5. $L = 0$

...	$T = 4$	$T = 3$	$T = 2$	$T = 1$	$T = 0$	T/N
					$[0]_2(0,0)$	$N = 0$
				$[2]_2(2,0)$		$N = 2$
			$[4]_2(4,0)$		$[0]_2(0,2)$	$N = 4$
		$[6]_2(6,0)$		$[2]_2(2,2)$		$N = 6$
	$[8]_2(8,0)$		$[4]_2(4,2)$		$[0]_2(0,4)$	$N = 8$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 6. $L = 2$

...	$T = 4$	$T = 3$	$T = 2$	$T = 1$	$T = 0$	T/N
				$[2]_2(2,0)$		$N = 2$
			$[4]_2(4,0)$	$[2]_2(2,1)$	$[0]_2(0,2)$	$N = 4$
		$[6]_2(6,0)$	$[4]_2(4,1)$	$2\times[2]_2(2,2)$		$N = 6$
	$[8](8,0)$	$[6]_2(6,1)$	$2\times[4]_2(4,2)$	$2\times[2]_2(2,3)$	$[0]_2(0,4)$	$N = 8$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

respective irrep $[k]_2$ appears for the specified value of N. This multiplicity is exactly equal to the multiplicity of the appearance of the considered value of L in the reduction (86) of the corresponding $su(3)$ irrep (λ, μ) to the $so(3)$ values of L [26] (see $N = 6$ and $N = 8$ rows of Table 6).

7.2. Energy Distribution of States with Fixed L. As established above, because of the correspondence (132) and the relation between the $SU(3)$ and $SU(2)$ second order Casimir operators [38], the Hamiltonian (88) and the bases in this case are equivalent to the ones in the $U(6)$ limit of the model (82). As a result, the eigenvalues of the Hamiltonian (88) for the states with a fixed L are the energies, given by (89). Obviously in (89) the dependence of the energies of the collective states on the number of phonons (vector bosons) N is parabolic. The rest of the quantum numbers T, T_0 defining the states with fixed L in this case are expressed in terms of N by means of the reduction procedure described above. The parity of the states is defined [38] in the same way, as $\pi = (-1)^T$. Although we use the same labels for the basis states, it must be kept in mind, that the states that we consider in this limit are ordered in different from the $U(6)$ case, sequences of $SU(3)$ irreps, reflecting their positions among the other states with the same angular momenta. This not only requires their interpretation in terms of N, but also gives a different physical meaning of the parameters of the Hamiltonian, hence they are fitted again in this limit, but in respect to the variable N.

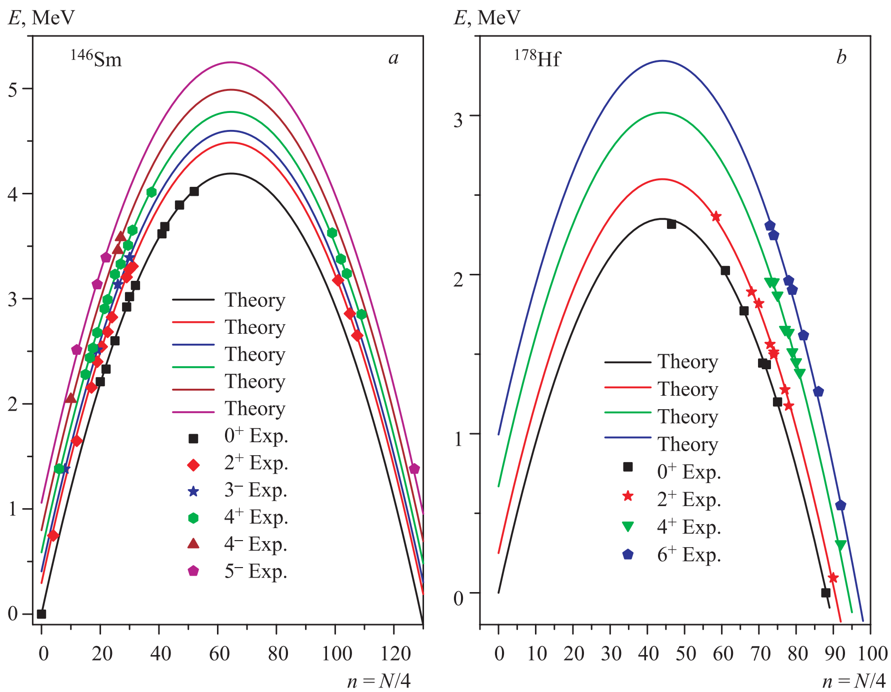


Fig. 10. Comparison of the theoretical and experimental energy distributions of states with fixed L in the spectra of ^{146}Sm (a) and ^{178}Hf (b)

Here, we first evaluate the parameter of inertia β_3 in front of the term $L(L+1)$ in (89) by fitting the energies of the ground states band (GSB) with $J^\pi = 0^+_1, 2^+_1, 4^+_1, 6^+_1, \ldots$ to their experimental values in each nuclei. Further, the values of N_{L_i} corresponding to the experimentally observed $E^{\rm exp}_{L_i}$ and the values of the parameters in (89) are evaluated in a multistep χ-square fitting procedure making use only of the three sets of states with $J^\pi_i = 0^+_i, 2^+_i, 4^+_i$. We choose nuclei with enough of these states to have good statistics in the fit. Most of these states are band heads (all the 0^+, some of the 2^+ and 4^+) of collective bands and as a result the situation of the whole band depends on them. The set of N_{L_i} with minimal value of χ^2 determines the distribution (the parameters of the Hamiltonian) of the L^+_i states energies with respect to the number of bosons N_{L_i} that build them. For the set of 0^+ states $(L=0)$ we chose both $T = T_0 = 0$, so the parameters a and b in (89) are evaluated and fixed. Further, for the 2^+ we use even $T > 0$, $T_0 = 0$ and so we get the α_3 parameter, and finally for the 4^+ we determine $T > 0$ and also a possible value of $T_0 > 0$ in order to obtain α_1. Sets of states with other values of $L = 1, 3, 5, 6\ldots$ and/or with negative parity

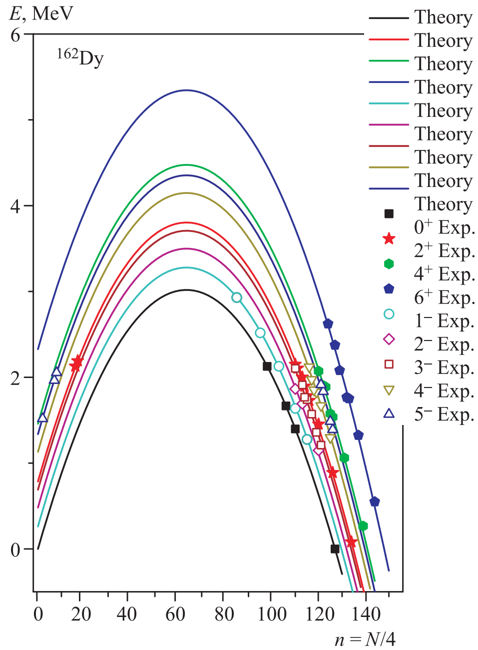


Fig. 11. The same as in Fig. 6 for ^{162}Dy

(T-odd) can be included in the consideration only by determining in a convenient way the values of T, T_0 and finding the sequences of N_{L_i} corresponding to the observed experimental energies. The first two parameters a and b ($b < 0$) of the Hamiltonian actually determine the form of the parabolas, and they are the same for each set of states with fixed L. The rest of them with the values of the quantum numbers T, T_0, and L, only shift the curves in respect to each other. The predicting power of the model in this limit is related to the possibility to describe any other set of states, once all the parameters of the Hamiltonian are evaluated from the distributions of 0^+, 2^+, and 4^+ states, without involving additional parameters.

The results of the treatment described above for the collective spectra of three even–even nuclei from the rare-earth region are illustrated in Figs. 10 and 11. The theoretical distribution of the energies with respect to the attributed values of N_{L_i} and the good agreement with the experimental ones can be clearly seen. Additionally, N_{ini} — the initial value of N by means of which the lowest state with the considered J^π is constructed, the used values of T, T_0, the values obtained for the Hamiltonian parameters β_3, a, b, α_3, and α_1 with their respective χ^2 are presented in Table 7. The numbers s in the first column give the number of the experimentally observed states with the respective J^π.

Table 7. **Values of the parameters of the model Hamiltonian (88) of the considered nuclei obtained in the fitting of energies of the set of states with given J^π to their experimental values**

Nucleus	s	J^π	$n_{\rm ini} = N_{\rm ini}/4$	T	T_0	χ^2	Parameters
^{146}Sm	11	0^+	0	0	0	0.0004	$a = 0.03243$ $b = -0.00006$
	13	2^+	4	4	0	0.0008	$\alpha_3 = -0.00161$
	16	4^+	6	8	8	0.0008	$\alpha_1 = -0.00553$
	4	3^-	8	5	5	0.0004	$\beta_3 = 0.04134$
	3	4^-	10	3	3	0.0013	
	4	5^-	127	7	7	0.0017	
^{178}Hf	4	0^+	88	0	0	$2 \cdot 10^{-6}$	$a = 0.02376$ $b = -0.00040$
	12	2^+	90	4	0	0.0006	$\alpha_3 = 0.03543$
	7	4^+	92	8	8	0.0025	$\alpha_1 = -0.02110$
	7	6^+	93	10	10	0.0022	$\beta_3 = 0.02879$
^{162}Dy	4	0^+	127	0	0	10^{-6}	$a = 0.02376$ $b = -0.00005$
	11	2^+	134	4	0	0.0006	$\alpha_3 = 0.03543$
	7	4^+	139	8	8	0.0025	$\alpha_1 = -0.02110$
	7	6^+	144	10	10	0.0023	$\beta_3 = 0.01288$
	5	1^-	115	3	3	0.0004	
	3	2^-	120	3	1	0.0009	
	7	3^-	121	5	5	0.0009	
	6	4^-	125	5	3	0.0017	
	7	5^-	126	7	7	0.0010	

The examples chosen for the present application are nuclei for which there is experimental data for the energies of more than 5 states with each of the angular momenta $L = 0, 2, 4$ in the low-lying spectra. One of these nuclei, ^{146}Sm, has a typical vibrational spectra and the other one, ^{178}Hf, has typical rotational character. This is confirmed by the obtained values for the inertia parameter, β_3, given in Table 7. It is well known that the main distinction of these two types of spectra is the position of the first excited 2_1^+ state of the ground state band, which for vibrational nuclei is rather high (over 1 MeV), but for the well-deformed ones lies low around 0.07 MeV.

For the nucleus with vibrational spectra ^{146}Sm we apply the procedure described above with values of T that differ quite significantly ($\Delta T = 4$) for the sets with $L = 0, 2, 4$. This corresponds to rather large changes in the values of the initial $N_{\rm min} = 2T$ which allows one, according to the reduction rules for the basis, to place them on the left-hand side of the symmetric parabolas. As a result (see Fig. 10), the values of N_{L_i} increase with increasing energy of these states.

For the spectra of ^{146}Sm, in addition to the $L = 0^+, 2^+, 4^+$ we have included the states with $L = 3^-, 4^-, 5^-$.

With the procedure employed, the ordering of the states into different bands can be recognized. The GSB is formed from the lowest states with $L = 0^+, 2^+, 4^+$, which are almost equidistant for the case of vibrational nuclei with very close values of n for the states belonging to a given band. The almost degenerate $0^+, 2^+, 4^+$ triplets of states, characteristic for the harmonic quadrupole vibrations, can also be observed on the theoretical energy curves, and are characterized with almost equal difference between their corresponding values of N. In order to avoid nearly degeneracies of the energies in respect to N_{L_i}, we use the symmetric feature of the second order curves and place some of the states on the right side of the parabolas.

The low-lying states of the ground band, $L = 0_1^+, 2_1^+, 4_1^+$, of the typical rotational nuclei [1] ^{162}Dy require small changes in the number of quanta N_{L_1} that build the corresponding initial states in each sequence. Making use of the latter and the symmetric feature of the second order curves the states with a given L in the rotational spectra are placed on the right-hand side of the theoretical curves. This corresponds to the second solution $N''_{0_1} = -a/b$ of equation (89) for the ground state with $T = T_0 = 0$, defining the maximum N_{0_1} that builds it. This can be used as a restriction on the values of N_{L_i}. On a parabola, specified for a fixed L, the number of bosons that build the states will decrease with increasing energies. Hence, if the number of quanta that is required to build a collective state is taken as a measure of collectivity, the states from a rotational spectra are much more collective than vibrational ones, which is the expected result. In this example, one can also observe the structure of collective bands that are formed by sets of states from the different curves and in particular the ground band, the first two excited β and γ bands (see Figs. 10 and 11).

This result confirms the empirical investigation [108] of the energy distribution of the states with fixed angular momentum. It is rather well described [83] by the simple phenomenological formula $E_L(n) = An - Bn^2$, where $A > 0$ and $B > 0$ are fitting parameters and n is an integer number corresponding to each one of the states with given L. A relation $N = 4n$ between the total number of vector bosons N and the introduced in [109] number of ideal monopole bosons n is valid in this application. In [83] the importance of the value of n in particular for the description of the distribution of the 0^+ is revealed by its introduction as an additional variable in the inertia parameter β_3 of the model Hamiltonian (88) for the considered bands.

8. GENERALIZED REDUCTION SCHEME FOR THE IVBM

The use of symplectic geometry in the investigation of the nuclear collective motion, relates the later to its microscopic structure. A further elaboration in the problem can be achieved if we consider the nuclear many-body system as consisting of two interacting proton and neutron subsystems. This motivation is behind the algebraic construction of the phenomenological IVBM [14], where $\mathrm{Sp}\,(12, R)$ — the group of linear canonical transformation in a 12-dimensional phase space [99] appears as the group of dynamical symmetry of the model. The three considered above dynamical symmetries can be unified in the reduction scheme (134).

In mathematical terms the established relations are based on the appearance of the physically important $U(2)$ [38] group of the «T spin» as the maximal compact subgroup of $\mathrm{Sp}\,(4, R)$ [43], as well as its noncompact counterpart $SU(1,1)$ [42]. By means of this vertical structure, the dynamical symmetries describing the ground and excited bands are connected with the dynamical symmetry describing the sets of states with fixed angular momentum, most of which are band-head configurations. This reveals the important role of their structure, in particular the number of bosons from which the band heads are build for the development of the excited bands, a feature that is due to the consideration of the symplectic extension of the model. Another very important model characteristic leading to the correct description of the experimental energies is the interaction between the proton and neutron subsystems, yielded in the symplectic extension of the model, but still retaining the exact analytic solutions in each of the considered applications.

This generalized reduction scheme relates the presented applications of the three dynamical symmetries in their physical interpretations and the conclusions following from each one of them. It was established that the two reduction schemes, that describe the developments of collective bands in various types of nuclear spectra, the one through $U(6)$ and the one through $SO(6)$, although using different realizations of the basis states and the Hamiltonians, because of their connection through the content of the $\mathrm{Sp}\,(4, R)$, yield very similar applications for the description of the ground bands and the excited positive and negative parity bands. A common feature of these applications is the possibility to mix with varying strength the two main collective modes — vibrational and rotational, which results in the accurate description even of nuclei at the critical points of phase/shape transitions in the framework of these exactly solvable cases. The success of this approach is due not only to the easy evaluation of the relatively small number of model parameters by means of fitting to the experiment, moreover only of the first lowest bands like the ground band and the immediately following one or two excited bands. The important predicting power of the model is related to the symplectic extension, which allows one (only by correctly finding the number of bosons that build the band-head configurations of the other observed excited bands) to evaluate the energies of all states that belong to them. In this respect the energy distributions of the states with fixed angular momenta and in particular of the low-lying ones with $L = 0_i^+, 2_i^+, 4_i^+$, most of which are band-head configurations, obtained in the connecting $\mathrm{Sp}\,(4, R)$, give the tool to obtain their corresponding N_{L_i} values, on which the excited bands can be build [83]

$$\begin{array}{lllll}
U(6) & \supset & U(2) & \otimes & SU(3) \\
[N] & & T & & \ (\lambda,\mu) \\
\cap & & \cap & & \cup \quad K \\
\mathrm{Sp}\,(12,R) & \supset & \mathrm{Sp}\,(4,R) & \otimes & SO(3) \\
\cup & & \cup & & \cap \\
SO(6) & \otimes & \mathrm{Sp}\,(2,R) & & \\
\cup \quad \omega & & (SU(1,1)) & & \\
\overline{SU(3)} & \otimes & O(2) & & \\
(\overline{\lambda},\overline{\mu}) & & \nu & &
\end{array} \qquad (134)$$

In order to obtain even greater predicting power of the symplectic extension of the IVBM, we need a further systematic investigation of the behavior of the model parameters as functions of the specific nuclear characteristics [110], as well as of the energy distributions of the band-head states in sequences of nuclei, which is our future aim.

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