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DYNAMICS AND POTENTIALS[1]

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A dynamics (i.e., a one-parameter group of automorphisms) of a system described by a C^*-algebra with a local structure in terms of C^*-subalgebras $\mathcal{A}(I)$ for local domains I of the physical space (a discrete lattice) is normally constructed out of potentials $P(I)$, each of which is a self-adjoint element of the subalgebra $\mathcal{A}(I)$, such that the the first time derivative of the dynamical change of any local observable A is i times the convergent sum of the commutator $[P(I), A]$ over all finite regions I. We will invert this relation under the assumption (obviously assumed in the usual approach) that local observables all have the first time derivative; i.e., we prove the existence of potentials for any given dynamics satisfying the above-stated condition. Furthermore, by imposing a further condition for the potential $P(I)$ to be chosen for each I that it does not have a portion which can be shifted to potentials for any proper subset of I, we also show (1) the existence, (2) uniqueness, (3) an automatic convergence property for the sum over I, and (4) a quite convenient property for the chosen potential. The so-obtained properties (3) and (4) are not assumed and are very useful, though they were never noticed nor used before.

We consider a system of finite kinds of finite spins and fermions on a discrete lattice, local regions being all finite subsets of the lattice and all local algebras being full matrix algebras of finite dimensions. For all dynamics for which all elements of any local algebra is once time differentiable, we prove that there exist a system of potentials which describe the time derivative of the given dynamics by a convergent sum stated above.

The fundamental technique for finding such potentials is a noncommutative expectation which is defined on the basis of a product state of the algebra. For each choice of a product state, we obtain one expectation which produces one set of potentials, all of which satisfy what we call the standardness condition and the convergence condition. We call this family of potentials standard potentials (corresponding to any specific choice of the product state). The standard potentials corresponding to different product states are different but produce the same time derivative, known as equivalent potentials.

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1. OUTLINE

The System under consideration consists of Fermions and spins on a discrete lattice $\mathbb{L}$ (e.g., $\mathbb{Z}^d$), described by the Algebra of Observables $\mathcal{A}$, which is a C^*-algebra with the following structure:

[1]Based on a joint work [2] with Hajime Moriya.

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(i) **Grading**: $\Theta \in \mathrm{Aut}\,\mathcal{A}$, $\Theta^2 = \mathrm{id}$.
$\mathcal{A} = \mathcal{A}_+ + \mathcal{A}_-$, $\mathcal{A}_\pm = \{A \in \mathcal{A}; \Theta(A) = \pm A\}$.
(Θ counts Fermion even-oddness.)

(ii) **Local Structure**: For each finite subset I of $\mathbb{L}$ (denoted $I \Subset \mathbb{L}$), a C^*-subalgebra $\mathcal{A}(I)$ of $\mathcal{A}$
($\{A(I); I \Subset \mathbb{L}\}$ satisfying natural mutual relations).

(iii) **Graded Commutation Relations**:
If $I_1 \cap I_2 = \emptyset$, $A_i \in \mathcal{A}_{\sigma_i}(I_i)$, $\sigma_i = +$ or $-$,
then $A_1 A_2 = \epsilon(\sigma_1, \sigma_2) A_2 A_1$,

$$\epsilon(\sigma_1, \sigma_2) = \begin{cases} - \text{ if } \sigma_1 = \sigma_2 = -, \\ + \text{ otherwise.} \end{cases}$$

The Basic Assumption is $\mathcal{A}(I) \approx M_{d|I|}(\mathbb{C})$, i.e., a finite dimensional full matrix algebra ($|I| =$ the number of points in I).

In the Traditional Approach, one starts with a Potential (satisfying a condition chosen for the sake of successful proof) and derives a Dynamics.

Our Basic Result is obtained in the opposite direction. For any given Dynamics, which satisfy a basic assumption that local observable has the time derivative (automatically satisfied in traditional appoach of constructing dynamics out of potential), Potential for the dynamics is shown to exist and to satisfy the following two properties automatically, and the potential satisfying these properties is unique for each given potential. (Existence and uniqueness.)

(i) Convergence condition (new).
(ii) Properties under a conditional expectation (new and useful).

The following Applications will be explained in Sec. 4 (Theorems 3 and 2):
(1) Energy estimates (consequence of the property (ii)).
(2) Useful for equivalence proof of equilibrium conditions.

2. DYNAMICS AND POTENTIAL

A Dynamics is a one-parameter group $t \in \mathbb{R} \to \alpha_\approx \in \mathrm{Aut}\,\mathcal{A}$. Our basic Assumption about dynamics is as follows:

If $I \Subset L$ and $A \in \mathcal{A}(I)$, then $(\dot{A} =)\delta_\alpha(A) = \left.\dfrac{d}{dt}\alpha_t(A)\right|_{t=0}$ exists.

So defined δ_α is a $*$-derivation on a dense domain

$$\mathcal{A}_0 \equiv \bigcup \{\mathcal{A}(I); I \Subset L\}.$$

A Potential is the function: $I \Subset L \to P(I) = P(I)^* \in \mathcal{A}(I)_+$. (Physically it is the Interaction energy among local objects in I.)

In Normal approach the following two steps are taken:

- construction of δ_α out of $\{P(I)\}$
  $\delta_\alpha(A) = i\sum_I [P(I), A]$ ($I \cap \operatorname{supp} A = \emptyset \Rightarrow [P(I), A] = 0$)
  (Formally, $\dot{A} = i[H, A]$ with Hamiltonian $H = \sum_I P(I)$.);
- construction of α_t out of δ_α.

We take opposite steps and obtain the following New result:

Theorem 1. *Existence of $P(I)$ for any $*$-derivation δ on $\mathcal{A}_0$, satisfying*

(1) Natural convergence condition (new):
$H_I = \sum_J \{P(I); J \cap I \neq \emptyset\}$ *convergent.*
$\Longrightarrow (\dot{A}) = i[H_I, A]$ *for* $A \in \mathcal{A}(I)$*;*

(2) Standardness (new):
$$E_J^\varphi(P_\varphi(I)) = \begin{cases} 0 & \text{if } J \not\supset I, \\ P_\varphi(I) & \text{if } J \supset I. \end{cases}$$

3. SUPPLEMENTARY EXPLANATION

(1) Equivalent Potentials

Definition 1. *Two potentials $\{P_1(I)\}$ and $\{P_2(I)\}$ are equivalent if δ_α derived from them are the same.*

Example of Equivalent Potentials. Let $I_2 \subset I_3$ be finite subsets with $|I_2| = 2$, and $|I_3| = 3$. For a given potential $\{P_1(I)\}$ with $P_1(I_2) \neq 0$, define another potential P_2 by
$$P_2(I) = P_1(I) \quad \text{if} \quad I \neq I_2, I_3,$$
$$P_2(I_2) = 0, \;\; P_2(I_3) = P_1(I_3) + P_1(I_2).$$
Then P_1 and P_2 satisfy the minimal conditions for potential ($P(I) = P(I)^* \in \mathcal{A}(I)_+$), $P_1 \neq P_2$ and they give the same time derivative δ_α. So they are equivalent potentials.

The standardness condition (2) in Theorem 1 is for the purpose of avoiding the situation described in the above example.

(2) (Noncommutative) Conditional Expectation E_J^φ

A state φ is a Θ-even product state, i.e.,
$$\varphi(\Theta(A)) = \varphi(A), \quad \varphi(AB) = \varphi(A)\varphi(B)$$

for any $A \in \mathcal{A}(I)$, $B \in \mathcal{A}(J)$, $I \cap J = \emptyset$. (Such product states for the fermion and spin algebra exist in abundance. The tracial state τ and the fermion vacuum state are examples.)

E_I^φ is a linear, positive, unital projection from $\mathcal{A}$ to $\mathcal{A}(I)$, characterized by $E_J^\varphi(A) \in \mathcal{A}(J)$ and the following set of equations:

$$\varphi(CAB) = \varphi(CE_J^\varphi(A)B) \quad (B, C \in \mathcal{A}(J)).$$

It has the following property:

$$E_I^\varphi E_J^\varphi = E_{I\cap J}^\varphi, \qquad E_I^\varphi E_{I^c}^\varphi = E_{I^c}^\varphi E_I^\varphi = \varphi,$$
$$\lim_\nu E_{K_\nu}^\varphi(A) = E_K^\varphi(A) \quad \text{if } K_\nu \longrightarrow K,$$
$$\lim_{\nu \nearrow \mathbb{L}} E_{K_\nu}^\varphi(A) = A.$$

Here $K_\nu \longrightarrow K$ for a net K_ν means

$$K = \bigcap_\mu \left(\bigcup_{\nu \geqslant \mu} K_\nu \right) = \bigcup_\mu \left(\bigcap_{\nu \geqslant \mu} K_\nu \right).$$

4. TRANSLATION COVARIANT SYSTEM

For $\mathbb{L} = \mathbb{Z}^d$, it is an additive group, which we call translation group. There is a representation of this group: $n \in \mathbb{L} \to \tau_n \in \operatorname{Aut} \mathcal{A}$ such that

$$\tau_n(\mathcal{A}(I)) = \mathcal{A}(I+n), \ \tau_n\Theta = \Theta\tau_n \text{ for all } \ n \in \mathbb{L}.$$

A potential P is translation covariant if

$$\tau_n(P(I)) = P(I+n) \text{ for all } \ I \Subset \mathbb{L}.$$

Theorem 2. *The real linear space of all translation covariant standard potentials forms a separable Banach space with the norm*

$$||P|| = ||H_P(n)|| \quad (\textit{independent of } n).$$

This theorem provides a tool for Application (2) in Sec. 1.
We have introduced earlier $H_P(I)$ = the total interaction enregy of the partial system in I as an open system where open system refers to the fact that the interactions between I and outside (I^c) are all included. We also define

$$U_P(I) = \sum_{J \subset I} P(J)$$

for the total interaction energy of the system I as a closed system. We have the following Energy Estimate:

Theorem 3.

$$||U_P(I)|| \leqslant ||H_P(I)|| \leqslant ||P|| \cdot |I|.$$

The first inequality is immediate from $||E_I^{\varphi}|| = 1$ and $U_P(I) = E_I^{\varphi} H_P(I)$.

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