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GENERATING FUNCTION FOR EXTENDED JACOBI POLYNOMIALS, NONCOMMUTATIVE DIFFERENTIAL CALCULUS AND RELATIVISTIC ENERGY AND MOMENTUM OPERATORS

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It is shown that the generating function for the matrix elements of irreps of Lorentz group is the common eigenfunction of the interior derivatives of the noncommutative differential calculus over the commutative algebra generated by the coordinate functions. These derivatives commute and can be interpreted as the quantum mechanical operators of the relativistic momentum corresponding to the half of the non-Euclidean distance from the origin in the Lobachevsky momentum space.

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We show that the realization of the Poincare group Lie algebra in the framework of the Quantum Mechanics in the relativistic configuration space (RQM) naturally requires to consider the noncommutative differential calculus over the associative commutative algebra of the coordinate functions. In particular the generators of translations, which are the momentum operators in accordance with the basic principles of Quantum Mechanics, are expressed in terms of the noncommutative derivatives.

Referring the reader to the original paper [1] and more recent articles [2–5] we stress here only that the concept of RQM is based on the expansion in matrix elements of the unitary representations of the Lorentz group [6]. In this paper we consider the case of two-dimensional space with the signature $(1,2)$ so that the transitivity surface of the Lorentz group $SO(2,1)$ is the mass shall of the particle

$$p_\mu p^\mu = {p^0}^2 - \tilde{p}^2 = m^2c^2, \quad p^0 > 0, \quad \mu = 0,1,2. \tag{1}$$

This group can also be considered as a factor group of $SU(1,1)$ by its center.

The role of the relativistic plane wave in the RQM formalism plays the kernel of Gelfand–Graev transformation:

$$\langle \tilde{\rho} | \tilde{p} \rangle = \left(\frac{p^0 - \tilde{p} \cdot \tilde{n}}{mc} \right)^{-\frac{1}{2} - i\rho}, \quad \tilde{n}^2 = 1, \tag{2}$$

$\langle \tilde{\rho} | \tilde{p} \rangle$ is the wave function of the free particle, i.e., of the state with definite value of energy and momentum. The parameter ρ is relativistic-invariant whose limits are the same as for the nonrelativistic radius-vector: $0 \leqslant \rho < \infty$. The vectors $\tilde{\rho} = \rho \tilde{n}, \tilde{n} = (\cos \psi, \sin \psi)$ span the 2-dimensional relativistic configuration space. In the nonrelativistic limit $\langle \tilde{\rho} | \tilde{p} \rangle \longrightarrow \mathrm{e}^{i \frac{\tilde{\rho} \tilde{p}}{\hbar}}$. In what follows we put $c = m = \hbar = 1$. We also use the hyperspherical coordinates in the momentum space:

$$p^0 = \cos h\chi, \quad \tilde{p} = \sin h\chi \tilde{n}_p, \quad \tilde{n}_p = (\cos \phi, \sin \phi). \tag{3}$$

The relativistic plane wave (2) is the generating function for the matrix elements of the unitary representations of the s.c. principal series which are equal up to the normalization factor to the associate Legendre functions [6]:

$$\langle \tilde{\rho} | \tilde{p} \rangle = \sum_{m=-\infty}^{\infty} (-1)^m \frac{\Gamma(-i\rho + 1/2)}{\Gamma(-i\rho + 1/2 + m)} P^m_{-\frac{1}{2} - i\rho}(\cosh \chi) \, \mathrm{e}^{im(\psi - \phi)}. \tag{4}$$

The commuting momentum operators are

$$\begin{aligned} \hat{p}^0 &= \cosh i\partial_\rho + \frac{i}{2\rho} \sin hi\partial_\rho - \frac{1}{2\rho(\rho + i/2)} (\partial_\psi)^2 \mathrm{e}^{\partial_\rho}, \\ \hat{p}^\pm &= \frac{\mathrm{e}^{\pm i\psi}}{2} \left\{ \hat{p}^0 - \mathrm{e}^{\partial_\rho} \pm \frac{1}{(\rho + i/2)} \mathrm{e}^{\partial_\rho} \partial_\psi \right\}, \\ \hat{p}^\pm &= \frac{\hat{p}^1 \pm i\hat{p}^2}{2}, \quad \partial_\rho = \frac{\partial}{\partial \rho}, \quad \partial_\psi = \frac{\partial}{\partial \psi}. \end{aligned} \tag{5}$$

In the nonrelativistic limit these operators transfer to the standard quantum mechanical energy and momentum operators. Relativistic momentum operators commute

$$[\hat{p}^\mu, \hat{p}^\nu] = 0, \ \mu, \nu = 0, 1, 2. \tag{6}$$

Despite the commutativity (6) of the momentum operators in the relativistic configuration ρ representation (5) they cannot be considered as the generators of the Poincare group. Because the operator $\hat{p}^0$ enters into the expressions for $\hat{p}^1, \hat{p}^2$, the last operators cannot be the exterior derivatives in any linear (generalized) differential calculus. But identification of the momentum operator with the generator of the translation in linear representation of the Poincare group is the fundamental physical requirement of the Quantum Theory. The solution to this problem can be found if we transfer from «old» momentum operators $\hat{p}^\mu$ to the new momenta $\hat{q}^\mu$, corresponding to the «half distance» in the Lobachevsky momentum space (1) as follows. In the momentum space $\hat{q}$-operators are given by (cf. (3))

$$q^0 = \cosh\frac{\chi}{2}, \quad \tilde{q} = \sin h\frac{\chi}{2}\tilde{n}_p. \tag{7}$$

We recall that χ is the non-Euclidean distance from the origin. The «half distance» or kinetic momenta possess the remarkable properties. For example, the relativistic energy E in the nonrelativistic limit $\|\tilde{p}\| \ll mc$ can be presented approximately as $E = p^0 c = \sqrt{m^2c^4 + \tilde{p}^2c^2} \simeq mc^2 + \tilde{p}^2/2m$. In terms of the new momentum, the relativistic formula has exactly the nonrelativistic form without any approximation:

$$E - mc^2 = mc^2\sinh^2\frac{\chi}{2} = \frac{\tilde{k}^2}{2m}, \tag{8}$$

where $\tilde{k} = 2\tilde{q}$. We show here that the corresponding operators of kinetic momentum do exist in the relativistic ρ representation and belong to the noncommutative differential calculus [5, 7]. To derive the corresponding momentum operators in the relativistic ρ representation we must consider instead of the plane wave (2) the generating function for the Jacobi functions of more general form:

$$\begin{aligned}\langle\tilde{\rho}, 0|\tilde{p}\rangle &= \left(\cosh\frac{\chi}{2} - \sinh\frac{\chi}{2}\mathrm{e}^{i(\psi-\phi)}\right)^{-i\rho-\frac{1}{2}+n} \times \\ &\quad\times \left(\cosh\frac{\chi}{2} - \sinh\frac{\chi}{2}\mathrm{e}^{-i(\psi-\phi)}\right)^{-i\rho-\frac{1}{2}-n}, \\ \langle\tilde{\rho}, 0|\tilde{p}\rangle &= \langle\tilde{\rho}|\tilde{p}\rangle \; \langle\tilde{\rho}, n|\tilde{p}\rangle = \sum_{m=-\infty}^{\infty} P_{mn}^{-\frac{1}{2}-i\rho}(\cosh\chi)\mathrm{e}^{i(n-m)(\psi-\phi)},\end{aligned} \tag{9}$$

m, n are simultaneously integer or semi-integer numbers. The desired operators $\hat{\tilde{q}} = (\tilde{q}^1, \tilde{q}^2)$ in configuration ρ, ψ, n representation are obtained using formulae (7)–(9) of Subsec. 6.7.3 of the book [6]:

$$\hat{q}^\pm = \mathrm{e}^{\pm i\psi}\left\{\frac{-i\rho - \dfrac{1}{2} \pm n}{i\rho}\sin h\frac{i}{2}\partial_\rho \pm \frac{1}{2\rho}\partial_\psi \mathrm{e}^{\frac{i}{2}\partial_\rho}\right\}\mathrm{e}^{\mp\frac{1}{2}\partial_n}, \tag{10}$$

$$\left[\hat{q}^1, \hat{q}^2\right] = \frac{i}{2}\left[\hat{q}^+, \hat{q}^-\right] = 0, \quad \hat{q}^\pm = \frac{\hat{q}^1 \pm i\hat{q}^2}{2}. \tag{11}$$

The generalized plane waves (9) are the common eigenfunctions for $\hat{q}^\pm$:

$$\hat{q}^\pm \; \langle\tilde{\rho}, n|\tilde{p}\rangle = q^\pm \; \langle\tilde{\rho}, n|\tilde{p}\rangle = \sin h\frac{\chi}{2}\mathrm{e}^{\pm i\phi} \; \langle\tilde{\rho}, n|\tilde{p}\rangle \; . \tag{12}$$

Also the operators exist corresponding to the eigenvalue $\cosh \chi/2$:

$$\begin{aligned} \tilde{c}_{\pm} &= -\frac{1}{2i\rho} \mathrm{e}^{\mp \frac{1}{2}\partial_n} \{(-i\rho \mp n)\, \mathrm{e}^{\frac{i}{2}\partial_\rho} + (-i\rho \pm n)\, \mathrm{e}^{-\frac{i}{2}\partial_\rho} \pm i\partial_\psi \, \mathrm{e}^{\frac{i}{2}\partial_\rho}\} \\ &= \left(\mathrm{e}^{-i\psi} \hat{q}^{\pm} + \mathrm{e}^{\frac{i}{2}\partial_\rho} \mathrm{e}^{\mp \frac{1}{2}\partial_n}\right), \quad \hat{c}_{\pm} \, \langle \tilde{\rho}, n|\tilde{p}\rangle = \cosh \frac{\chi}{2} \, \langle \tilde{\rho}, n|\tilde{p}\rangle \, . \end{aligned} \tag{13}$$

Now we consider the noncommutative differential calculus over the associative algebra A over $\mathbb{C}$. Finite linear combinations of elements of A and their finite products are again elements of A. A differential calculus over A is a $\mathbb{Z}$-graded associative algebra over $\mathbb{C}$:

$$\Omega(A) = \sum_r = 0_{\otimes} \Omega^r(A). \tag{14}$$

The elements of $\Omega^r(A)$ are called r forms. There exists an exterior derivative $\hat{d}$ which satisfies the following conditions:

$$\hat{d}^2 = 0, \qquad \hat{d}(\omega\omega') = (\hat{d}\omega)\omega' + (-1)^r \omega \hat{d}\omega', \tag{15}$$

where ω and ω' are r and r' forms, respectively. All products in this formula are the wedge products, but we omit the symbol $\wedge$ in formulas. In our case A is the commutative algebra generated by the coordinate-x_i functions $f(x_1, \dots, x_n), \quad i = 1, \dots, n$. We can develop an explicit construction defining $\hat{d}$ as an operator valued 1-form. Differentiation is given by

$$\hat{d}\omega = [\hat{d}, \omega]_\wedge = \hat{d}\omega - (-1)^r \omega \hat{d}. \tag{16}$$

Important is the difference between the standard differential calculus and the relativistic one. In the first one, the differential and the coordinate function commute $[dx_k, x_i] = 0$, because in the standard case x_i and dx_k are the independent numerical parameters. But the relativistic differential calculus is noncommutative: $[dx_k, x_i] \neq 0$.

To establish the relativistic differential calculus we must determine $\hat{d}$. First we recall the simple connection existing between the nonrelativistic $\hat{d}$ and momentum operators: $\hat{d} = dx_1 \partial x_1 + dx_2 \partial x_2 = i(dx_+ \hat{k}_- + dx_- \hat{k}_+) = d\rho \partial_\rho + d\psi \partial_\psi$, where $dx_\pm = dx_1 \pm dx_2 = \mathrm{e}^{\pm i\psi}(d\rho \pm i\rho d\psi)$. In the relativistic case the differentials $dx_\pm$ are modified and gain the operator valued form:

$$\hat{d}x_\pm = \mathrm{e}^{\pm i\psi} \left(d\rho \pm \mathrm{e}^{-\frac{i}{2}\partial_\rho} (i\rho \mp n) d\psi\right) \mathrm{e}^{-\frac{1}{2}\partial_n}. \tag{17}$$

The identity is satisfied

$$i\left(\hat{d}x_+ \hat{q}^- + \hat{d}x_- \hat{q}^+\right) = d\rho(-2i \sin h\frac{i}{2}\partial_\rho) + d\psi \partial_\psi \longrightarrow d\rho \partial_\rho + d\psi \partial_\psi. \tag{18}$$

It is easily seen that this expression is incomplete. It must be extended to involve the «$\cosh\chi/2$» terms or operators $\hat{c}_\pm$ in order to satisfy the generalized Leibnitz rule of the noncommutative derivation:

$$\hat{d} = i\left(\hat{d}x_+\hat{q}^- + \hat{d}x_-\hat{q}^+ + \right.$$
$$\left. + d\rho_+\left(\mathrm{e}^{\frac{1}{2}\partial_n}\hat{c}_+ + \mathrm{e}^{-\frac{1}{2}\partial_n}\hat{c}_- - 2\right) + d\rho_-\left(\mathrm{e}^{\frac{1}{2}\partial_n}\hat{c}_+ - \mathrm{e}^{-\frac{1}{2}\partial_n}\hat{c}_-\right)\right). \quad (19)$$

The following identity can be easily proved $\mathrm{e}^{\frac{1}{2}\partial_n}\,\hat{c}_+ - \mathrm{e}^{-\frac{1}{2}\partial_n}\,\hat{c}_- = \mathrm{e}^{\frac{1}{2}\partial_n}\mathrm{e}^{-i\psi}\,\hat{q}^+ - \mathrm{e}^{-\frac{1}{2}\partial_n}\mathrm{e}^{i\psi}\,\hat{q}^-$. We see that expressions multiplied by $d\rho_+$ and $d\rho_-$ are bound. This lets us to put $d\rho_- = 0$ without limiting the generality. And $\hat{d}$ takes the form

$$\hat{d} = d\rho(-2i\sinh\frac{i}{2}\partial_\rho) + d\rho_+(2i\cosh\frac{i}{2}\partial_\rho - 2) + d\psi\partial_\psi$$
$$= d\rho_\rightarrow\overrightarrow{\partial} + d\rho_\leftarrow\overleftarrow{\partial} + d\psi\partial_\psi, d\rho_\rightarrow = \frac{d\rho_+ - d\rho_-}{2}, d\rho_\leftarrow = \frac{d\rho_+ + d\rho}{2}, \quad (20)$$

where the identity has been taken into account $\partial_n\hat{c}_+ = \mathrm{e}^{-\frac{1}{2}\partial_n}\,\hat{c}_- = 2\cosh i/2\partial_\rho$.

Operators $\overrightarrow{\partial} = \dfrac{\mathrm{e}^{\frac{i}{2}\partial_\rho} - 1}{\frac{i}{2}}$ and $\overleftarrow{\partial} = \dfrac{\mathrm{e}^{-i\frac{i}{2}\partial_\rho} - 1}{-\frac{i}{2}}$ are the left and right interior derivatives of the noncommutative differential calculus above. It is sufficient to keep only one of the $\overleftrightarrow{\partial}$ derivatives and we put $d\rho_\leftarrow = 0$ (and omit the subscript $\rightarrow$: $d\rho_\rightarrow = d\rho$)

$$\hat{d} = d\rho\,\overrightarrow{\partial} + d\psi\,\partial_\psi. \quad (21)$$

In the relativistic differential calculus the differential $d\rho$ does not commute with the coordinate ρ. We recall that in the standard calculus ρ and $d\rho$, of course, commute $[d\rho, \rho] = 0$. In the relativistic calculus

$$\hat{d}\rho = [\hat{d}, \rho] = [\hat{d}\rho\partial_\rho, \rho] = d\rho\frac{\mathrm{e}^{\frac{i}{2}\partial_\rho} - 1}{i/2}\rho - \rho d\rho\frac{\mathrm{e}^{\frac{i}{2}\partial_\rho} - 1}{i/2} = d\rho\mathrm{e}^{\frac{i}{2}\partial_\rho}, \quad (22)$$

and we obtain

$$[\hat{d}\rho, \rho] = \frac{i}{2}\hat{d}\rho. \quad (23)$$

It follows directly from (23)

$$[\hat{d}\rho, f(\rho)] = \frac{i}{2}(\overrightarrow{\partial}f(\rho))\hat{d}\rho = \frac{i}{2}\hat{d}\rho(\overleftarrow{\partial}f(\rho)). \quad (24)$$

Free Hamiltonian has a form (8) $\hat{H}_0 - mc^2 = \dfrac{2\hat{q}^+\hat{q}^-}{m}$. Potential $V(\rho)$ is introduced (see [1]–[5]) as an addition to the free Hamiltonian and we come to the relativistic

Schrödinger equation which does not differ by form from the nonrelativistic one

$$\left(\hat{H}_0 + V(\rho)\right)\psi(\tilde{\rho}) = E\psi(\tilde{\rho}). \tag{25}$$

At last, we write down the momentum operators corresponding to the half of the distance in the Lobachevsky p space in terms of the relativistic noncommutative differential calculus

$$\hat{q}^{\pm} = \mathrm{e}^{\pm i\psi}\left(\frac{-i\rho - 1/2 \pm n}{4\rho}(\overrightarrow{*} + \overleftarrow{*})\,\hat{d}\rho - \frac{1}{2\rho}\left(1 + \frac{i}{2}\overrightarrow{*}\hat{d}\rho\right) * \hat{d}_{\psi}\right), \tag{26}$$

where $\overrightarrow{*}$ and $\overleftarrow{*}$ are left and right noncommutative Hodge symbols introduced in [5]; $*$ is the standard Hodge symbol corresponding to the commutative differentiation in ψ.

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