ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
2012. Т. 43. ВЫП. 5

BRST RENORMALIZATION

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We consider the renormalization of general gauge theories on curved space-time background, with the main assumption being the existence of a gauge-invariant and diffeomorphism invariant regularization. Using the Batalin–Vilkovisky (BV) formalism, one can show that the theory possesses gauge-invariant and diffeomorphism invariant renormalizability at quantum level, up to an arbitrary order of the loop expansion.

PACS: 04.62.+v

INTRODUCTION

The quantum field theory (QFT) in curved space is an important ingredient of our general understanding of the quantum description of nature. One of the most important aspects of the modern QFT is the theory of gauge fields and their perturbative renormalization. It is quite interesting to know whether the existing methods to analyze renormalizability of gauge theories are working well in curved space. In the previous considerations of the problem [1, 2] (see also [3]), it has been assumed that the gauge-invariant renormalization of the theory is indeed possible, due to the existence of both gauge-invariant and diffeomorphism invariant regularization, such as, for instance, dimensional one. Starting from this point, it is possible to establish the prescription for constructing the renormalizable theories of interacting matter fields on curved background [1, 4].

Here we consider, in a more formal way than it was done before, the issue of gauge-invariant renormalizability in curved space-time. For this end, we are going to apply the BV formalism [5]. It is well known that this formalism enables one to prove the gauge-invariant renormalizability of general gauge theories in a situation when all fields under consideration are quantum ones [6]. It is of

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course important to generalize these considerations to the case when the QFT is defined in the presence of external conditions, in particular, in curved space-time. In this case, one has to take care about both gauge symmetries and general covariance. The last symmetry involves both quantum and external fields, making the consideration more complicated. Our main purpose is to consider the general features of renormalization of the theory of quantum matter fields in curved space-time, using the powerful BV formalism.

1. GENERAL GAUGE THEORIES IN CURVED SPACE

Let us consider a theory of gauge fields A^i in an external gravitational field $g_{\mu\nu}$. The classical theory is described by the action which depends on both dynamical fields and external metric,

$$S_0 = S_0(A, g). \tag{1}$$

Here and below we use the condensed notation $g \equiv g_{\mu\nu}$ for the metric, when it is an argument of some functional or function. The action (1) is assumed to be gauge-invariant,

$$S_{0,i} R^i_a = 0, \quad \delta A^i = R^i_a(A, g)\lambda^a, \quad \lambda^a = \lambda^a(x) \quad (a = 1, 2, \ldots, n), \tag{2}$$

as well as covariant,

$$\delta_g S_0 \;\; = \;\; \frac{\delta S_0}{\delta A^i}\delta_g A^i + \frac{\delta S_0}{\delta g_{\mu\nu}}\delta_g g_{\mu\nu} = 0, \tag{3}$$

where λ^a are independent parameters of the gauge transformation, corresponding to the symmetry group of the theory. The diffeomorphism transformation of the metric in Eq. (3) has the form

$$\begin{aligned} \delta_g g_{\mu\nu} = -g_{\mu\alpha}\partial_\nu\xi^\alpha - g_{\nu\alpha}\partial_\mu\xi^\alpha - \partial_\alpha g_{\mu\nu}\xi^\alpha = \\ = -g_{\mu\alpha}\nabla_\nu\xi^\alpha - g_{\nu\alpha}\nabla_\mu\xi^\alpha = -\nabla_\mu\xi_\nu - \nabla_\nu\xi_\mu. \end{aligned} \tag{4}$$

Here ξ^α are the parameters of the coordinates transformation,

$$\xi^\alpha = \xi^\alpha(x) \quad (\alpha = 1, 2, \ldots, d). \tag{5}$$

As usual, an explicit expression for $\delta_g A^i$ depends on tensor (or spinor) properties of A^i. For example, in the case of a scalar field A one has $\delta_g A = -\partial_\alpha A \xi^\alpha$, while in the case of a vector field A^μ the transformation rule is $\delta_g A^\mu = A^\nu \nabla_\nu \xi^\mu - \xi^\nu \nabla_\nu A^\mu$, etc. In general, our interest is to explore the renormalization properties of the theories which include all three kind of fields (fermions, vectors and scalars), such that, for instance, the Standard Model and its extensions, including Grand Unified Theories (GUTs), would be covered. Therefore, the notation A^i in (2) and (3) means the set of fields with the different transformation rules.

The generating functional $Z(J, \phi^*, g)$ of the Green functions can be constructed in the form of the functional integral

$$\mathcal{Z}(J, \phi^*, g) = \int d\phi \exp\left\{ \frac{i}{\hbar}\Big[S_\psi(\phi, \phi^*, g) + J_A \phi^A \Big] \right\}. \tag{6}$$

Here* $\phi^A = (A^i, B^a, C^a, \bar{C}^a)$ represents the full set of fields of the complete configuration space of the theory under consideration and $\phi^*_A = (A^*_i, B^*_a, C^*_a, \bar{C}^*_a)$ are corresponding antifields. Finally, $S_\psi(\phi, \phi^*, g)$ is the quantum action constructed with the help of the solution $S = S(\phi, \phi^*, g)$ of the master equation (for details see [5])

$$(S, S) = 0, \quad S(\phi, \phi^*, g)|_{\phi^*=0} = S_0(A, g) \tag{7}$$

in the form

$$S_\psi(\phi, \phi^*, g) = S\left(\phi, \phi^* + \frac{\delta \Psi(\phi, g)}{\delta \phi}, g \right). \tag{8}$$

In the last equation, (8), $\Psi(\phi, g)$ is a gauge fixing functional. Note that S_ψ satisfies the master equation

$$(S_\psi, S_\psi) = 0. \tag{9}$$

From the gauge invariance of initial action (2), in the usual manner one can derive the BRST symmetry and the Ward identities for generating functionals (see [6]).

A solution to the master equation (7) can be always found in form of a series in antifields ϕ^* (see [5]),

$$S(\phi, \phi^*, g) = S_0(A, g) + A^*_i R^i_a(A, g) C^a + \bar{C}^*_a B^a + \dots, \tag{10}$$

where dots mean higher order terms in fields B^a, C^a. We assume that every term in (10) is transformed as a scalar under arbitrary local transformations of coordinates $x^\mu \to x^\mu + \xi^\mu(x)$. It means the general covariance of $S = S(\phi, \phi^*, g)$,

$$\delta_g S(\phi, \phi^*, g) = \frac{\delta S}{\delta \phi^A} \delta_g \phi^A + \delta_g \phi^*_A \frac{\delta S}{\delta \phi^*_A} + \frac{\delta S}{\delta g_{\mu\nu}} \delta_g g_{\mu\nu} = 0. \tag{11}$$

*We restrict ourselves to the case of irreducible close gauge theories only, in order to simplify the description of the configuration space.

Let us choose the gauge fixing functional $\Psi = \psi(\phi, g)$ in a covariant form

$$\delta_g \Psi = 0, \tag{12}$$

then the quantum action $S_\psi = S_\psi(\phi, \phi^*, g)$ obeys the general covariance too,

$$\delta_g S_\psi = 0. \tag{13}$$

From Eq. (13) and the assumption that the term with the sources J_A in (6) is covariant

$$\delta_g (J_A \phi^A) = (\delta_g J_A)\phi^A + J_A(\delta_g \phi^A) = 0, \tag{14}$$

follows the general covariance of $\mathcal{Z} = \mathcal{Z}(J, \phi^*, g)$. Indeed,

$$\begin{aligned} \delta_g \mathcal{Z}(J, \phi^*, g) = \\ = \frac{i}{\hbar} \int d\phi \Big[\delta_g \Phi^*_A \frac{\delta S_\psi(\phi, \phi^*, g)}{\delta \phi^*_A} + \frac{\delta S_\psi(\phi, \phi^*, g)}{\delta g_{\mu\nu}} \delta_g g_{\mu\nu} + (\delta_g J_A)\phi^A \Big] \times \\ \times \exp\Big\{ \frac{i}{\hbar} \Big[S_\psi(\phi, \phi^*, g) + J_A \phi^A \Big] \Big\}. \end{aligned} \tag{15}$$

Making change of integration variables in the functional integral, (15),

$$\phi^A \quad \rightarrow \quad \phi^A + \delta_g \phi^A, \tag{16}$$

we arrive at the relation

$$\begin{aligned} \delta_g \mathcal{Z}(J, \phi^*, g) = \\ = \frac{i}{\hbar} \int d\Phi \Big[\delta_g S_\psi + \delta_g (J_A \phi^A) \Big] \exp\Big\{ \frac{i}{\hbar} \Big[S_\psi(\phi, \phi^*, g) + J_A \phi^A \Big] \Big\} = 0. \end{aligned} \tag{17}$$

From (17) it follows that the generating functional of connected Green functions $\mathcal{W}(J, \phi^*, g)$,

$$\mathcal{W}(J, \Phi^*, g) = \frac{i}{\hbar} \ln Z(J, \phi^*, g), \tag{18}$$

obeys the property of the general covariance as well

$$\delta_g \mathcal{W}(J, \phi^*, g) = 0. \tag{19}$$

Consider now the generating functional of vertex functions $\Gamma = \Gamma(\Phi, \Phi^*, g)$,

$$\Gamma(\phi, \phi^*, g) = \mathcal{W}(J, \phi^*, g) - J_A \phi^A, \tag{20}$$

where

$$\phi^A = \frac{\delta \mathcal{W}(J, \phi^*, g)}{\delta J_A}, \quad J_A = -\frac{\delta \Gamma(\phi, \phi^*, g)}{\delta \phi^A}. \tag{21}$$

From the definition of ϕ^A (21) and the general covariance of $W(J, \phi^*, g)$, we can conclude the general covariance of $J_A \phi^A$. Therefore,

$$\delta_g \Gamma(\phi, \phi^*, g) = \delta_g \mathcal{W}(J, \phi^*, g) = 0. \tag{22}$$

2. GAUGE-INVARIANT RENORMALIZATION IN CURVED SPACE-TIME

Up to now we have considered non-renormalized generating functionals of Green functions. The next step is to prove the general covariance for renormalized generating functionals. For this end, let us first consider the one-loop approximation for $\Gamma = \Gamma(\phi, \phi^*, g)$,

$$\Gamma = S_\psi + \bar{\Gamma}^{(1)} = S_\psi + \hbar\big[\bar{\Gamma}^{(1)}_{\text{div}} + \bar{\Gamma}^{(1)}_{\text{fin}}\big] + O(\hbar^2), \tag{23}$$

where $\bar{\Gamma}^{(1)}_{\text{div}}$ and $\bar{\Gamma}^{(1)}_{\text{fin}}$ denote the divergent and finite parts of the one-loop approximation for Γ. The divergent local term $\bar{\Gamma}^{(1)}_{\text{div}}$ gives the first counterpart in the one-loop renormalized action $S_{\psi 1}$,

$$S_\psi \to S_{\psi 1} = S_\psi - \hbar \bar{\Gamma}^{(1)}_{\text{div}}. \tag{24}$$

From (13) and (22) it follows that in the one-loop approximation we have

$$\delta_g\big[\bar{\Gamma}^{(1)}_{\text{div}} + \bar{\Gamma}^{(1)}_{\text{fin}}\big] = 0, \tag{25}$$

and therefore $\bar{\Gamma}^{(1)}_{\text{div}}$ and $\bar{\Gamma}^{(1)}_{\text{fin}}$ obey the general covariance independently:

$$\delta_g \bar{\Gamma}^{(1)}_{\text{div}} = 0, \quad \delta_g \bar{\Gamma}^{(1)}_{\text{fin}} = 0. \tag{26}$$

In its turn, the one-loop renormalized action $S_{\psi 1}$ (i.e., classical action, renormalized at the one-loop level) is covariant:

$$\delta_g S_{\psi 1} = 0. \tag{27}$$

Constructing the generating functional of the one-loop renormalized Green functions $\mathcal{Z}_1(J, \phi^*, g)$, with the action $S_{\psi 1} = S_{\psi 1}(\phi, \phi^*, g)$, and repeating the arguments given above, we arrive at the relation

$$\delta_g \mathcal{Z}_1 = 0, \quad \delta_g W_1 = 0, \quad \delta_g \Gamma_1 = 0. \tag{28}$$

In the last equation we have introduced the new useful notation for the effective action Γ_1 renormalized up to the one-loop order. This functional includes the contributions of one-loop and also higher loop orders, however, only the one-loop divergences are removed by renormalization. This means that Γ_1 is finite in the $\mathcal{O}(\hbar)$ order, but may be divergent starting from $\mathcal{O}(\hbar^2)$ and beyond.

The generating functional of vertex functions $\Gamma_1 = \Gamma_1(\phi, \phi^*, g)$, which is finite in the one-loop approximation, can be presented in the form

$$\Gamma_1 = S_\psi + \hbar\bar{\Gamma}^{(1)}_{\rm fin} + \hbar^2\big[\bar{\Gamma}^{(2)}_{1,{\rm div}} + \bar{\Gamma}^{(2)}_{1,{\rm fin}}\big] + O(\hbar^3). \tag{29}$$

Indeed, this functional contains a divergent part $\bar{\Gamma}^{(2)}_{1,{\rm div}}$ and defines renormalization of the action S_ψ in the two-loop approximation:

$$S_\psi \to S_{\psi 2} = S_{\psi 1} - \hbar^2\bar{\Gamma}^{(2)}_{1,{\rm div}}. \tag{30}$$

Starting from (26), (27) and (28), we derive

$$\delta_g\bar{\Gamma}^{(2)}_{1,{\rm div}} = 0, \quad \delta_g\bar{\Gamma}^{(2)}_{1,{\rm fin}} = 0. \tag{31}$$

The last equation means that the general covariance condition is satisfied separately for the divergent and finite parts of $\bar{\Gamma}_1$ in the two-loop approximation. As a consequence, the two-loop renormalized action $S_{\psi 2} = S_{\psi 2}(\Phi, \Phi^*, g)$ is a covariant functional

$$\delta_g S_{\psi 2} = 0. \tag{32}$$

Applying the induction method, we can repeat the procedure to an arbitrary order of the loop expansion. In this way, we arrive at the following results:

a) The full renormalized action, $S_{\psi R} = S_{\psi R}(\Phi, \Phi^*, g)$,

$$S_{\psi R} = S_\psi - \sum_{n=1}^{\infty}\hbar^n\bar{\Gamma}^{(n)}_{n-1,{\rm div}}, \tag{33}$$

which is local in each finite order in $\hbar$, obeys the general covariance

$$\delta_g S_{\psi R} = 0. \tag{34}$$

b) The renormalized generating functional of vertex functions, $\Gamma_R = \Gamma_R(\Phi, \Phi^*, g)$),

$$\Gamma_R = S_\psi + \sum_{n=1}^{\infty}\hbar^n\bar{\Gamma}^{(n)}_{n-1,{\rm fin}}, \tag{35}$$

which is finite in each finite order in $\hbar$, is covariant:

$$\delta_g\Gamma_R = 0. \tag{36}$$

It was proved in [6] that the renormalized action $S_{\psi R}$ satisfies the master equation

$$(S_{\psi R}, S_{\psi R}) = 0 \tag{37}$$

and the Ward identities for non-renormalized and renormalized generating functionals of vertex functions have the form

$$(\Gamma, \Gamma) = 0, \quad (\Gamma_R, \Gamma_R) = 0. \tag{38}$$

The last equations mean that the gauge-invariant renormalizability (38) of a quantum field theory takes place in the presence of an external gravitational field, such that the general covariance of Effective Action (36) is also preserved. In order to use this important result, we have to perform an additional consideration and check how the covariance is preserved in case when we use apparently non-covariant techniques, e.g., related to the representation of the metric as a sum of the flat one and perturbation. This subject will be treated in the next section.

3. NON-COVARIANT GAUGES

In many cases, it is interesting to consider the renormalization of quantum field theory in curved space using the non-covariant gauge fixing functionals. Let us see how the non-covariant gauge fixing can be implemented in the quantum theory.

Our purpose is to investigate the problem of general covariant renormalizability for general gauge theories in the presence of an external gravitational field, when one uses non-covariant gauge fixing functional $\Psi = \Psi(\phi, g)$,

$$\delta_g \Psi \neq 0. \tag{39}$$

As before, we assume that the classical action of the theory $S = S(\phi, \phi^*, g)$ is covariant, i.e., $\delta_g S = 0$, but now the action $S_\psi = S_\psi(\phi, \phi^*, g) = S(\phi, \phi^* + \delta\Psi/\delta\phi, g)$ is not covariant, $\delta_g S_\psi \neq 0$. Our consideration will be essentially based on the known formalism for investigating the gauge dependence in general gauge theories, given in [6]. Non-covariance of S_ψ can be described in the form of anticanonical infinitesimal transformation with the odd generating functional

$$X(\phi, \phi^*, g) = \phi^*_A \phi^A + \delta_g \Psi(\phi, g), \tag{40}$$

$$\Phi^A = \frac{\delta X(\phi', \phi^*, g)}{\delta \phi^*_A} = \Phi^{A'}, \quad \phi^{*'}_A = \frac{\delta X(\phi', \phi^*, g)}{\delta \phi'_A} = \phi^*_A + \frac{\delta \delta_g \Psi}{\delta \phi^A}, \tag{41}$$

when

$$\delta_g S_\psi = \frac{\delta \delta_g \Psi}{\delta \phi^A} \frac{\delta S_\psi}{\delta \phi^*_A} = (\delta_g \Psi, S_\psi). \tag{42}$$

The variation of S_ψ leads to the variations of generating functionals of the Green functions $\mathcal{Z} = \mathcal{Z}(J, \phi^*, g)$, connected Green functions $\mathcal{W} = \mathcal{W}(J, \phi^*, g)$ and vertex functions $\Gamma = \Gamma(\phi, \phi^*, g)$ in the form

$$\delta_g \mathcal{Z} = \frac{i}{\hbar} J_A \frac{\delta}{\delta \phi^*_A} \delta_g \Psi \left(\frac{\hbar}{i} \frac{\delta}{\delta J}, g \right) \mathcal{Z}, \quad \delta_g \mathcal{W} = J_A \frac{\delta}{\delta \phi^*_A} \langle \delta_g \Psi \rangle,$$
$$\delta_g \Gamma = (\langle\langle \delta_g \Psi \rangle\rangle, \Gamma),$$

where the notations

$$\langle \delta_g \Psi \rangle = \delta_g \Psi \left(\frac{\delta \mathcal{W}}{\delta J} + \frac{\hbar}{i} \frac{\delta}{\delta J}, g \right), \quad \langle\langle \delta_g \Psi \rangle\rangle = \delta_g \Psi \left(\phi + i\hbar (\Gamma'')^{-1} \frac{\delta_l}{\delta \phi}, g \right),$$
$$\Gamma''_{AB} = \frac{\delta_l}{\delta \phi^A} \frac{\delta}{\delta \phi^B} \Gamma$$

were used. These results can be immediately reproduced in the renormalized theory [6]. Namely, for the variation (42), the corresponding variation of renormalized action $\delta_g S_{\psi R}$ can be presented in the form

$$\delta_g S_{\psi R} = (\delta_g \Psi_R, S_{\psi R}) \tag{43}$$

of the anticanonical transformation with local generating functional $X = \phi^*_A \phi^A + \delta_g \Psi_R$,

$$\delta_g \Psi_R(\phi, \phi^*, g) = \delta_g \Psi(\phi, g) - \sum_{n=1}^{\infty} \hbar^n \delta_g \Psi^{(n)}_{n-1,\mathrm{div}}(\phi, \phi^*, g), \tag{44}$$

while the variation of renormalized vertex generating functional $\delta_g \Gamma_R$ has the form

$$\delta_g \Gamma_R = (\langle\langle \delta_g \Psi_R \rangle\rangle_R, \Gamma_R), \tag{45}$$

which corresponds to finite anticanonical transformation with generating function

$$X = \phi^*_A \phi^A + \langle\langle \delta_g \Psi_R \rangle\rangle_R, \quad \langle\langle \delta_g \Psi_R \rangle\rangle_R = \delta_g \Psi(\phi, g) + \sum_{n=1}^{\infty} \hbar^n \delta_g \Psi^{(n)}_{n-1,\mathrm{fin}}. \tag{46}$$

In the formulas presented above, we have used the notations $\delta_g \Psi^{(n)}_{n-1,\mathrm{div}}$ and $\delta_g \Psi^{(n)}_{n-1,\mathrm{fin}}$ for the divergent and finite terms, respectively, of the n-loop approximation for the generating function of an anticanonical transformation which is finite in $(n-1)$th order approximation and is constructed on the basis of the theory with the action $S_{\psi(n-1)}$.

The interpretation of the relations (45) and (46) is that the theory with external gravitational field may have non-covariance in the renormalized effective action, but it comes only from the possible non-covariance of the arguments. Therefore, the violation of the general coordinate symmetry which can occur because of the non-covariant gauge-fixing can be always included into the arguments. As a consequence, one can always define some special set of arguments, in terms of which the quantum dynamics is described in a completely covariant way. One important aspect of this feature is that we can actually perform general considerations or make practical calculations in non-covariant gauges. After that we can always restore the covariance, using those parts of effective action which are not affected by gauge transformation. Practical examples of this technique can be found in many publications, but here we constructed a theoretical background for its consistent description. In the next sections we will also see that this result opens the way for a practical construction of renormalizable gauge theories in curved space-time.

Note that there exists another interpretation of the gauge dependence of effective action (see [7]). Namely, it can be proved that dependence on the gauge of effective action is proportional to its extremals, i.e., physical quantities calculated on shell do not depend on the gauge.

CONCLUSIONS

We have considered the general scheme of gauge-invariant and covariant renormalization of the quantum gauge theory of matter fields in curved space-time. Using the Batalin–Vilkovisky formalism, we have shown that in the theory which admits gauge-invariant and diffeomorphism invariant regularization, these two symmetries hold in the counterterms to all orders of the loops expansion. Starting from a renormalizable theory in flat space-time and using a standard prescription [1, 4], one can always arrive at the theory which is renormalizable in curved space-time as well.

Acknowledgements. The authors are grateful to I. L. Buchbinder for useful discussions.

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