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NONLOCAL GRAVITATIONAL MODELS AND EXACT SOLUTIONS

*S. Yu. Vernov**

Instituto de Ciencias del Espacio, Institut d'Estudis Espacials de Catalunya
Campus UAB, Facultat de Ciències, Bellaterra (Barcelona), Spain
Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow

A nonlocal gravity model with a function $f(\Box^{-1}R)$, where $\Box$ is the d'Alembert operator, is considered. The algorithm, allowing one to reconstruct $f(\Box^{-1}R)$, corresponding to the given Hubble parameter and the state parameter of the matter, is proposed. Using this algorithm, we find the functions $f(\Box^{-1}R)$, corresponding to de Sitter solutions.

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1. NONLOCAL GRAVITATIONAL MODELS

In this paper we consider nonlocal gravity models, which are described by the action

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\Big(1+f\left(\Box^{-1}R\right)\Big)-2\Lambda\right]+\mathcal{L}_{\rm matter}\right\}, \qquad (1)$$

where $\kappa^2 \equiv 8\pi/{M_{\rm Pl}}^2$, the Planck mass being $M_{\rm Pl} = 1.2\cdot 10^{19}$ GeV. We use the signature $(-,+,+,+)$, g is the determinant of the metric tensor $g_{\mu\nu}$, Λ is the cosmological constant, f is a differentiable function, and $\mathcal{L}_{\rm matter}$ is the matter Lagrangian. Note that the modified gravity action (1) does not include a new dimensional parameter. This nonlocal model has a local scalar-tensor formulation. Introducing two scalar fields, η and ξ, we can rewrite action (1) in the following local form:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}\left[R\left(1+f(\eta)-\xi\right)+\xi\Box\eta-2\Lambda\right]+\mathcal{L}_{\rm matter}\right\}. \qquad (2)$$

By varying the action (2) over ξ, we get $\Box\eta = R$. Substituting $\eta = \Box^{-1}R$ into action (2), one reobtains action (1). Varying action (2) with respect to the metric

*E-mail: vernov@ieec.uab.es, svernov@theory.sinp.msu.ru

tensor $g_{\mu\nu}$, one gets

$$\frac{1}{2}g_{\mu\nu}\left[R(1+f(\eta)-\xi)-\partial_\rho\xi\partial^\rho\eta-2\Lambda\right]-R_{\mu\nu}(1+f(\eta)-\xi)+$$

$$+\frac{1}{2}\left(\partial_\mu\xi\partial_\nu\eta+\partial_\mu\eta\partial_\nu\xi\right)-\left(g_{\mu\nu}\Box-\nabla_\mu\partial_\nu\right)(f(\eta)-\xi)+\kappa^2T_{\mathrm{matter}\,\mu\nu}=0, \quad (3)$$

where ∇_μ is the covariant derivative and $T_{\mathrm{matter}\,\mu\nu}$ the energy-momentum tensor of matter.

Variation of action (2) with respect to η yields $\Box\xi+f'(\eta)R=0$, where the prime denotes derivative with respect to η. If the scalar fields η and ξ depend on time only, then in the spatially flat Friedmann–Lemaître–Robertson–Walker metric with the interval

$$ds^2=-dt^2+a^2(t)(dx_1^2+dx_2^2+dx_3^2), \quad (4)$$

Eq. (3) are equivalent to the following ones:

$$-3H^2(1+\Psi)+\frac{1}{2}\dot{\xi}\dot{\eta}-3H\dot{\Psi}+\Lambda+\kappa^2\rho_m=0, \quad (5)$$

$$(2\dot{H}+3H^2)(1+\Psi)+\frac{1}{2}\dot{\xi}\dot{\eta}+\ddot{\Psi}+2H\dot{\Psi}-\Lambda+\kappa^2P_m=0, \quad (6)$$

where $\Psi(t)=f(\eta(t))-\xi(t)$, $H=\dot{a}/a$ is the Hubble parameter, differentiation with respect to time t is denoted by a dot. For a perfect matter fluid, we have $T_{\mathrm{matter}\,00}=\rho_m(t)$ and $T_{\mathrm{matter}\,ij}=P_m(t)g_{ij}$. The equation of state (EoS) is

$$\dot{\rho}_m=-3H(P_m+\rho_m). \quad (7)$$

The equations of motion for the scalar fields η and ξ are as follows:

$$\ddot{\eta}+3H\dot{\eta}=-6(\dot{H}+2H^2), \quad (8)$$

$$\ddot{\xi}+3H\dot{\xi}=6(\dot{H}+2H^2)f'(\eta). \quad (9)$$

Note that the considered system of equations does not include the function η, but only $f(\eta)$, $f'(\eta)$ and time derivatives of η. Also, one can add a constant to $f(\eta)$ and the same constant to ξ, without changing equations. So, $f(\eta)$ can be determined up to a constant.

Our goal is to demonstrate how one can reconstruct $f(\eta)$ and get a model with the exact solution for the given Hubble parameter $H(t)$ and the state parameter $w_m(t)=P_m(t)/\rho_m(t)$. We show that to do this it is enough to solve only linear equations.

The algorithm is as follows:
- Assume the explicit form of $H(t)$ and $w_m(t)$.
- Solve (7) and get $\rho_m(t)$.
- Solve (8) and get $\eta(t)$.
- Subtracting equation (5) from equation (6), get a linear differential equation

$$\ddot{\Psi} + 5H\dot{\Psi} + (2\dot{H} + 6H^2)(1 + \Psi) - 2\Lambda + \kappa^2(w_m - 1)\rho_m = 0, \qquad (10)$$

- Using the known $H(t)$, $w_m(t)$, and $\rho_m(t)$, solve (10) and get $\Psi(t)$.
- Substituting $\xi(t) = f(\eta(t)) - \Psi(t)$ into Eq. (9), we get a linear differential equation for $f(\eta)$

$$f''(\eta)\dot{\eta}^2 - 12(\dot{H} + 2H^2)f'(\eta) = \ddot{\Psi} + 3H\dot{\Psi}. \qquad (11)$$

To get (11) we also use the inverse function $t(\eta)$. Note that Eq. (11) is a necessary condition that the model has the solutions in the given form.
- Solve (11) and get the sought-for function $f(\eta)$.
- Substitute the obtained function $f(\eta)$ to Eq. (5) and Eq. (6) to check the existence of the solutions in the given form.

2. NONLOCAL MODELS WITH DE SITTER SOLUTIONS

To demonstrate how the algorithm works we seek such $f(\eta)$ that the model has a de Sitter solution, in other words, the Hubble parameter is a nonzero constant: $H = H_0$. In this case, Eq. (8) has the following general solution:

$$\eta(t) = -4H_0(t - t_0) - \eta_0\, \mathrm{e}^{-3H_0(t-t_0)}, \qquad (12)$$

with integration constants t_0 and η_0. All equations are homogeneous. If a solution exists at $t_0 = 0$, then it exists at an arbitrary t_0. So, without loss of generality we can set $t_0 = 0$.

Note that Eq. (11) has been obtained without any restrictions on solutions and the perfect matter fluid. To demonstrate how one can get $f(\eta)$, which admits the existence of de Sitter solutions, in the explicit form, we restrict ourselves to the case $\eta_0 = 0$. In this case, Eq. (11) has the following form:

$$16H_0^2 f''(\eta) - 24H_0^2 f'(\eta) = \Phi(\eta), \qquad (13)$$

where $\Phi(\eta) = \Phi(-4H_0 t) \equiv \ddot{\Psi} + 3H_0\dot{\Psi}$. We get the following solution

$$f(\eta) = \frac{1}{16H_0^2}\int^{\eta}\left\{\int^{\zeta}\Phi(\tilde{\zeta})\,\mathrm{e}^{-3\tilde{\zeta}/2}\,d\tilde{\zeta} + 16C_3H_0^2\right\}\mathrm{e}^{3\zeta/2}d\zeta + C_4, \qquad (14)$$

where C_3 and C_4 are arbitrary constants. We can fix C_4 without loss of generality.

Following [1], we consider the matter with the state parameter $w_m \equiv P_m/\rho_m$ to be a constant, not equal to -1. Thus, equation (7) has the following general solution

$$\rho_m = \rho_0\, \mathrm{e}^{-3(1+w_m)H_0 t}, \tag{15}$$

where ρ_0 is an arbitrary constant. Equation (10) has the following general solution:

• At $w_m \neq 0$ and $w_m \neq -1/3$,

$$\Psi_1(t) = C_1\, \mathrm{e}^{-3H_0 t} + C_2\, \mathrm{e}^{-2H_0 t} - 1 + \frac{\Lambda}{3H_0^2} - \frac{\kappa^2\rho_0(w_m-1)}{3H_0^2 w_m(1+3w_m)}\, \mathrm{e}^{-3H_0(w_m+1)t},$$

• At $w_m = -1/3$,

$$\Psi_2(t) = C_1\, \mathrm{e}^{-3H_0 t} + C_2\, \mathrm{e}^{-2H_0 t} - 1 + \frac{\Lambda}{3H_0^2} + \frac{4\kappa^2\rho_0}{3H_0}\, \mathrm{e}^{-2H_0 t} t,$$

• At $w_m = 0$,

$$\Psi_3(t) = C_1\, \mathrm{e}^{-3H_0 t} + C_2\, \mathrm{e}^{-2H_0 t} - 1 + \frac{\Lambda}{3H_0^2} - \frac{\kappa^2\rho_0}{H_0}\, \mathrm{e}^{-3H_0 t} t,$$

where C_1 and C_2 are arbitrary constants.

Substituting the explicit form of $\Psi(t)$, we get

$$f_1(\eta) = \frac{C_2}{4}\, \mathrm{e}^{\eta/2} + C_3\, \mathrm{e}^{3\eta/2} + C_4 - \frac{\kappa^2\rho_0}{3(1+3w_m)H_0^2}\, \mathrm{e}^{3(w_m+1)\eta/4} \quad \text{at} \quad w_m \neq -\frac{1}{3}, \tag{16}$$

$$\tilde{f}_1(\eta) = \frac{C_2}{4}\, \mathrm{e}^{\eta/2} + C_3\, \mathrm{e}^{3\eta/2} + C_4 + \frac{\kappa^2\rho_0}{4H_0^2}\left(1 - \frac{1}{3}\eta\right) \mathrm{e}^{\eta/2}, \quad \text{at} \quad w_m = -\frac{1}{3}, \tag{17}$$

where C_3 and C_4 are arbitrary constants. Note that C_2 is an arbitrary constant as well.

One can see that the key ingredient of all functions $f_i(\eta)$ is an exponent function. For the models with $f(\eta)$ equal to an exponential function or a sum of exponential functions, particular de Sitter solutions have been found in [1, 2]. de Sitter solutions in the case of the exponential function f have been generalized and their stability has been analysed in [3].

CONCLUSION

Exact solutions play an important role in modern cosmological models, in particular, in nonlocal cosmological models [1–6]. The main result of this paper is the algorithm, using which one can reconstruct $f(\Box^{-1}R)$, corresponding to the given Hubble parameter and the state parameter of the matter. We have found that the function f corresponding to de Sitter solution is an exponential function or a sum of exponential functions*. In the case of the exponential function f, expanding universe solutions $a \sim t^n$ have been found in [2, 6]. We plan to analyse possible forms of the corresponding function f in future investigations.

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*If the model includes the perfect fluid with $w_m = -1/3$, the form of f is more complicated (formula (17)).