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THE PROPERTY OF MAXIMAL TRANSCENDENTALITY: CALCULATION OF ANOMALOUS DIMENSIONS IN THE $\mathcal{N}=4$ SYM AND MASTER INTEGRALS

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We review results for the universal anomalous dimension $\gamma_{\rm uni}(j)$ of Wilson twist-2 operators in the $\mathcal{N}=4$ Supersymmetric Yang–Mills theory, having the property of maximal transcendentality. It is shown that a similar property is observed in the results for master integrals.

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INTRODUCTION

This paper deals with the study of the properties of the Balitsky–Fadin–Kuraev–Lipatov (BFKL) [1] and Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) [2] equations in the $\mathcal{N}=4$ Supersymmetric Yang–Mills (SYM) model [3].

The BFKL and DGLAP equations resum, respectively, the most important contributions $\sim \alpha_s \ln(1/x)$ and $\sim \alpha_s \ln(Q^2/\Lambda^2)$ in different kinematical regions of the Bjorken variable x and the «mass» Q^2 of the virtual photon in the deep inelastic lepton–hadron scattering (DIS) (see Fig. 1 for the muon–nucleon case) and, thus, they are the cornerstone in analyses of the experimental data from lepton–nucleon and nucleon–nucleon scattering processes.

In the supersymmetric cases the equations are simplified drastically. In the $\mathcal{N}=4$ SYM they become to be related to each other for the nonphysical values of Mellin moments j, as has been proposed by Lipatov in [4].

The purpose of this paper is to show similar properties of the results for the anomalous dimension matrix of the twist-2 Wilson operators and the results for the so-called master integrals.

The anomalous dimensions govern the Bjorken scaling violation for parton distributions in the framework of QCD. These quantities are given by the Mellin

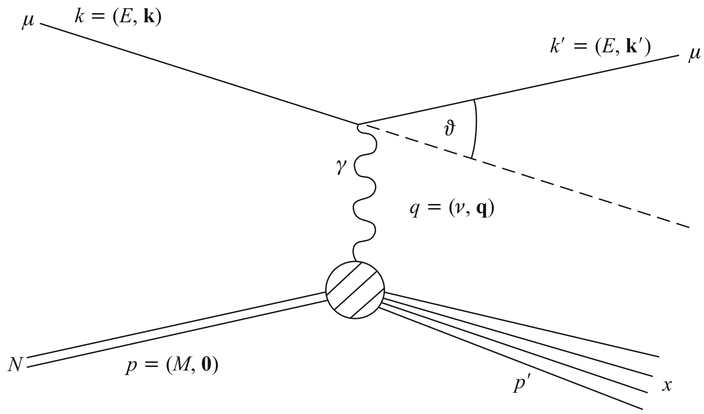

Fig. 1. The deep inelastic muon–nucleon scattering, where k, q and p are the muon, photon and nucleon momenta, respectively. In the DIS kinematics, $p^2 = M^2 \to 0$, where M is the nucleon mass. The standard variables are $Q^2 = -q^2 > 0$ and the Bjorken variable $x = Q^2/(2pq)$, where Q^2 is the «mass» of the virtual photon and x is the part of the nucleon momentum carried by the colliding parton (quark or gluon)

transformation (the symbol $\sim$ is used for spin-dependent case and $a_s = \alpha_s/(4\pi)$)

$$\begin{aligned} \gamma_{ab}(j) &= \int\limits_0^1 dx\, x^{j-1} W_{b\to a}(x) = \sum_{k=0}^{\infty} \gamma_{ab}^{(k)}(j) a_s^{k+1}, \\ \tilde{\gamma}_{ab}(j) &= \int\limits_0^1 dx\, x^{j-1} \tilde{W}_{b\to a}(x) = \sum_{k=0}^{\infty} \tilde{\gamma}_{ab}^{(k)}(j) a_s^{k+1} \end{aligned} \tag{1}$$

of the splitting kernels $W_{b\to a}(x)$ and $\tilde{W}_{b\to a}(x)$ for the DGLAP equation [2] which evolves the parton densities $f_a(x, Q^2)$ and $\tilde{f}_a(x, Q^2)$ (hereafter $a = \lambda$, g, ϕ for the spinor, vector and scalar particles, respectively*) as follows:

$$\begin{aligned} \frac{d}{d\ln Q^2} f_a(x, Q^2) &= \int\limits_x^1 \frac{dy}{y} \sum_b W_{b\to a}(x/y)\, f_b(y, Q^2), \\ \frac{d}{d\ln Q^2} \tilde{f}_a(x, Q^2) &= \int\limits_x^1 \frac{dy}{y} \sum_b \tilde{W}_{b\to a}(x/y)\, \tilde{f}_b(y, Q^2). \end{aligned} \tag{2}$$

*In the spin-dependent case $a = \lambda$, g.

The anomalous dimensions and splitting kernels in QCD are known up to the next-to-next-to-leading order (NNLO)* of the perturbation theory (see [5] and references therein).

The QCD expressions for anomalous dimensions can be transformed to the case of the $\mathcal{N}$-extended Supersymmetric Yang–Mills theories (SYM) if one will use for the Casimir operators C_A, C_F, T_f the following values: $C_A = C_F = N_c$, $T_f n_f = \mathcal{N} N_c/2$. For $\mathcal{N} = 2$ and $\mathcal{N} = 4$-extended SYM the anomalous dimensions of the Wilson operators also get additional contributions coming from scalar particles [4]. These anomalous dimensions were calculated in the next-to-leading order [4, 6] for the $\mathcal{N} = 4$ SYM.

However, it turns out that the expressions for eigenvalues of the anomalous dimension matrix in the $\mathcal{N} = 4$ SYM [3] can be derived directly from the QCD anomalous dimensions without tedious calculations by using a number of plausible arguments. The method elaborated in [4] for this purpose is based on special properties of the integral kernel for the BFKL equation [1, 7, 8] in this model and a new relation between the BFKL and DGLAP equations (see [4]). In the NLO approximation this method gives the correct results for anomalous dimensions eigenvalues, which was checked by *direct calculations* in [6]. Using the results for the NNLO corrections to anomalous dimensions in QCD [5] and the method of [4], we derive the eigenvalues of the anomalous dimension matrix for the $\mathcal{N} = 4$ SYM in the NNLO approximation [9].

Starting from four loops, i.e., above existing QCD calculations, the corresponding results for the anomalous dimensions can be obtained (see [10–12]) from the long-range asymptotic Bethe equations together with some additional terms, the so-called *wrapping corrections*, coming in agreement with Luscher approach**.

The obtained result is very important for the verification of the various assumptions (see recent reviews [14–16] and references therein) coming from the investigations of the properties of conformal operators in the context of AdS/CFT correspondence [17].

The paper is organized as follows. In Sec. 1, we discuss the BFKL equation, the leading-order (LO) anomalous dimensions of Wilson operators and propose the method of obtaining the eigenvalues of the anomalous dimension matrix above the leading order. Section 2 contains the calculations of some Feynman diagrams by a similar method. In Sec. 3, we consider three-loop results for the universal anomalous dimension taken from the corresponding calculations in

*It is in spin-average case. The corresponding functions in the spin-dependent case are calculated now only in the next-to-leading order (NLO).

**The three- and four-loop results for the universal anomalous dimension have been reproduced (see [13]) also by solution of the so-called Baxter equation, which can be obtained from the long-range asymptotic Bethe equations.

QCD. Four-loop corrections to the universal anomalous dimension are considered in Sec. 4.

1. EVOLUTION EQUATION IN $\mathcal{N} = 4$ SYM

The reason to investigate the BFKL and DGLAP equations in the case of supersymmetric theories is related to a common belief that the high symmetry may significantly simplify their structure. Indeed, it was found in the leading-order approximation [18] that the so-called quasi-partonic operators in $\mathcal{N} = 1$ SYM are unified in supermultiplets with anomalous dimensions obtained from some universal anomalous dimension by shifting its argument by an integer number. Further, the anomalous dimension matrices for twist-2 operators are fixed by the superconformal invariance [18]. Calculations in the maximally extended $\mathcal{N} = 4$ SYM, where the coupling constant is not renormalized, give even more remarkable results. Namely, it turns out that here all twist-2 operators enter in the same multiplet, their anomalous dimension matrix is fixed completely by the superconformal invariance, and its universal anomalous dimension in LO is proportional to $\Psi(j-1) - \Psi(1)$ (see Subsec. 1.2), which means that the evolution equations for the matrix elements of quasi-partonic operators in the multicolor limit $N_c \to \infty$ are equivalent to the Schrödinger equation for an integrable Heisenberg spin model [19, 20]. In QCD, the integrability remains only in a small sector of these operators [21] (see also [22]). In the case of $\mathcal{N} = 4$ SYM the equations for other sets of operators are also integrable [23, 24].

Similar results related to the integrability of the multicolor QCD were obtained earlier in the Regge limit [25]. Moreover, it was shown [4] that in the $\mathcal{N} = 4$ SYM there is a deep relation between the BFKL and DGLAP evolution equations. Namely, the j-plane singularities of anomalous dimensions of the Wilson twist-2 operators in this case can be obtained from the eigenvalues of the BFKL kernel by their analytic continuation. The NLO calculations in $\mathcal{N} = 4$ SYM demonstrated [4] that some of these relations are also valid in higher orders of perturbation theory. In particular, the BFKL equation has the property of the Hermitian separability, the linear combinations of the multiplicatively renormalized operators do not depend on the coupling constant, the eigenvalues of the anomalous dimension matrix are expressed in terms of the universal function $\gamma_{\rm uni}(j)$ which can also be obtained from the BFKL equation [4].

1.1. BFKL. To begin with, we review shortly the results of [7, 8], where the QCD radiative corrections to the BFKL integral kernel at $t = 0$ were calculated*. We discuss only the formulae important for our analysis.

The total cross section $\sigma(s)$ for the high-energy scattering of colorless particles A, B written in terms of their impact factors $\Phi_i(q_i)$ and the t-channel partial

*The $t \neq 0$ case can be found in the recent papers [26].

wave $G_\omega(q, q')$ for the gluon–gluon scattering is

$$\sigma(s) = \int \frac{d^2q\, d^2q'}{(2\pi)^2\, q^2\, q'^2} \Phi_A(q)\, \Phi_B(q') \int\limits_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{s_0}\right)^{\omega} G_\omega(q, q'), \quad s_0 = |q||q'|. \tag{3}$$

Here, q and q' are transverse momenta* of virtual gluons and $s = 2p_Ap_B$ is the squared invariant mass for the colliding particle momenta p_A and p_B.

Using the dimensional regularization in the $\overline{\mathrm{MS}}$ scheme to remove ultraviolet and infrared divergences in intermediate expressions, the BFKL equation for $G_\omega(q, q')$ can be written in the following form:

$$\omega G_\omega(q, q_1) = \delta^{D-2}(q - q_1) + \int d^{D-2}q_2\, K(q, q_2)\, G_\omega(q_2, q_1), \tag{4}$$

where

$$K(q_1, q_2) = 2\omega(q_1)\, \delta^{D-2}(q_1 - q_2) + K_r(q_1, q_2) \tag{5}$$

and the space-time dimension $D = 4 - 2\varepsilon$ for $\varepsilon \to 0$. The gluon Regge trajectory $\omega(q)$ and the integral kernel $K_r(q_1, q_2)$ related to the real particle production have been calculated in [27–29].

As was shown in [7, 8], a complete and orthogonal set of eigenfunctions of the homogeneous BFKL equation in LO is

$$G_{n,\gamma}(q/q', \theta) = \left(\frac{q^2}{q'^2}\right)^{\gamma-1} \mathrm{e}^{in\theta}. \tag{6}$$

The BFKL kernel in this representation is diagonalized up to the effects related with the running coupling constant $a_s(q^2)$:

$$\omega_{\overline{\mathrm{MS}}}^{\mathrm{QCD}} = 4a_s(q^2)\Big[\chi(n, \gamma) + \delta_{\overline{\mathrm{MS}}}^{\mathrm{QCD}}(n, \gamma)a_s(q^2)\Big]. \tag{7}$$

Applying the formulae of [8], we obtain the following results for eigenvalues (7):

$$\chi(n, \gamma) = 2\Psi(1) - \Psi\Big(\gamma + \frac{n}{2}\Big) - \Psi\Big(1 - \gamma + \frac{n}{2}\Big), \tag{8}$$

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*To simplify equations, hereafter we omit arrows in the notation of transverse momenta $\overrightarrow{q}, \overrightarrow{q'}, \overrightarrow{q_1}, \overrightarrow{q_2}, \ldots$, i.e., in our formulae the momenta $\overrightarrow{q}, \overrightarrow{q'}, \overrightarrow{q_1}, \overrightarrow{q_2}, \ldots$ will be represented as $q, q', q_1, q_2, \ldots$, respectively. Note, however, that the momenta p_A and p_B are D-space momenta.

$$\delta^{\rm QCD}_{\overline{\rm MS}}(n,\gamma) = \left(\frac{67}{9} - 2\zeta(2) - \frac{10}{9}\,\frac{n_f}{N_c}\right)\chi(n,\gamma) + 6\zeta(3) + \Psi''\Big(\gamma + \frac{n}{2}\Big) + \\ + \Psi''\Big(1-\gamma+\frac{n}{2}\Big) - 2\Phi(n,\gamma) - 2\Phi(n,1-\gamma) - \left(\frac{11}{3} - \frac{2}{3}\,\frac{n_f}{N_c}\right)\frac{1}{2}\chi^2(n,\gamma) + \\ + \frac{\pi^2\cos(\pi\gamma)}{\sin^2(\pi\gamma)(1-2\gamma)}\left\{\left(1+\frac{\tilde{n}_f}{N_c^3}\right)\frac{\gamma(1-\gamma)}{2(3-2\gamma)(1+2\gamma)}\delta_n^2 - \right. \\ \left. - \left(3 + \left(1+\frac{\tilde{n}_f}{N_c^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right)\delta_n^0\right\}, \quad (9)$$

where δ_n^m is the Kronecker symbol, and $\Psi(z)$, $\Psi'(z)$ and $\Psi''(z)$ are the Euler Ψ function and its derivatives. The function $\Phi(n,\gamma)$ is given below:

$$\Phi(n,\gamma) = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k+\gamma+n/2}\Big[\Psi'(k+n+1) - \Psi'(k+1) + \\ + (-1)^k\big(\beta'(k+n+1)+\beta'(k+1)\big) - \frac{1}{k+\gamma+n/2}\big(\Psi(k+n+1)-\Psi(k+1)\big)\Big] \quad (10)$$

and

$$\beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right].$$

Adding contributions of scalars and transforming fermions from fundamental to adjoint representation, we can obtain the BFKL form (7) in $\mathcal{N}=4$ SYM in $\overline{\rm DR}$ scheme [30]:

$$\delta^{N=4}_{\overline{\rm DR}}(n,\gamma) = 6\zeta(3) + \Psi''\Big(\gamma+\frac{n}{2}\Big) + \Psi''\Big(1-\gamma+\frac{n}{2}\Big) - \\ - 2\Phi(n,\gamma) - 2\Phi(n,1-\gamma) - 2\zeta(2)\chi(n,\gamma), \quad (11)$$

where the $\overline{\rm DR}$ coupling constant $\hat{a}_s$ is related [31] with the $\overline{\rm MS}$ one a_s as

$$\hat{a}_s = a_s + \frac{1}{3}a_s^2. \quad (12)$$

Note that the sum $\Phi(n,\gamma)+\Phi(n,1-\gamma)$ can be rewritten (see [4]) as a combination of functions with argument dependent on $\gamma+n/2 \equiv M$ and $1-\gamma+n/2 \equiv \tilde{M}$. Indeed,

$$\Phi(n,\gamma)+\Phi(n,1-\gamma) = \chi(n,\gamma)\big(\beta'(M)+\beta'(1-\widetilde{M})\big) + \Phi_2(M) - \beta'(M)\times \\ \times[\Psi(1)-\Psi(M)] + \Phi_2(1-\widetilde{M}) - \beta'(1-\widetilde{M})[\Psi(1)-\Psi(1-\widetilde{M})],$$

where $\chi(n,\gamma)$ is given by Eq. (8) and

$$\Phi_2(M) = \sum_{k=0}^{\infty} \frac{\left(\beta'(k+1) + (-1)^k \Psi'(k+1)\right)}{k+M} - \\ - \sum_{k=0}^{\infty} \frac{(-1)^k \left(\Psi(k+1) - \Psi(1)\right)}{(k+M)^2}. \quad (13)$$

So, this transformation leads to the Hermitian separability of BFKL equation in $\mathcal{N}=4$ SYM (see [4] and discussions therein).

1.2. Leading-Order Anomalous Dimension Matrix in $\mathcal{N}=4$ SYM. In the $\mathcal{N}=4$ SYM theory [3], one can introduce the following color and $SU(4)$ singlet local Wilson twist-2 operators [4, 6]:

$$\mathcal{O}^g_{\mu_1,\ldots,\mu_j} = \hat{S} G^a_{\rho\mu_1} \mathcal{D}_{\mu_2} \mathcal{D}_{\mu_3} \cdots \mathcal{D}_{\mu_{j-1}} G^a_{\rho\mu_j}, \quad (14)$$

$$\tilde{\mathcal{O}}^g_{\mu_1,\ldots,\mu_j} = \hat{S} G^a_{\rho\mu_1} \mathcal{D}_{\mu_2} \mathcal{D}_{\mu_3} \cdots \mathcal{D}_{\mu_{j-1}} \tilde{G}^a_{\rho\mu_j}, \quad (15)$$

$$\mathcal{O}^\lambda_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\lambda}^a_i \gamma_{\mu_1} \mathcal{D}_{\mu_2} \cdots \mathcal{D}_{\mu_j} \lambda^{a\,i}, \quad (16)$$

$$\tilde{\mathcal{O}}^\lambda_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\lambda}^a_i \gamma_5 \gamma_{\mu_1} \mathcal{D}_{\mu_2} \cdots \mathcal{D}_{\mu_j} \lambda^{a\,i}, \quad (17)$$

$$\mathcal{O}^\phi_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\phi}^a_r \mathcal{D}_{\mu_1} \mathcal{D}_{\mu_2} \cdots \mathcal{D}_{\mu_j} \phi^a_r, \quad (18)$$

where $\mathcal{D}_\mu$ are covariant derivatives. The spinors λ_i and field tensor $G_{\rho\mu}$ describe gluinos and gluons, respectively, and ϕ_r are the complex scalar fields. For all operators in Eqs. (14)–(18) the symmetrization of the tensors in the Lorentz indices $\mu_1,\ldots,\mu_j$ and a subtraction of their traces is assumed.

The elements of the LO anomalous dimension matrix in the $\mathcal{N}=4$ SYM have the following form (see [20]):

for tensor twist-2 operators:

$$\begin{aligned}
&\gamma^{(0)}_{gg}(j) = 4\left(\Psi(1) - \Psi(j-1) - \frac{2}{j} + \frac{1}{j+1} - \frac{1}{j+2}\right), \\
&\gamma^{(0)}_{\lambda g}(j) = 8\left(\frac{1}{j} - \frac{2}{j+1} + \frac{2}{j+2}\right), \qquad \gamma^{(0)}_{\varphi g}(j) = 12\left(\frac{1}{j+1} - \frac{1}{j+2}\right), \\
&\gamma^{(0)}_{g\lambda}(j) = 2\left(\frac{2}{j-1} - \frac{2}{j} + \frac{1}{j+1}\right), \qquad \gamma^{(0)}_{q\varphi}(j) = \frac{8}{j}, \\
&\gamma^{(0)}_{\lambda\lambda}(j) = 4\left(\Psi(1) - \Psi(j) + \frac{1}{j} - \frac{2}{j+1}\right), \quad \gamma^{(0)}_{\varphi\lambda}(j) = \frac{6}{j+1}, \\
&\gamma^{(0)}_{\varphi\varphi}(j) = 4\left(\Psi(1) - \Psi(j+1)\right), \qquad \gamma^{(0)}_{g\varphi}(j) = 4\left(\frac{1}{j-1} - \frac{1}{j}\right);
\end{aligned} \quad (19)$$

for the pseudo-tensor operators:

$$
\begin{aligned}
\widetilde{\gamma}_{gg}^{(0)}(j) &= 4\left(\Psi(1) - \Psi(j+1) - \frac{2}{j+1} + \frac{2}{j}\right),\\
\widetilde{\gamma}_{\lambda g}^{a,(0)}(j) &= 8\left(-\frac{1}{j} + \frac{2}{j+1}\right), \qquad \widetilde{\gamma}_{g\lambda}^{(0)}(j) = 2\left(\frac{2}{j} - \frac{1}{j+1}\right),\\
\widetilde{\gamma}_{\lambda\lambda}^{(0)}(j) &= 4\left(\Psi(1) - \Psi(j+1) + \frac{1}{j+1} - \frac{1}{j}\right).
\end{aligned}
\tag{20}
$$

The matrices, based on the anomalous dimensions (19) and (20), can be diagonalized [4, 20]. They have the following remarkable form:

$$
\left[D\Gamma D^{-1}\right]_{\rm unpol}^{N=4} = \begin{vmatrix} -4S_1(j-2) & 0 & 0 \\ 0 & -4S_1(j) & 0 \\ 0 & 0 & -4S_1(j+2) \end{vmatrix},
$$

$$
\left[D\Gamma D^{-1}\right]_{\rm pol}^{N=4} = \begin{vmatrix} -4S_1(j-1) & 0 \\ 0 & -4S_1(j+1) \end{vmatrix},
$$

where $S_1(j)$ is defined below in (25).

Thus, the LO anomalous dimensions of all multiplicatively renormalized operators can be extracted through one universal function

$$
\gamma_{\rm uni}^{(0)}(j) = -4S(j-2) \equiv -4\Big(\Psi(j-1) - \Psi(1)\Big) \equiv -4\sum_{r=1}^{j-2}\frac{1}{r}.
$$

1.3. Method of Obtaining the Eigenvalues of the Anomalous Dimension Matrix in $\mathcal{N} = 4$ SYM. As was already pointed out in the Introduction, the universal anomalous dimension can be extracted directly from the QCD results without finding the scalar particle contribution. This possibility is based on the deep relation between the DGLAP and BFKL dynamics in the $\mathcal{N} = 4$ SYM [4,8].

To begin with, the eigenvalues of the BFKL kernel are the analytic functions of the conformal spin $|n|$ at least in two first orders of perturbation theory (see Eqs. (7), (8) and (11)). Further, in the framework of the $\overline{\rm DR}$ scheme [30], one can obtain from (8) and (9) that there is no mixing among the special functions of different transcendentality levels i^*; i.e., all special functions at the NLO correction contain only sums of the terms $\sim 1/\gamma^i$ $(i = 3)$. More precisely, if we introduce the transcendentality level i for the eigenvalues $\omega(\gamma)$ of integral kernels

*Similar arguments were also used in [32] to obtain analytic results for contributions of some complicated massive Feynman diagrams without direct calculations (see also Sec. 2).

of the BFKL equations in accordance with the complexity of the terms in the corresponding sums

$$\Psi \sim 1/\gamma, \quad \Psi' \sim \beta' \sim \zeta(2) \sim 1/\gamma^2, \quad \Psi'' \sim \beta'' \sim \Phi \sim \zeta(3) \sim 1/\gamma^3,$$

then for the BFKL kernel in LO and in NLO the corresponding levels are $i = 1$ and $i = 3$, respectively.

Because in $\mathcal{N} = 4$ SYM there is a relation between the BFKL and DGLAP equations (see [4,8]), similar properties should be valid for the anomalous dimensions themselves; i.e., the basic functions $\gamma^{(0)}_{\rm uni}(j)$, $\gamma^{(1)}_{\rm uni}(j)$ and $\gamma^{(2)}_{\rm uni}(j)$ are assumed to be of the types $\sim 1/j^i$ with the levels $i = 1$, $i = 3$ and $i = 5$, respectively. An exception could be for the terms appearing at a given order from previous orders of the perturbation theory. Such contributions could be generated and/or removed by an approximate finite renormalization of the coupling constant. But these terms do not appear in the $\overline{\rm DR}$ scheme.

It is known that at the LO and NLO approximations (with the SUSY relation for the QCD color factors $C_F = C_A = N_c$) the most complicated contributions (with $i = 1$ and $i = 3$, respectively) are the same for all LO and NLO anomalous dimensions in QCD [5] and for the LO and NLO scalar–scalar anomalous dimensions [6]. This property allows one to find the universal anomalous dimensions $\gamma^{(0)}_{\rm uni}(j)$ and $\gamma^{(1)}_{\rm uni}(j)$ without knowing all elements of the anomalous dimensions matrix [4], which was verified by the exact calculations in [6].

Using the above arguments, we conclude that at the NNLO level there is only one possible candidate for $\gamma^{(2)}_{\rm uni}(j)$. Namely, it is the most complicated part of the QCD anomalous dimensions matrix (with the SUSY relation for the QCD color factors $C_F = C_A = N_c$). Indeed, after the diagonalization of the anomalous dimensions matrix, its eigenvalues should have this most complicated part as a common contribution, because they differ from each other only by a shift of the argument and their differences are constructed from less complicated terms. The nondiagonal matrix elements of the anomalous dimensions matrix also contain only less complicated terms (see, for example, anomalous dimensions exact expressions at LO and NLO approximations in [5] for QCD and [6] for $\mathcal{N} = 4$ SYM) and, therefore, they cannot generate the most complicated contributions to the eigenvalues of anomalous dimensions matrix.

Thus, the most complicated part of the NNLO QCD anomalous dimensions should coincide (up to color factors) with the universal anomalous dimension $\gamma^{(2)}_{\rm uni}(j)$.

2. CALCULATION OF FEYNMAN INTEGRALS

Similar arguments give a possibility to calculate a large class of Feynman diagrams, the so-called master integrals [33]. Let us consider it in some details.

Application of the integration-by-part (IBP) procedure [34] to loop internal momenta leads to relations between different Feynman integrals (FI) and, thus, to necessity to calculate only some of them, which in a sense, are independent (see [35]). These independent diagrams (which were chosen quite arbitrarily, of course) are called the master integrals [33].

The application of the IBP procedure [34] to the master integrals themselves leads to the differential equations [36, 37] for them with the inhomogeneous terms (ITs) containing less complicated diagrams*. The application of the IBP procedure to these diagrams leads to the new differential equations for them with the new ITs containing even farther less complicated diagrams. Repeating the procedure several times, at the last step one can obtain the ITs containing only tadpoles which can be calculated in turn very easily.

Solving the differential equations at this last step, one can reproduce the diagrams for ITs of the differential equations at the previous step. Repeating the procedure several times, one can obtain the results for the initial Feynman diagram.

This scheme has been used successfully for calculation of two-loop two-point [35, 39] and three-point diagrams [32, 39, 40] with one nonzero mass. This procedure is very powerful but quite complicated. There are, however, some simplifications, which are based on the series representations of Feynman integrals.

Indeed, the inverse-mass expansion of two-loop two-point (see Fig. 2) and three-point diagrams (see Fig. 3)** with one nonzero mass can be considered as

$$\begin{aligned}\mathrm{FI} = \frac{\hat{N}}{q^{2\alpha}} \sum_{n=1} C_n \frac{(\eta x)^n}{n^c} \Biggl\{ F_0(n) + \left[\ln(-x) F_{1,1}(n) + \frac{1}{\varepsilon} F_{1,2}(n) \right] + \\ + \left[\ln^2(-x)\, F_{2,1}(n) + \frac{1}{\varepsilon} \ln(-x) F_{2,2}(n) + \frac{1}{\varepsilon^2} F_{2,3}(n) + \zeta(2) F_{2,4}(n) \right] + \\ + \Biggl[\ln^3(-x)\, F_{3,1}(n) + \frac{1}{\varepsilon} \ln^2(-x) F_{3,2}(n) + \frac{1}{\varepsilon^2} \ln(-x) F_{3,3}(n) + \\ + \frac{1}{\varepsilon^3} F_{3,4}(n) + \zeta(2) \ln(-x) F_{3,5}(n) + \zeta(3) F_{3,6}(n) \Biggr] + \ldots \Biggr\}, \qquad (21)\end{aligned}$$

where $x = q^2/m^2$, $\eta = 1$ or -1, $c = 0$, 1 and 2, and $\alpha = 1$ and 2 for two-point and three-point cases, respectively.

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*The «less complicated diagrams» usually contain less number of propagators and sometimes they can be represented as diagrams with less number of loops and with some «effective masses» (see, for example, [38] and references therein).

**We consider only three-point diagrams with independent upward momenta q_1 and q_2, which obey the conditions $q_1^2 = q_2^2 = 0$ and $(q_1 + q_2)^2 \equiv q^2 \neq 0$, where q is downward momentum.

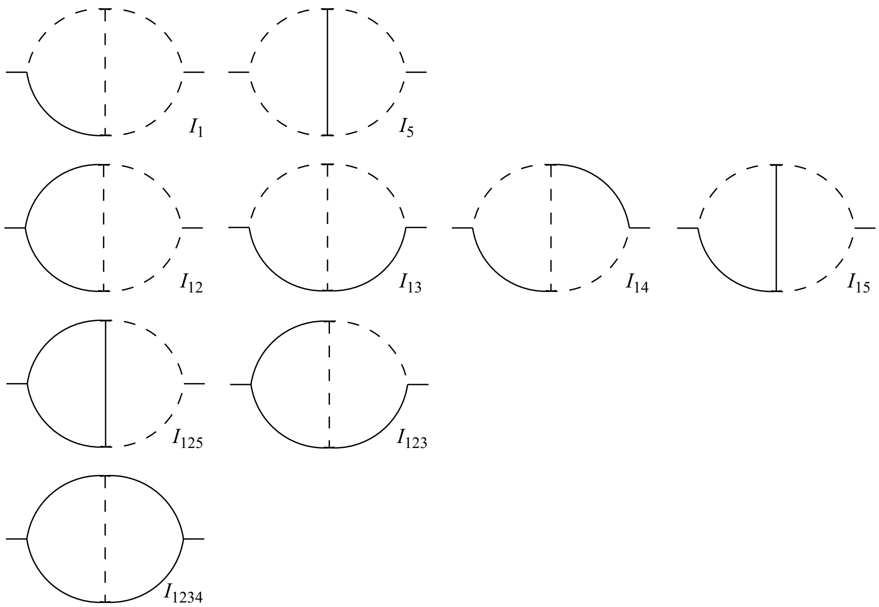

Fig. 2

Here the normalization $\hat{N} = (\overline{\mu}^2/m^2)^{2\varepsilon}$, where $\overline{\mu} = 4\pi\, \mathrm{e}^{-\gamma_E}\mu$ is in the standard $\overline{\mathrm{MS}}$ scheme and γ_E is the Euler constant. Moreover, the space-time dimension is $D = 4 - 2\varepsilon$ and

$$C_n = 1 \tag{22}$$

for diagrams with one-massive-particle cuts (m cuts) and

$$C_n = 1 \quad \text{and} \quad C_n = \frac{(n!)^2}{(2n)!} \equiv \hat{C}_n \tag{23}$$

for diagrams with two-massive-particle cuts ($2m$ cuts).

For m-cut case, the coefficients $F_{N,k}(n)$ should have the form

$$F_{N,k}(n) \sim \frac{S_{\pm a,\dots}}{n^b}. \tag{24}$$

In this section, $S_{\pm a} \equiv S_{\pm a}(j-1)$, $S_{\pm a,\pm b} \equiv S_{\pm a,\pm b}(j-1)$, $S_{\pm a,\pm b,\pm c} \equiv S_{\pm a,\pm b,\pm c}(j-1)$ are harmonic sums

$$S_{\pm a}(j) = \sum_{m=1}^{j} \frac{(\pm 1)^m}{m^a}, \quad S_{\pm a,\pm b,\pm c,\cdots}(j) = \sum_{m=1}^{j} \frac{(\pm 1)^m}{m^a}\, S_{\pm b,\pm c,\cdots}(m), \tag{25}$$

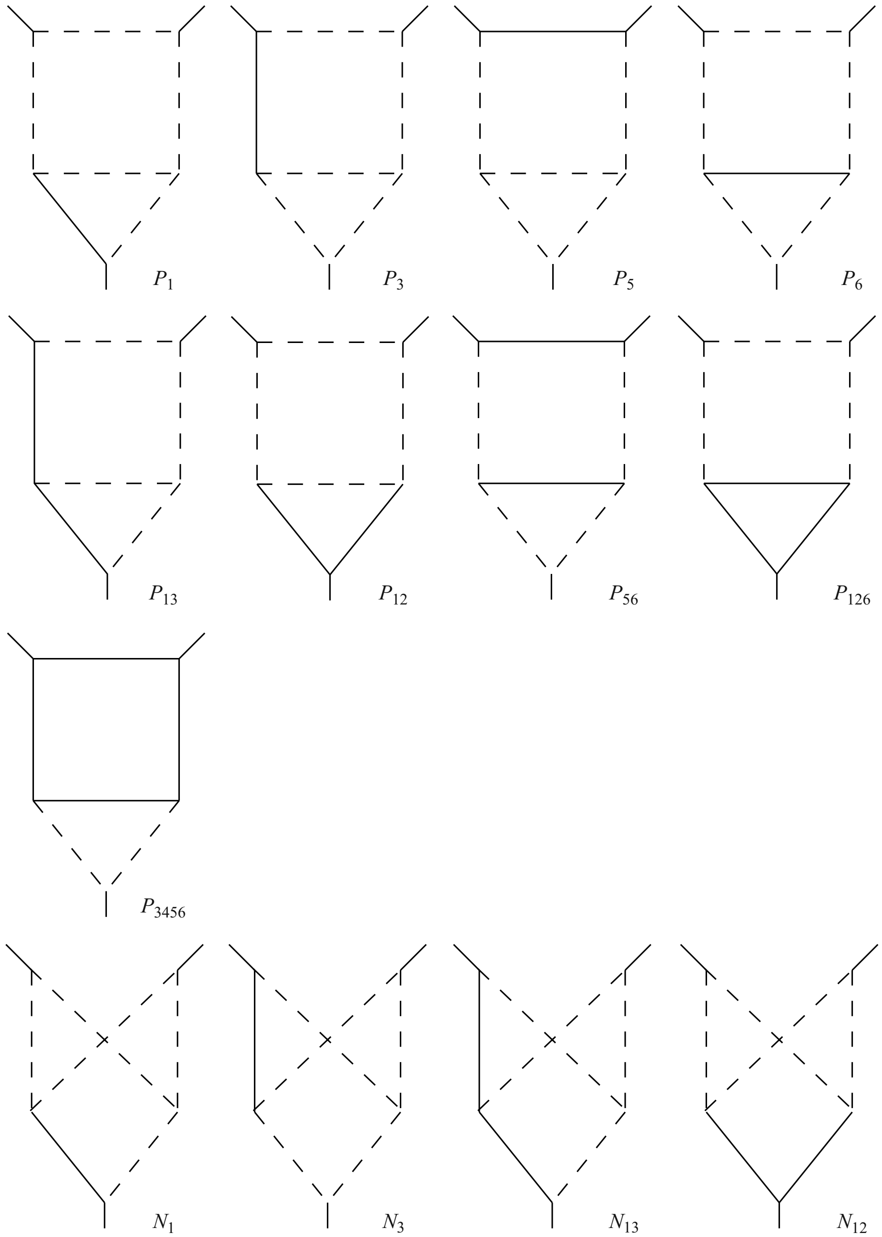

Fig. 3

For $2m$-cut case, the coefficients $F_{N,k}(n)$ should have the form*

$$F_{N,k}(n) \sim \frac{S_{\pm a,\dots}}{n^b}, \quad \frac{V_{a,\dots}}{n^b}, \quad \frac{W_{a,\dots}}{n^b}, \tag{26}$$

where

$$V_a(j) = \sum_{m=1}^{j} \frac{\hat{C}_m}{m^a}, \qquad V_{a,b,c,\cdots}(j) = \sum_{m=1}^{j} \frac{\hat{C}_m}{m^a}\, S_{b,c,\cdots}(m), \tag{27}$$

$$W_a(j) = \sum_{m=1}^{j} \frac{\hat{C}_m^{-1}}{m^a}, \quad W_{a,b,c,\cdots}(j) = \sum_{m=1}^{j} \frac{\hat{C}_m^{-1}}{m^a}\, S_{b,c,\cdots}(m). \tag{28}$$

The terms $\sim V_{a,\dots}$ and $\sim W_{a,\dots}$ can come only together with the coefficients $C_n = 1$ and $C_n = \hat{C}_n$, respectively. The terms $\sim S_{\pm a,\dots}$ can appear in combination with both C_n values. The origin of the appearance of the terms $\sim V_{a,\dots}$ and $\sim W_{a,\dots}$ in the $2m$-cut case is the product of series (21) with the different values of the coefficients $C_n = 1$ and $C_n = \hat{C}_n$

As examples, consider two-loop two-point diagrams I_1, I_5 and I_{12} shown in Fig. 2 and studied in [32]:

$$I_1 = \frac{\hat{N}}{q^2} \sum_{n=1} \frac{x^n}{n} \left\{ \frac{1}{2} \ln^2(-x) - \frac{2}{n} \ln(-x) + \zeta(2) + 2S_2 - 2\frac{S_1}{n} + \frac{3}{n^2} \right\}, \tag{29}$$

$$I_5 = \frac{\hat{N}}{q^2} \sum_{n=1} \frac{(-x)^n}{n} \left\{ -\ln^2(-x) + \frac{2}{n} \ln(-x) - 2\zeta(2) - 4S_{-2} - \frac{2}{n^2} - 2\frac{(-1)^n}{n^2} \right\}, \tag{30}$$

$$I_{12} = \frac{\hat{N}}{q^2} \sum_{n=1} \frac{x^n}{n^2} \left\{ \frac{1}{n} + \frac{(n!)^2}{(2n)!} \left(-2\ln(-x) - 3W_1 + \frac{2}{n} \right) \right\}. \tag{31}$$

From (29) and (30) one can see that the corresponding functions $F_{N,k}(n)$ have the form

$$F_{N,k}(n) \sim \frac{1}{n^{2-N}} \quad (N \geqslant 2), \tag{32}$$

if we introduce the following complexity of the sums $\left(\sum_{i=1}^{m} a_i = a\right)$:

$$\Phi_{\eta a} \sim \Phi_{\eta a_1, \eta a_2} \sim \Phi_{\eta a_1, \eta a_2, \cdots, \eta a_m} \sim \zeta_a \sim \frac{1}{n^a}, \tag{33}$$

where $\Phi = (S, V, W)$.

*Really, there are even more complicated terms than ones in Eqs. (58) and (59) of [32], which come from other η values in (21). However, they are outside of our present consideration.

In Eq. (31),

$$F_{N,k}(n) \sim \frac{1}{n^{1-N}} \quad (N \geqslant 1), \tag{34}$$

since now the factor $1/n^2$ has already been extracted.

So, Eqs. (29)–(31) show that the functions $F_{N,k}(n)$ should have the following form:

$$\frac{1}{n^c}\, F_{N,k}(n) \sim \frac{1}{n^{3-N}} \quad (N \geqslant 2) \tag{35}$$

and the number $3-N$ defines the level of transcendentality (or complexity) of the coefficients $F_{N,k}(n)$. The property reduces strongly the number of the possible elements in $F_{N,k}(n)$. The level of transcendentality decreases if we consider the singular parts of diagrams and/or coefficients in front of ζ functions and of logarithm powers.

Other I-type integrals in [32] have similar form. They have been calculated exactly by the differential equation method [36, 37].

Now we consider two-loop three-point diagrams P_1, P_5, P_6, P_{13} and P_{12} shown in Fig. 3 and considered in [32]:

$$\begin{aligned} P_1 = \frac{\hat{N}}{(q^2)^2} \sum_{n=1} \frac{x^n}{n} \Biggl\{ -\frac{1}{2\varepsilon^3} - \frac{S_1}{\varepsilon^2} + \frac{1}{2\varepsilon} \times \\ \times \left[5S_2 - S_1^2 + \frac{2}{n^2} - \frac{2}{n}\ln(-x) + \frac{1}{2}\ln^2(-x) - \zeta(2) \right] - \\ - \frac{8}{3}\zeta_3 - \left(S_1 + \frac{1}{n}\right)\zeta_2 + \frac{8}{3}S_3 + \frac{9}{2}S_1 S_2 + \frac{5}{6}S_1^3 + 4\frac{S_2}{n} + 2\frac{S_1}{n^2} + \frac{3}{n^3} + \\ + \left(\zeta_2 - 4S_2 - 2\frac{S_1}{n} - \frac{3}{n^2}\right)\ln(-x) + \left(S_1 + \frac{3}{2n}\right)\ln^2(-x) - \frac{1}{2}\ln^3(-x) \Biggr\}, \end{aligned} \tag{36}$$

$$\begin{aligned} P_5 = \frac{\hat{N}}{(q^2)^2} \sum_{n=1} \frac{(-x)^n}{n} \Biggl\{ -6\zeta_3 + 2(S_1\zeta_2 + 6S_3 - 2S_1S_2 + 4\frac{S_2}{n} - \frac{S_1^2}{n} + 2\frac{S_1}{n^2} + \\ + \left(-4S_2 + S_1^2 - 2\frac{S_1}{n}\right)\ln(-x) + S_1\ln^2(-x) \Biggr\}, \end{aligned} \tag{37}$$

$$\begin{aligned} P_6 = \frac{\hat{N}}{(q^2)^2} \sum_{n=1} \frac{(-x)^n}{n} \Biggl\{ -\frac{1}{\varepsilon^2}\left[\ln(-x) - \frac{1}{n}\right] + \frac{1}{\varepsilon}\Biggl[\zeta_2 - 3S_2 - 4S_{-2} - 3\frac{S_1}{n} - \frac{3}{n^2} + \\ + \left(3S_1 + \frac{3}{n}\right)\ln(-x) - \frac{3}{2}\ln^2(-x)\Biggr] + 2\zeta_3 + \left(7S_1 + \frac{2}{n}\right)\zeta_2 - 2S_3 - 9S_1S_2 + \end{aligned}$$

$$+ 10S_{-3} - 12S_{-2,1} - 4S_1S_{-2} - \frac{7}{2}\frac{S_2}{n} - \frac{9}{2}\frac{S_1^2}{n} - 5\frac{S_1}{n^2} - \frac{7}{n^3} + \left(\frac{7}{2}S_2 - \frac{9}{2}S_1^2 + \right.$$

$$\left. + 5\frac{S_1}{n} + \frac{7}{n^2} - 2\zeta_2\right)\ln(-x) + \frac{1}{2}\left(7S_1 + \frac{7}{n}\right)\ln^2(-x) + \frac{7}{6}\ln^3(-x)\Bigg\}, \quad (38)$$

$$P_{13} = \frac{\hat{N}}{(q^2)^2}\sum_{n=1} x^n \left\{ -\frac{S_2}{2\varepsilon^2} - \frac{1}{2\varepsilon}\left[S_3 + 4S_{1,2} - 4\frac{S_2}{n}\right] + \frac{S_2}{2}\zeta_2 - \right.$$

$$\left. - S_{1,3} - 3S_{3,1} + 3S_{1,1,2} + 3S_{1,2,1} - S_2^2 + \Big(7S_3 - 8S_{1,2}\Big)S_1 + \frac{5}{2}S_1^2S_2 \right\}, \quad (39)$$

$$P_{12} = \frac{\hat{N}}{q^2}\sum_{n=1}\frac{x^n}{n^2}\frac{(n!)^2}{(2n)!}\left\{ \frac{2}{\varepsilon^2} + \frac{2}{\varepsilon}\left(S_1 - 3W_1 + \frac{1}{n} - \ln(-x)\right) + 12W_2 - 18W_{1,1} - \right.$$

$$\left. - 13S_2 + S_1^2 - 6S_1W_1 + 2\frac{S_1}{n} + \frac{2}{n^2} - 2\left(S_1 + \frac{1}{n}\right)\ln(-x) + \ln^2(-x) \right\}. \quad (40)$$

Now the coefficients $F_{N,k}(n)$ have the form

$$\frac{1}{n^c}F_{N,k}(n) \sim \frac{1}{n^{4-N}} \quad (N \geqslant 3). \quad (41)$$

The diagrams P_1, P_5 and P_6 (and also P_3 in [32]) have been calculated exactly by the differential equation method [36, 37].

To find the results for P_{13} and P_{12} (and also all others in [32]), we have used the knowledge of the several n terms in the inverse-mass expansion (21) (usually less than $n = 100$) and the following arguments (see [40] and discussions therein):

• The coefficients should have the structure (41) with the rule (33). The condition (41) reduces strongly the number of possible harmonic sums. It should be related with the specific form of the differential equations for the considered master integrals, like

$$\left(\overline{k}\varepsilon + m^2\frac{d}{dm^2}\right)\ \mathrm{FI} = \text{less complicated diagrams},$$

with some $\overline{k}$ values. We note that for many other master integrals (for example, for sunsets with two massive lines in [35, 41]) the property (41) is violated: the coefficients $F_{N,k}(n)$ contain sums with different levels of complexity*.

*Really, Refs. [35, 41] contain the Nilson polylogarithms, whose sum of indices relates directly to the level of transcendentality $(4 - N)$. The representation of the series (29)–(31) and (36)–(40), containing $S_{\pm a,\ldots}$, as polylogarithms can be found in [32] for m-cut case and in [42] for $2m$-cut one, respectively.

• If a two-loop two-point diagram with «similar topology» (for example, I_1 for P_1 and P_3, I_5 for P_5 and P_6, I_{12} for P_{12} and so on) has already been calculated, we should consider a similar set of basic elements for the corresponding $F_{N,k}(n)$ of two-loop three-point diagrams but with the higher level of complexity.

• Let the considered diagram contain singularities and/or powers of logarithms.

Because in front of the leading singularity, or the largest power of logarithm, or the largest ζ function the coefficients are very simple, they can be often predicted directly from the first several terms of expansion.

Moreover, often we can calculate the singular part using another technique (see [32] for extraction of $\sim W_1(n)$ part). Then we should expand the singular parts, find the basic elements and try to use them (with the corresponding increase of the level of complexity) to predict the regular part of the diagram. If we have to find the ε-suppressed terms, we should increase the level of complexity for the corresponding basic elements.

Later, using the ansatz for $F_{N,k}(n)$ and several terms (usually, less than 100) in the above expression, which can be calculated exactly, we obtain a system of algebraic equations for the parameters of the ansatz. Solving the system, we can obtain analytical results for FI without exact calculations. To check the results, it is needed only to calculate a few more terms in the above inverse-mass expansion (21) and compare them with the predictions of our ansatz with the above fixed coefficients.

The arguments give a possibility to find the results for many complicated two-loop three-point diagrams without direct calculations. Some variations of the procedure have been successfully used for calculating the Feynman diagrams for many processes (see [32, 38, 39, 43]).

Note that the properties similar to (35) and (41) have recently been observed [44] in the so-called double operator-product-expansion limit of some four-point diagrams. These diagrams encode the quantum corrections to the four-point correlator and have been considered in [44] up to three-loop level of accuracy.

3. UNIVERSAL ANOMALOUS DIMENSION FOR $\mathcal{N} = 4$ SYM

The final three-loop result* for the universal anomalous dimension $\gamma_{\rm uni}(j)$ for $\mathcal{N} = 4$ SYM is [9]

$$\gamma(j) \equiv \gamma_{\rm uni}(j) = \hat{a}\gamma^{(0)}_{\rm uni}(j) + \hat{a}^2\gamma^{(1)}_{\rm uni}(j) + \hat{a}^3\gamma^{(2)}_{\rm uni}(j) + \ldots, \quad \hat{a} = \frac{\alpha N_c}{4\pi}, \tag{42}$$

*Note that, in accordance with [7], our normalization of $\gamma(j)$ contains the extra factor $-1/2$ in comparison with the standard normalization (see [4]) and differs by sign in comparison with the one from [5].

where

$$\frac{1}{4}\,\gamma_{\text{uni}}^{(0)}(j+2) = -S_1, \tag{43}$$

$$\frac{1}{8}\,\gamma_{\text{uni}}^{(1)}(j+2) = (S_3 + \overline{S}_{-3}) - 2\,\overline{S}_{-2,1} + 2\,S_1(S_2 + \overline{S}_{-2}), \tag{44}$$

$$\begin{aligned}\frac{1}{32}\,\gamma_{\text{uni}}^{(2)}(j+2) = {}& 2\overline{S}_{-3}\,S_2 - S_5 - 2\overline{S}_{-2}\,S_3 - 3\,\overline{S}_{-5} + 24\,\overline{S}_{-2,1,1,1} + \\ &+ 6(\overline{S}_{-4,1} + \overline{S}_{-3,2} + \overline{S}_{-2,3}) - 12(\overline{S}_{-3,1,1} + \overline{S}_{-2,1,2} + \overline{S}_{-2,2,1}) - \\ &- (S_2 + 2\,S_1^2)(3\overline{S}_{-3} + S_3 - 2\overline{S}_{-2,1}) - S_1(8\overline{S}_{-4} + \overline{S}_{-2}^2 + \\ &+ 4S_2\overline{S}_{-2} + 2S_2^2 + 3S_4 - 12\overline{S}_{-3,1} - 10\overline{S}_{-2,2} + 16\overline{S}_{-2,1,1})\end{aligned} \tag{45}$$

and $S_a \equiv S_a(j)$, $S_{a,b} \equiv S_{a,b}(j)$, $S_{a,b,c} \equiv S_{a,b,c}(j)$ are harmonic sums (see Eq. (25)) and

$$\overline{S}_{-a,b,c,\cdots}(j) = (-1)^j\,S_{-a,b,c,\cdots}(j) + S_{-a,b,c,\cdots}(\infty)(1-(-1)^j). \tag{46}$$

The expression (46) is defined for all integer values of arguments (see [4, 45, 46]) but can be easily analytically continued to real and complex j by the method of [45–47].

3.1. The Limit $j \to 1$**.** The limit $j \to 1$ is important for the investigation of the small-x behavior of parton distributions (see review [48] and references therein). Especially it became popular recently because there are new experimental data at small x produced by the H1 and ZEUS collaborations in HERA [49].

Using asymptotic expressions for harmonic sums at $j = 1+\omega \to 1$ (see [4,9]), we obtain for the $\mathcal{N}=4$ universal anomalous dimension $\gamma_{\text{uni}}(j)$ in Eq. (42)

$$\gamma_{\text{uni}}^{(0)}(1+\omega) = \frac{4}{\omega} + \mathcal{O}(\omega^1), \tag{47}$$

$$\gamma_{\text{uni}}^{(1)}(1+\omega) = -32\zeta_3 + \mathcal{O}(\omega^1), \tag{48}$$

$$\gamma_{\text{uni}}^{(2)}(1+\omega) = 32\zeta_3\,\frac{1}{\omega^2} - 232\zeta_4\,\frac{1}{\omega} - 1120\zeta_5 + 256\zeta_3\zeta_2 + \mathcal{O}(\omega^1) \tag{49}$$

in agreement with the predictions for $\gamma_{\text{uni}}^{(0)}(1+\omega)$, $\gamma_{\text{uni}}^{(1)}(1+\omega)$ and also for the first term of $\gamma_{\text{uni}}^{(2)}(1+\omega)$ coming from an investigation of BFKL equation at NLO accuracy in [8].

3.2. The Limit $j \to 4$**.** The investigation of the integrability in $\mathcal{N}=4$ SYM for BMN operators [50] gives a possibility to find the anomalous dimension of a Konishi operator [24, 51], which has the anomalous dimension coinciding with our expression (42) for $j = 4$:

$$\gamma_{\text{uni}}(j)\Big|_{j=4} = -6\,\hat{a}_s + 24\,\hat{a}_s^2 - 168\,\hat{a}_s^3. \tag{50}$$

It is also confirmed by direct calculation in two- [6,52] and three-loop [53] orders. The four- and five-loop corrections to the anomalous dimension of a Konishi operator have also been calculated recently in [54, 55] and [56, 57], respectively (see the recent review [16] and references therein).

3.3. The Limit $j \to \infty$. In the limit $j \to \infty$ the results (43)–(45) are simplified significantly. Note that this limit is related to the study of the asymptotics of structure functions and cross sections at $x \to 1$ corresponding to the quasi-elastic kinematics of the deep-inelastic ep scattering.

We obtain the following asymptotics for the $\mathcal{N} = 4$ universal anomalous dimension $\gamma_{\rm uni}(j)$ in Eq. (42):

$$\gamma_{\rm uni}^{(0)}(j) = -4(\ln j + \gamma_E) + \mathcal{O}(j^{-1}), \tag{51}$$

$$\gamma_{\rm uni}^{(1)}(j) = 8\zeta_2(\ln j + \gamma_E) + 12\zeta_3 + \mathcal{O}(j^{-1}), \tag{52}$$

$$\gamma_{\rm uni}^{(2)}(j) = -88\zeta_4(\ln j + \gamma_E) - 16\zeta_2\zeta_3 - 80\zeta_5 + \mathcal{O}(j^{-1}), \tag{53}$$

where γ_E is Euler constant (see also the normalization in Eq. (21)).

3.3.1. Resummation of $\gamma_{\rm uni}$ and the AdS/CFT Correspondence. Over the last few years there has been a great progress in the investigation of the $\mathcal{N} = 4$ SYM theory in the framework of the AdS/CFT correspondence [17], where the strong-coupling limit $\hat{a}_s \to \infty$ is described by a classical supergravity in the anti-de Sitter space $AdS_5 \times S^5$. In particular, a very interesting prediction [58] (see also [59]) was obtained for the large-j behavior of the anomalous dimension for twist-2 operators

$$\gamma(j) = a(z)\,\ln j, \quad z = \frac{\alpha N_c}{\pi} = 4\hat{a}_s \tag{54}$$

in the strong-coupling regime (see [60] for asymptotic corrections):

$$\lim_{z\to\infty} a = -z^{1/2} + \frac{3\ln 2}{8\pi} + \mathcal{O}(z^{-1/2}). \tag{55}$$

On the other hand, the results for $\gamma_{\rm uni}(j)$ in Eqs. (42) and (51)–(53) allow one to find three first terms of the small-z expansion of the coefficient $a(z)$

$$\lim_{z\to 0} a = -z + \frac{\pi^2}{12}\,z^2 - \frac{11}{720}\pi^4 z^3 + \ldots \tag{56}$$

For resummation of this series, Lipatov suggested the following equation for the approximation $\tilde{a}$ [6]:

$$z = -\widetilde{a} + \frac{\pi^2}{12}\,\widetilde{a}^2, \tag{57}$$

interpolating between its weak-coupling expansion up to NNLO

$$\tilde{a} = -z + \frac{\pi^2}{12}\,z^2 - \frac{1}{72}\pi^4 z^3 + \mathcal{O}(z^4) \tag{58}$$

and strong-coupling asymptotics

$$\tilde{a} = -\frac{2\sqrt{3}}{\pi}\, z^{1/2} + \frac{6}{\pi^2} + \mathcal{O}(z^{-1/2}) \approx -1.1026\, z^{1/2} + 0.6079 + \mathcal{O}(z^{-1/2}). \quad (59)$$

It is remarkable that the predictions for NNLO based on the above simple equation and obtained before the NNLO results (45) and (53) are valid with the accuracy $\sim 10\%$. It means that this extrapolation seems to be good for all values of z^*.

3.3.2. Beisert–Eden–Staudacher Equation. Recently the integral Beisert–Eden–Staudacher (BES) equation has been proposed in [61] for some function $f(x)$, which is related with $a(z)$ of (54) at $x = 0$, i.e., $f(0) = a(z)$.

At small coupling constant z, this equation gives a lot of coefficients c_m of the expansion

$$f(0) = \sum_{m=0} c_m z^m.$$

These coefficients c_m obey the transcendentality principle, i.e., $c_m \sim \zeta(2m)$ for $m > 0$ (or products of ζ function with the sum of indices equal to $2m$). Moreover, up to 4-loop, the coefficients are in agreement numerically with the ones obtained directly from calculations of Feynman diagrams [62, 63].

The most important purpose, however, is to find the $z \to \infty$ limit from the BES equation, i.e., to try to reproduce the Polyakov et al. asymptotics $\sim z^{1/2}$ (see the r.h.s. of (55)). The study was performed and the asymptotics were reproduced numerically [64] and analytically [65].

Recently the authors of [66] found a method to evaluate the $\tilde{c}_m$ coefficients of the expansion

$$f(0) = \sum_{m=0} \tilde{c}_m\, z^{(1-m)/2}$$

of the BES equation and calculated several of them. The first three coefficients are in agreement with the results of exact calculations performed in [58], [60] and [67], respectively. Moreover, the results of [66] agree well with the transcendentality principle: $\tilde{c}_1 \sim \ln 2$ and $\tilde{c}_m \sim \zeta(m)$ for $m > 1$ (or products of ζ function with the sum of indices equal to m).

4. BETHE ANSATZ AND FOUR-LOOP UNIVERSAL ANOMALOUS DIMENSION

The long-range asymptotic Bethe equations for twist-two operators have the form

$$\left(\frac{x_k^+}{x_k^-}\right)^2 = \prod_{m=1, m\neq k}^{M} \frac{x_k^- - x_m^+}{x_k^+ - x_m^-} \frac{(1 - g^2/x_k^+ x_m^-)}{(1 - g^2/x_k^- x_m^+)} \exp\left(2i\theta(u_k, u_j)\right), \quad \prod_{k=1}^{\hat{M}} \frac{x_k^+}{x_k^-} = 1. \quad (60)$$

---

*Some improvement of (57) can be found in [62].

These are $\hat{M}$ equations for $k = 1, \ldots, \hat{M}$ Bethe roots u_k, which need to be solved for the Bethe roots u_k. The variables $x_k^{\pm}$ are related to u_k through Zhukovsky map:

$$x_k^{\pm} = x(u_k^{\pm}), \quad u^{\pm} = u \pm \frac{i}{2}, \quad x(u) = \frac{u}{2}\left(1 + \sqrt{1 - 4\,\frac{g^2}{u^2}}\right). \qquad (61)$$

The dressing phase $\theta \sim \zeta(3)$ is a rather intricate function conjectured in [61], and its exact form is not so important for the present consideration.

Once the $\hat{M}$ Bethe roots are determined from the above equations for the state of interest, its asymptotic all-loop anomalous dimension is given by

$$\gamma^{ABA}(g) = 2\,g^2 \sum_{k=1}^{\hat{M}} \left(\frac{i}{x_k^+} - \frac{i}{x_k^-}\right). \qquad (62)$$

The above equations can be solved recursively order by order in g at arbitrary values of $\hat{M}$ once the one-loop solution for a given state is known.

This technical problem can nevertheless be surmounted. Assuming the maximum transcendentality principle [4] at four-loop order, one may derive the corresponding expression for the anomalous dimension by making an appropriate ansatz with unknown coefficients multiplying the nested harmonic sums, and subsequently fixing these constants. The latter is done by fitting to the exact anomalous dimensions for a sufficiently large list of specific values of $\hat{M}$ as calculated from the Bethe ansatz*.

Luckily, at one loop the exact solution of the Baxter equation is known [68] and is given by a Hahn polynomial. Knowing the one-loop roots, one can then expand equation (60) in the coupling constant g order by order in perturbation theory. The equations for the quantum corrections to the one-loop roots are of course linear, and thus numerically solvable with high precision.

The result for the four-loop asymptotic dimension has the form [10] ($\hat{M} = j + 2$)

$$\frac{1}{256}\,\gamma_{\text{uni}}^{ABA}(j+2) = 4S_{-7} + 6S_7 + \ldots + -\zeta(3)S_1(S_3 - S_{-3} + 2\,S_{-2,1}), \qquad (63)$$

where the symbol «...» marks large set of the nested sums of degree seven.

It is possible to analytically continue the expression in the r.h.s. of (63) to the vicinity of the pomeron pole at $M = -1 + \omega$. An explanation for how this is done may be found in [46].

*The study is similar to the one considered in Sec. 2 and used for calculations of Feynman integrals.

Harmonic sums of degree seven may lead to poles not higher than seventh order in ω. In fact, it is known that none of the sums in the r.h.s. of (63) can produce such a high-order pole except for the two sums S_7 and S_{-7}. Their residues at $1/\omega^7$ are of opposite sign. Thus, one immediately sees that the sum of the two residues does *not* cancel.

However, from BFKL calculations [4,8], it is possible to conclude that at the vicinity of the pomeron pole at $\hat{M} = -1+\omega$ the four-loop anomalous dimensions

$$\gamma_{\rm uni}(1+\omega) \sim 1/\omega^4. \tag{64}$$

It proves that the above result is not full and there are so-called wrapping corrections.

The contribution of the wrapping corrections has been added in [11]. So, the full result has the following form:

$$\gamma_{\rm uni}(j+2) = \gamma_{\rm uni}^{ABA}(j+2) + \gamma_{\rm uni}^{\rm wr}(j+2),$$

$$\frac{1}{256}\,\gamma_{\rm uni}^{\rm wr}(j+2) = \frac{1}{2}\,S_1^2\,\big[2S_{-5} + 2S_5 + 4\big(S_{4,1} - S_{3,-2} + \\ + S_{-2,-3} - 2S_{-2,-2,1}\big) - 4S_{-2}\zeta(3) - 5\zeta(5)\big].$$

This result is in full agreement with BFKL predictions (64).

We note that, by using similar technique and a property of reciprocity (see [69] and references therein), the five-loop corrections to universal anomalous dimensions have been found in [12].

CONCLUSION

In this review we presented the anomalous dimension $\gamma_{\rm uni}(j)$ for the $\mathcal{N}=4$ supersymmetric gauge theory up to the next-to-next-to-next-to-leading approximation. All the results have been obtained with the use of the *transcendentality principle*. At the first three orders, the universal anomalous dimensions have been extracted from the corresponding QCD calculations. The results for four and five loops have been obtained from the long-range asymptotic Bethe equations together with some additional terms, the so-called *wrapping corrections*, coming in agreement with the Luscher approach.

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