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FACTORIZATION METHOD FOR SCHRÖDINGER EQUATION IN RELATIVISTIC CONFIGURATION SPACE AND q-DEFORMATIONS

R. M. Mir-Kasimov

Joint Institute for Nuclear Research, Dubna

Namık Kemal University, Tekirdag, Turkey

Review paper is devoted to the relativistic configuration space (RCS) concept, a version of the relativistic Quantum Mechanics in RCS, the generalization of the Dirac–Infeld–Hall factorization method in the framework of the noncommutative differential calculus natural for RCS, different versions of the deformed oscillators, emerging as the generalization of the harmonic oscillator for RCS.

In the formulation of the Newton–Wigner postulates for the relativistic localized states, the hypothesis of commutativity of the position operator components is silently accepted as an evident fact. In the present work, it is shown that commutativity is not necessary condition and the alternative (noncommutative) approach to the relativistic position operator and localization concept can be realized in the framework of the physically as well as mathematically comprehensive scheme.

The different generalizations of the Dirac–Infeld–Hall factorization method for this case are constructed. This method enables us to find out all possible generalizations of the most important nonrelativistic integrable case — the harmonic oscillator. It is also shown that the relativistic oscillator = q-oscillator.

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INTRODUCTION

The concept of the relativistic configuration space is based on two premises:

1. The Newton–Wigner notion of the relativistic localized states. In the original formulation, the condition of the commutativity of components of the position operator is accepted without saying [1–3]. First premise is to admit the noncommuting position operators without any change of the postulates of Newton and Wigner.

2. Consideration of the closure of the algebra of noncommutative position operators including all commutators which is the Lie algebra of the Lorentz group $SO(3,1)$. Transfer to the maximal commuting subalgebra of the corresponding enveloping algebra, whose elements form the *commutative* relativistic configuration representation. The «price» of all these modifications is the necessity to

change the differential calculus, from the standard to the deformed (noncommutative) one*.

Newton–Wigner Position Operator. In the formulation of the Newton–Wigner postulates for the relativistic localized states, the hypothesis of commutativity of the position operator components is silently accepted as an evident fact. In the present work, it is shown that commutativity is not necessary condition and the alternative (noncommutative) approach to the relativistic position operator and localization concept can be realized in the framework of the physically as well as mathematically comprehensive scheme.

The concept of localization is one of the most important and most intriguing problems in quantum theory. The idea of localizability plays principal role in the physical interpretation of the theory. We cannot avoid this concept when comparing the measurement results with the theory predictions and considering the uncertainty relations. The localizability is necessary element in constructing the initial and final states when describing the collision phenomena. In the case of the two-body problem, the question of the localizability of the bound state must be solved in a transparent way.

In the nonrelativistic case for the potentials $V(r)$ depending only on relative distance between interacting particles $r = |\mathbf{r}|$, the coordinates of the center of mass $\mathbf{R}$ and $\mathbf{r}$ are separated. The Galilean invariance of the motion of the system as a whole is respected, the free motion of the bound state (of the system as the whole) is described by the irreducible unitary representation of the Galilean group. From this point of view, it would be natural to call the spherically symmetric potentials $V(r)$ the Galilean potentials. The internal motion of the system is reduced to the motion of the effective particle with the reduced mass $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ in the field of potential. We can call the two-body systems with the spherically symmetric potentials $V(r)$ the Galilean elementary systems by evident analogy with the relativistic particle localization concept of E. Wigner [1]. Let us remember also that nonrelativistic compound quark models of hadrons (with spherically symmetric potentials) are very efficient, and they describe the bound states which are the elementary systems. But there are no doubts that these compound systems are actually relativistic and it is necessary to find the comprehensive relativistic potential theory standing behind.

We stress that keeping in mind the relativistic potential models we seek for such relativistic analog of the relative coordinate $\mathbf{r}$ on which the interaction po-

*This review is based on the papers [4–9]. The reader can find the references to preceding papers concerning the development of the «Snyder quantum space-time» which was the precursor of the approach considered in this paper. The list includes the names of H. Snyder, W. Pauli, Chen Ning Yang, I. E. Tamm, Yu. A. Gol'fand, V. G. Kadyshevsky and others.

tential depends so that the total relativistic invariance is respected in analogy with Galilean invariance of the nonrelativistic two-particle problem with the spherically symmetric potentials $V(r)$.

This discussion can be continued, but it is clear that old problem of finding the relativistic position operator still deserves to search for its solution. The basic ideas on this subject have been expressed by Newton and Wigner [1–3]. Their essential result is that for single particles a notion of the localizability and a corresponding *commuting* observables are uniquely determined by relativistic kinematics. On the other hand, no relativistic quantum theory of interaction based on these ideas was constructed. In the present contribution, we shall consider the possibility of introducing the concept of the *noncommuting* relativistic position operators obeying all Newton–Wigner postulates, having the transparent physical interpretation and admitting very simple quantum dynamical interpretation.

It must be stressed that the standard quantum–mechanical position operator $\mathbf{x} = i\hbar\nabla_{\mathbf{p}}$ is connected with the Euclidean structures in terms of which the localization of a particle is considered. Let us quote here [1]: «Existence and uniqueness of a notion of localizability for a physical system are properties which depend only on the transformation law of the system under Euclidean group, i.e., the group of all space translations and rotations. The analysis of localizability in the Lorentz and Galilei invariant cases is then just a matter of discussing what representations of the Euclidean group can arise there». Both groups — Galilean and Poincaré — contain the Euclidean group as their subgroup. But maybe there are another realizations of the Euclidean group in the framework of the representation theory which allow another definition of the position operator. We show here that the answer is positive.

The fact that the manifold of the physically realizable states contains only solutions with the positive energy has a number of consequences for the observables. Consider the solutions of the Klein–Gordon equation φ, ψ:

$$\varphi,\, \psi \in \{(+) : p^{\mu}p_{\mu} = (p^0)^2 - \tilde{p}^2 = m^2c^2, \quad p^0 \geqslant 0\}, \tag{1}$$

with the inner product

$$(\varphi, \psi) = \int\limits_{(+)} d\Omega_{\mathbf{p}}\, \overline{\varphi(\mathbf{p})}\psi(\mathbf{p}), \quad d\Omega_{\mathbf{p}} = \frac{d\mathbf{p}mc}{p^0}. \tag{2}$$

The standard position operator

$$\hat{\hat{\mathbf{x}}} = i\hbar\nabla_p \tag{3}$$

is non-Hermitian in the metric (2):

$$\left(\varphi, \hat{\hat{\mathbf{x}}}\psi\right) = \int_{(+)} d\Omega_{\mathbf{p}}\, \overline{\varphi(\mathbf{p})}\, i\hbar\nabla_p\, \psi(\mathbf{p}) =$$

$$= \int_{(+)} d\Omega_{\mathbf{p}}\, \overline{\left[\left(i\hbar\nabla_p - \frac{i\hbar\mathbf{p}}{\mathbf{p}^2 + m^2c^2}\right)\varphi(\mathbf{p})\right]}\, \psi(\mathbf{p}). \quad (4)$$

So the operator $i\hbar\nabla_p$ does not correspond to any observable and cannot be interpreted as a physical operator. It follows also that the Klein–Gordon wave function cannot be considered as a probability amplitude to find the particle at the point $\mathbf{x}$ at the moment of time x^0.

The simplest way to obtain the position operator is to accept that the position operator is the Hermitian part of $\hat{\mathbf{x}} = i\hbar\nabla_p$:

$$\hat{\mathbf{x}}_{\mathrm{NW}} = \frac{1}{2}\left[\hat{\mathbf{x}} + \hat{\mathbf{x}}^\dagger\right] = i\hbar\nabla_p - \frac{i\hbar}{2}\,\frac{\mathbf{p}}{\mathbf{p}^2 + m^2c^2}. \quad (5)$$

Newton and Wigner derived this operator on the basis of a number of conditions which localized states must satisfy (see [2], the papers [10, 12, 12] also might be useful).

For the wave function of the localized state (at the moment x^0) $\psi_{\mathbf{y}}(\mathbf{x})$ in the configurational space, a number of conclusions can be derived from the Newton–Wigner theory. We indicate two of them:

- The position operator components $\hat{x}^i_{\mathrm{NW}}$ commute

$$\left[\hat{x}^i_{\mathrm{NW}}, \hat{x}^j_{\mathrm{NW}},\right] = 0, \quad \left[\hat{x}^i_{\mathrm{NW}}, p^j,\right] = i\delta_{ij}. \quad (6)$$

- The localized eigenfunction is not $\delta(\mathbf{x}-\mathbf{y})$ as in the nonrelativistic theory; it is a function $\psi_{\mathbf{y}}(\mathbf{x})$ smeared in the spatial region of the size of the Compton wave length of the particle λ_0, because $\delta(\mathbf{x}-\mathbf{y})$ cannot be constructed from the positive frequency solutions only:

$$\psi_{\mathbf{y}}(\mathbf{x}) = \mathrm{const}\left(\frac{mc}{\hbar r}\right)^{5/4} K_{5/4}\left(\frac{r}{\lambda_0}\right), \quad r = |\mathbf{x}-\mathbf{y}|, \quad \lambda_0 = \frac{\hbar}{mc}, \quad (7)$$

$K_\nu(z)$ is the MacDonald function; λ_0 is the Compton wave length of the particle.

Alternative to Newton–Wigner Approach. Actually we do not consider an alternative. Rather we consider an independent perspective which is opened thanks to denial of the commutativity requirement. The Newton–Wigner theory uses essentially the momentum space. To determine the nonlocal operator $\hat{\mathbf{x}}_{\mathrm{NW}}$ (5) directly in the configurational space would be very difficult.

But there is another circumstance essential for formulating the main idea of the present paper. In [2], the wave functions localized at different points are connected by translation:

$$\mathbf{x} \longrightarrow \mathbf{x} + \mathbf{a}, \quad \mathrm{e}^{i\mathbf{k}(\mathbf{x}+\mathbf{a})} = \mathrm{e}^{i\mathbf{k}\mathbf{x}}\,\mathrm{e}^{i\mathbf{k}\mathbf{a}}. \tag{8}$$

The second relation has two mathematical meanings:

1) We consider, as in [2], the translations in the configurational space. Then the plane waves (exponentials) are the matrix elements of the irreducible unitary representations of the translation group numbered by the value of momentum $\mathbf{k}$. Fourier transformation is the expansion in matrix elements of the unitary irreducible representations of the translation group of the configurational space.

2) We consider (in addition to $[1, 2]$) the translations in the momentum $\mathbf{k}$-space. Then the same formula (8) describes the matrix element of the product of two irreps numbered by $\mathbf{x}$ and $\mathbf{a}$ correspondingly by the vector (of the momentum space) $\mathbf{k}$. The *inverse* Fourier transformation is the expansion in matrix elements of the unitary irreducible representations of the translation group of the momentum space.

Such a symmetry between transformation within the same representation and the product of the representations is specific to the Euclidean translations. In the nonrelativistic theory, the difference between 1 and 2 is formal and unimportant because the geometries of the configurational and momentum spaces are isomorphic (mathematically) and Euclidean. Physical sense of the configurational and momentum spaces is different of course. The translations of the momentum space correspond to Galilean transformations:

$$\begin{gathered} \mathbf{x} \longrightarrow \mathbf{x} + \mathbf{V}t, \quad \dot{\mathbf{x}} \longrightarrow \dot{\mathbf{x}} + \mathbf{V}, \\ m\dot{\mathbf{x}} \longrightarrow m\dot{\mathbf{x}} + m\mathbf{V} \quad \mathbf{p} \longrightarrow \mathbf{p} + \mathbf{k}, \quad \mathbf{p} = m\dot{\mathbf{x}}, \quad \mathbf{k} = m\mathbf{V}. \end{gathered} \tag{9}$$

The position operator (3) is the generator of translations of the momentum space.

Now we formulate the alternative to the Newton–Wigner concept. It is based on the simple observations.

1. From (1), (2) we conclude that the geometry of the momentum space, i.e., the manifold of realizable states of the relativistic particle of the positive frequency is the Lobachevsky space (1). We shall develop the one particle relativistic theory accepting this as the triggering point. Then we must substitute:

2. Galilean group $\longrightarrow$ Lorentz group,

3. Galilean boosts $\longrightarrow$ Lorentz boosts

$$\hat{\mathbf{x}} = i\hbar\nabla_p \longrightarrow \hat{\mathbf{x}}_{\mathrm{rel}} = i\hbar\sqrt{1 + \frac{\mathbf{p}^2}{m^2c^2}}\,\nabla_p. \tag{10}$$

Thereby, we consider the components of $\hat{\mathbf{x}}_{\mathrm{rel}}$ of (10) as the candidates for the relativistic position operators. But these are noncommuting operators. So proceeding along this geometrically natural way, first what we must do is to give up with the commutativity of components of the position operator. We stress that in [2], the commutativity requirement is tacitly contained in the list of basic natural requirements.

First of all, we note immediately that operators (10) are Hermitian with the norm (2). After lifting the commutativity condition we can say that the new position operators are (the simplest) Hermitian operators in this metric. Now in contrast with the commutative case, the components of the position operator cannot be measured. But no limitations exist that the very concept of the configurational space in the relativistic case can be modified as compared with the nonrelativistic theory. The consequence of such a modification must be the change of all the concept of the measuring the position, uncertainty relations, etc.

To make this statements more clear, let us return for a time being to the nonrelativistic case. As coordinates commute, we can diagonalize simultaneously all three components of it.

At the same time, many other operators of the universal enveloping algebra of the Euclidean Lie algebra are also diagonal; for example, the Casimir operator $\hat{\mathbf{x}}^2$ which is invariant operator of the Euclidean group of the *momentum space*

$$\begin{gathered}\left[\hat{\mathbf{x}}^2\right] \mathrm{e}^{i\mathbf{px}} = \triangle_{\mathbf{p}}\, \mathrm{e}^{i\mathbf{px}} = \mathbf{x}^2\, \mathrm{e}^{i\mathbf{px}}, \\ \hat{x}^i e^{i\mathbf{px}} = x^i e^{i\mathbf{px}}, \quad 0 \leqslant x < \infty, \quad -\infty < x^i < \infty. \end{gathered} \tag{11}$$

Important is that the common eigenfunctions of these operators $\mathrm{e}^{i\mathbf{px}}$ are the kernels of the Fourier transform connecting the Euclidean momentum space of the nonrelativistic quantum mechanics and *corresponding configurational space*.

In the relativistic case it is natural to consider as the momentum space, adequate from the physical point of view, the space given by (1), i.e., the Lobachevsky space of the physical solutions of the Klein–Gordon equation. Integration over this space (with the Lorentz-invariant volume element $d\Omega_{\mathbf{p}}$) is given by (2). If we wish to follow the concept presented in the previous paragraph, we should consider the universal enveloping algebra of the Lorentz group, determine the maximal set of mutually commuting operators, determine their common eigenfunctions (new plane waves) and spectrum. The Casimir operator of the Lorentz group Lie algebra can be chosen in the form

$$\hat{r}^2 = \hat{\mathbf{x}}^2_{\mathrm{rel}} - \frac{\mathbf{M}^2}{m^2c^2} - \frac{\hbar^2}{m^2c^2}, \tag{12}$$

where $\mathbf{M}$ is the angular momentum operator. The nonrelativistic limit of (12) is $\hat{\mathbf{x}}^2$ (see ()). Spectrum of r for the unitary representations of the Lorentz group takes continuous and discrete values [21]. All these representations find the applications in various models of relativistic interactions. We shall concentrate on the so-called principal series for which $0 \leqslant r < \infty$.

The eigenfunctions of $\hat{r}^2$ are the matrix elements of unitary irreducible representations of the Lorentz group or their generating functions — kernels of Gelfand–Graev transformations:

$$\hat{r}^2\langle\mathbf{p}|\mathbf{r}\rangle = r^2\langle\mathbf{p}|\mathbf{r}\rangle, \quad \langle\mathbf{r}|\mathbf{p}\rangle = \langle\mathbf{p}|\mathbf{r}\rangle^*. \tag{13}$$

They play the role of plane waves in the given relativistic formalism; explicitly

$$\langle\mathbf{r}|\mathbf{p}\rangle = \left(\frac{p^0 - \mathbf{pn}}{mc}\right)^{-1-irmc/\hbar}, \quad \mathbf{n}^2 = 1. \tag{14}$$

The unit vector $\mathbf{n}$ gives the sense to the symbol $\mathbf{r}$ — by definition

$$\mathbf{r} = r\mathbf{n}. \tag{15}$$

We shall call the space of vectors $\mathbf{r}$ the relativistic configurational space.

The partial expansion for the plane wave (14) is

$$\begin{gathered}\langle\mathbf{r}|\mathbf{p}\rangle = \sum_{l=0}^{\infty} i^l(2l+1)\, p_l(\cosh\,\chi, r)\, P_l(\mathbf{n}_p\cdot\mathbf{n}),\\ p^0 = \cosh\chi, \quad \mathbf{p} = \sinh\,\chi\mathbf{n}_p, \quad \mathbf{n}_p^2 = 1,\end{gathered} \tag{16}$$

where

$$p_l(\cosh\chi, r) = (-1)^l\sqrt{\frac{\pi}{2\sinh\,\chi}}\,\frac{\Gamma(ir+l+1)}{\Gamma(Ir+1)}\,P^{-1/2+Ir}_{-1/2+Ir}\cosh\,\chi. \tag{17}$$

The expansion (16) is analogous to the nonrelativistic one

$$\mathrm{e}^{i\mathbf{pr}} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(pr)P_l(\mathbf{n}_p\cdot\mathbf{n}), \tag{18}$$

where $j_l(pr) = \sqrt{\dfrac{\pi}{2pr}}\,J_{l+1/2}$ are the spherical Bessel functions. In the nonrelativistic limit

$$p_l(\cosh\chi, r) \longrightarrow j_l(pr). \tag{19}$$

The orthogonality and completeness conditions for the relativistic plane waves are

$$\begin{gathered}\frac{1}{(2\pi)^3}\int\langle\mathbf{r}|\mathbf{p}\rangle\,\langle\mathbf{p}|\mathbf{r}'\rangle d\Omega_{\mathbf{p}} = \delta(\mathbf{r}-\mathbf{r}'),\\ \frac{1}{(2\pi)^3}\int\langle\mathbf{p}|\mathbf{r}\rangle\,\langle\mathbf{r}|\mathbf{p}'\rangle d\mathbf{r} = \delta(\mathbf{p}(-)\mathbf{p}') = \frac{p^0}{mc}\,\delta(\mathbf{p}-\mathbf{p}').\end{gathered} \tag{20}$$

The relativistic configurational space is an example of the quantum 3-dimensional Euclidean space. The quantum nature of the $\mathbf{r}$-space is predefined by the fact that the Lie algebra of its isometry group is realized in the framework of noncommutative differential calculus. The momentum operators (generators of translations) are

$$\begin{aligned}
H_0 = \hat{p}^0 &= \cosh\left(i\,\frac{\partial}{\partial r}\right) + \frac{i}{r}\sinh\left(i\,\frac{\partial}{\partial r}\right) - \frac{\triangle_{\vartheta,\psi}}{2r^2}\,\exp\left(i\,\frac{\partial}{\partial r}\right),\\
\hat{p}^1 &= -\sin\vartheta\cos\psi\left[\exp\left(i\frac{\partial}{\partial r}\right) - H_0\right] -\\
&\qquad - i\left(\frac{\cos\vartheta\cos\psi}{r}\frac{\partial}{\partial\vartheta} - \frac{\sin\psi}{r\sin\vartheta}\frac{\partial}{\partial\psi}\right)\exp\left(i\,\frac{\partial}{\partial r}\right),\\
\hat{p}^2 &= -\sin\vartheta\sin\psi\left[\exp\left(i\frac{\partial}{\partial r}\right) - H_0\right] -\\
&\qquad - i\left(\frac{\cos\vartheta\sin\psi}{r}\frac{\partial}{\partial\vartheta} + \frac{\cos\psi}{r\sin\vartheta}\frac{\partial}{\partial\psi}\right)\exp\left(i\frac{\partial}{\partial r}\right),\\
\hat{p}^3 &= -\cos\vartheta\left[\exp\left(i\frac{\partial}{\partial r}\right) - H_0\right] + i\frac{\sin\vartheta}{r}\frac{\partial}{\partial\vartheta}\,\exp\left(i\frac{\partial}{\partial r}\right).
\end{aligned} \tag{21}$$

They play the role of inner derivatives in relevant differential calculi. These operators mutually commute

$$[\hat{p}^\mu, \hat{p}^\nu] = 0, \quad \mu,\nu = 0,1,2,3. \tag{22}$$

But the corresponding differentials of the coordinate functions do not commute with the coordinate functions themselves. For the details we refer the reader to [8, 9]. Note that the integration in the second formula in (20) is carried over with the Euclidean volume element $d\mathbf{r}$.

The common eigenfunctions of $\hat{p}^\mu$ are $\langle\mathbf{r}|\mathbf{p}\rangle$ (13)

$$\hat{p}^\mu\langle\mathbf{r}|\mathbf{p}\rangle = p^\mu\langle\mathbf{r}|\mathbf{p}\rangle, \tag{23}$$

from which we conclude that the «plane waves» (13) indeed describe the free relativistic motion with definite value of the 4-momentum. The Relativistic Schrödinger Equation (RSE) arising here belongs to the deformed (noncommutative) differential calculus [4, 9] (cf. [13, 14]). This is a new realization of the Lie algebra of the Euclidean group which we discussed in Introduction.

Operators $\hat{p}^\mu$ identically satisfy the relativistic relation between energy and momentum (1). Important is also to note that these operators solve the problem of «extracting the root square» in the relation $\hat{p}^\mu = \sqrt{\mathbf{p}^2 + m^2c^2}$:

$$\hat{p}^0\langle\mathbf{r}|\mathbf{p}\rangle = p^0\langle\mathbf{r}|\mathbf{p}\rangle = \sqrt{\mathbf{p}^2 + m^2c^2}\,\langle\mathbf{r}|\mathbf{p}\rangle. \tag{24}$$

In the nonrelativistic limit,

$$|\mathbf{p}| \ll mc, \quad p^0 \cong mc + \frac{\mathbf{p}^2}{2mc}, \quad r \gg \frac{\hbar}{mc}, \tag{25}$$

relativistic plane waves $\langle \mathbf{r}|\mathbf{p}\rangle$ transfer to usual plane waves

$$\langle \mathbf{r}|\mathbf{p}\rangle = \exp\left[-\left(1 + ir\frac{mc}{\hbar}\right)\ln\left(\frac{p^0 - \mathbf{pn}}{mc}\right)\right] \cong$$
$$\cong \exp\left[-\left(1 + ir\frac{mc}{\hbar}\right)\ln\left(1 - \frac{\mathbf{pn}}{mc} + \frac{\mathbf{p}^2}{2m^2c^2} + \ldots\right)\right] \cong$$
$$\cong \exp\left(i\,\frac{\mathbf{p}\cdot(r\mathbf{n})}{\hbar}\right) = \exp\left(i\,\frac{\mathbf{pr}}{\hbar}\right). \tag{26}$$

The wave function of the particle can be expanded in the Fourier integral in the relativistic plane waves

$$\psi(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int \langle \mathbf{r}|\mathbf{p}\rangle\,\psi(\mathbf{p})\,d\Omega_{\mathbf{p}}. \tag{27}$$

Particles are localized in the relativistic configurational space in a usual sense. The position operator $\hat{\mathbf{r}}$ in $\mathbf{r}$-representation acts on a wave function in a usual way

$$\hat{\mathbf{r}}\psi(\mathbf{r}) = \mathbf{r}\psi(\mathbf{r}). \tag{28}$$

The eigenfunctions $\psi_{\mathbf{r}_0}(\mathbf{r})$ of $\hat{\mathbf{r}}$ corresponding to the eigenvalue $\mathbf{r}_0$ are $\psi_{\mathbf{r}_0}(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_0)$ so that

$$\hat{\mathbf{r}}\psi_{\mathbf{r}_0}(\mathbf{r}) = \mathbf{r}\psi_{\mathbf{r}_0}\cdot(\mathbf{r}). \tag{29}$$

Eigenfunctions corresponding to different eigenvalues, i.e., the states localized at different points $\mathbf{r}_0$ and $\widetilde{\mathbf{r}}_0$, are orthogonal

$$\int \psi_{\mathbf{r}_0}\psi_{\widetilde{\mathbf{r}}_0}d\mathbf{r} = \delta(\widetilde{\mathbf{r}}_0 - \mathbf{r}_0), \tag{30}$$

which is the usual localization condition in the new relativistic configurational space.

1. ONE-DIMENSIONAL RELATIVISTIC QUANTUM MECHANICS

In the one-dimensional case, the Lobachevsky space is a hyperbola:

$$p_0^2 - p^2 = 1, \tag{31}$$

and the relativistic plane wave is the exponent

$$\langle x|p\rangle = (p_0 - p)^{-ix} = \mathrm{e}^{i\chi x}, \quad -\infty < x < \infty, \tag{32}$$

where x is the relativistic coordinate, χ is rapidity:

$$\begin{gathered} p_0 = \cosh\chi, \quad p = \sinh\chi, \\ \chi = \ln(p_0 + p) = \text{rapidity}, \quad -\infty < \chi < \infty, \\ d\Omega_p = \frac{dp}{p_0} = d\chi. \end{gathered} \tag{33}$$

We use the unit system, in which $\hbar = c = m = 1$. The wave function $\psi(x)$ is connected with its momentum space counterpart by Fourier transformation

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\Omega_p \langle x|p\rangle \psi(p). \tag{34}$$

The plane waves obey the following orthogonality and completeness conditions:

$$\begin{gathered} \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle x|p\rangle \, d\Omega_p \langle p|x'\rangle = \delta(x - x'), \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle p|x\rangle \, d\Omega_p \langle x|p'\rangle = \delta(\chi - \chi'). \end{gathered}$$

The free Hamiltonian and the momentum are a finite-difference operators:

$$H_0 = \cosh i\frac{d}{dx}, \quad p = -\sinh i\frac{d}{dx}.$$

The plane wave $\langle x|p\rangle$ obeys the free finite-difference RSE equation:

$$(H_0 - p_0)\langle x|p\rangle = 0. \tag{35}$$

We can write Eq. (35) in the «nonrelativistic» form, using the simple relation of the hyperbolic trigonometry, the «half-rapidity relation»:

$$\cosh\chi = 1 + 2\sinh^2\frac{\chi}{2}.$$

The operator of free relativistic kinetic energy takes the form (we restore for the time being the dimensional quantities)

$$h_0 = 2mc^2\,\sinh^2\frac{i\hbar}{2mc}\frac{d}{dx} = \frac{\hat{k}^2}{2m} = H_0 - mc^2, \tag{36}$$

where the corresponding momentum operator was introduced

$$\hat{k} = -2mc\,\sinh\frac{i\hbar}{2mc}\frac{d}{dx}, \quad k = 2mc\,\sinh\frac{\chi}{2}. \tag{37}$$

The RSE takes the form:

$$(h-e)\psi(x)=0, \tag{38}$$

where

$$h=\frac{\hat{k}^2}{2m}+V(x), \quad e=\frac{k^2}{2}. \tag{39}$$

In this form, RSE is indistinguishable with nonrelativistic Schrödinger equation, until the explicit form of the momentum operator is fixed. Obviously, in nonrelativistic limit we have

$$h\to-\frac{\hbar}{2m}\frac{d^2}{dx^2}. \tag{40}$$

2. THE NONRELATIVISTIC FACTORIZATION METHOD

One of the most important integrable cases of usual Schrödinger equation is the linear oscillator problem. We shall construct the relativistic Hamiltonian corresponding to this important nonrelativistic problem and see that there are several ways to do that. It is impossible to introduce the notion of the elastic forces in the relativistic configurational space. We shall proceed by analogy, trying to restore all important features of the nonrelativistic quantum mechanical linear oscillator properties:

1. The integrability. In particular, this property of nonrelativistic oscillator is reflected in factorizability of Hamiltonian. This will be exploited by us intensively.

2. The correct nonrelativistic limit.

3. The symmetry. We shall require the existence of both kinds of the symmetry: the symmetry of the Hamiltonian and the dynamical symmetry. Of course, it is obvious from the beginning that in the relativistic case these symmetries must be in some sense generalized. We see that in our case the «generalization» means going over to the q-deformed groups $SU_q(2)$ and $SU_q(1,1)$ with q given by (66). By other words, the relativization $=q$-deformation.

4. The existence of some generalization of the symmetry between the coordinates and momenta, which in the nonrelativistic case follows from the explicitly symmetric Hamiltonian:

$$H_{\mathrm{nonrel}}=\frac{P^2+Q^2}{2}. \tag{41}$$

Let us consider the nonrelativistic one-dimensional Hamiltonian

$$H=-\frac{1}{2}\frac{d^2}{dx^2}+V(x) \tag{42}$$

with the positive-definite ground state wave function

$$\psi_0(x) = \mathrm{e}^{-\rho(x)} \geqslant 0 \tag{43}$$

and the lowest energy level E_0

$$H\psi_0(x) = E_0\psi_0(x). \tag{44}$$

We can express $V(x)$ in terms of $\rho(x)$ and E_0:

$$V(x) = \frac{1}{2}\left[\left(\frac{d\rho(x)}{dx}\right)^2 - \frac{d^2\rho(x)}{dx^2}\right] - E_0. \tag{45}$$

Now Hamiltonian can be written in the factorized form

$$H - E_0 = a^+ a^-, \tag{46}$$

where

$$a^{\pm} = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{dx} + \frac{d\rho(x)}{dx}\right) = \mp\frac{1}{\sqrt{2}}\,\mathrm{e}^{\mp\rho(x)}\frac{d}{dx}\,\mathrm{e}^{\pm\rho(x)}. \tag{47}$$

The commutator of the creation and annihilation operators

$$[a^+, a^-] = \frac{d^2\rho(x)}{dx^2} \tag{48}$$

is simply a function of x and does not contain the derivative operator as a^+, a^- themselves do. We stress that in general case the commutator of two first-order differential operators is the differential operator of the second order. The absence of the derivatives in the right-hand side of (48) is the consequence of the specific construction (47) and, of course, of the Leibnitz rule for differentiation operation:

$$\frac{d}{dx}f_1(x)\cdot f_2(x) = \left(\frac{d}{dx}f_1(x)\right)\cdot f_2(x) + f_1(x)\cdot\left(\frac{d}{dx}f_2(x)\right). \tag{49}$$

The harmonic oscillator case is singled out. In this case, the right-hand side of (48) is not a function of x; but a constant ω, the frequency of oscillator:

$$\left[a^+, a^-\right] = \mathrm{const} = \omega. \tag{50}$$

The $\rho(x)$ takes the form

$$\rho(x) = \frac{\omega x^2}{2} = \frac{m\omega x^2}{2\hbar}, \tag{51}$$

and a^+, a^- are

$$a^{\pm} = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{d} + \omega x\right) = \mp\frac{i}{\sqrt{2}}\left(\hat{p} \pm i\omega x\right). \tag{52}$$

The Hamiltonian H could be written in different forms:

$$H = \frac{1}{2}\{a^+, a^-\} = \alpha a^+ a^- + (1-\alpha)\, a^- a^+ + (2\alpha - 1)\, E_0 = a^- a^+ - E_0,$$
$$E_0 = \frac{\omega}{2}, \quad \alpha \text{ — arbitrary number.} \tag{53}$$

The fact of integrability of the oscillator problem is reflected in commutation relations of the creation and annihilation operators with the Hamiltonian [5,7,16]:

$$[H, a^+] = \omega a^+, \quad [H, a^-] = -\omega a^-. \tag{54}$$

The relations (54) give us the simple way for constructing the spectrum and the eigenvectors of Hamiltonian H. If Ψ is an eigenvector of H ($H\Psi = E\Psi$), the functions $a^+\Psi$ and $a^-\Psi$ (provided that they are nonzero and belong to $L_2(R)$) and the new eigenvectors corresponding to the eigenvalues $E+\omega$ and $E-\omega$ are, respectively,

$$\begin{aligned} H(a^+\Psi) &= a^+(H+\omega)\Psi = (E+\omega)a^+\Psi, \\ H(a^-\Psi) &= a^-(H-\omega)\Psi = (E-\omega)a^-\Psi. \end{aligned} \tag{55}$$

In fact, the integrability is not strictly connected with commutators in left-hand sides of (54). Suppose we have instead of (54) some other linear combination of Ha^+ and a^+H of the form

$$AHa^+ - Ba^+H = \omega a^+. \tag{56}$$

Obviously, this relation could be used for the constructing the higher and lower eigenvectors of the Hamiltonian in the same way as (54) was used in (55). Of course, the spectrum of the Hamiltonian will be different. We shall have such a situation when we consider the factorization in the relativistic case. The Leibnitz rule for the finite-difference calculus is different from (49) and we shall see that integrability of the relativistic oscillator problem is provided by the relation of the type (56) rather than (55). Ultimately, this will lead us to the deformed symmetries for the relativistic oscillator [5].

3. THE RELATIVISTIC OSCILLATOR

Let $\psi_0(x)$ be the ground state wave function; and e_0, the lowest energy level in Eq. (38). We introduce the relativistic creation and annihilation operators ([5])

$$A^{\pm} = \mp i\sqrt{2}\alpha(x)\, \mathrm{e}^{\pm\rho(x)}\, \sinh \frac{i}{2}\frac{d}{dx}\, \mathrm{e}^{\mp\rho(x)}. \tag{57}$$

The difference between (47) and (57) is originated from the different Leibnitz rules for the finite-difference operators as compared with the differential one (49):

$$\begin{aligned}\sinh\frac{i}{2}\frac{d}{dx}\left(f_1(x)\cdot f_2(x)\right) &= \left(\sinh\frac{i}{2}\frac{d}{dx}f_1(x)\right)\left(\cosh\frac{i}{2}\frac{d}{dx}f_2(x)\right)+\\ &+\left(\cosh\frac{i}{2}\frac{d}{dx}f_1(x)\right)\left(\sinh\frac{i}{2}\frac{d}{dx}f_2(x)\right),\\ \cosh\frac{i}{2}\frac{d}{dx}\left(f_1(x)\cdot f_2(x)\right) &= \left(\cosh\frac{i}{2}\frac{d}{dx}f_1(x)\right)\left(\cosh\frac{i}{2}\frac{d}{dx}f_2(x)\right)+\\ &+\left(\sinh\frac{i}{2}\frac{d}{dx}f_1(x)\right)\left(\sinh\frac{i}{2}\frac{d}{dx}f_2(x)\right).\end{aligned} \tag{58}$$

This new Leibnitz rules naturally suggest a new construction — the «$q(x)$-mutator»:

$$\begin{gathered}[A^-,A^+]_{q(x)} = A^-\,\mathrm{e}^{a(x)}A^+ - A^+\,\mathrm{e}^{-a(x)}A^-,\\ q(x)=\mathrm{e}^{a(x)}.\end{gathered} \tag{59}$$

In the explicit form

$$\begin{aligned}[A^-,A^+]_{q(x)} = \frac{\alpha(x)}{2}\exp\left(\frac{i}{2}\frac{d}{dx}\right)\alpha(x)\,\sinh Z(x)\,\exp\left(\frac{i}{2}\frac{d}{dx}\right)+\\ +\exp\left(-\frac{i}{2}\frac{d}{dx}\right)\alpha(x)\,\sinh Z(x)\,\exp\left(-\frac{i}{2}\frac{d}{dx}\right)-\\ -\exp\left(\frac{i}{2}\frac{d}{dx}\right)\alpha(x)\,\sinh\left(Z(x)+2\rho_{s/2}(x)\right)\,\exp\left(-\frac{i}{2}\frac{d}{dx}\right)-\\ -\exp\left(-\frac{i}{2}\frac{d}{dx}\right)\alpha(x)\,\sinh\left(Z(x)-2\rho_{s/2}(x)\right)\,\exp\left(\frac{i}{2}\frac{d}{dx}\right),\end{aligned} \tag{60}$$

where the following notations were introduced:

$$\begin{gathered}Z(x)=2\rho(x)+a(x)-2\rho_{\frac{c}{2}}(x),\\ \rho_{\frac{s}{2}}(x)=\sinh\frac{i}{2}\frac{d}{dx}\rho(x),\quad \rho_{\frac{c}{2}}(x)=\cosh\frac{i}{2}\frac{d}{dx}\rho(x).\end{gathered} \tag{61}$$

The right-hand side of (60) is much more complicated than the right-hand side of the commutator of the nonrelativistic creation and annihilation operators a^+ and a^- (48). We see that the condition that there are no finite-difference differentiations in the right-hand side of (59) is:

$$Z(x)=0. \tag{62}$$

This gives us the relation connecting $q(x)$ and $\rho(x)$. For $Z(x)=0$, we have

$$[A^-,A^+]_{q(x)} = -\alpha(x)\,\sinh\,\frac{i}{2}\frac{d}{dx}\left[\alpha(x)\,\sinh\left(2\rho_{s/2}(x)\right)\right]. \tag{63}$$

Again, following the example of the nonrelativistic oscillator, we wonder whether it is possible that the expression in the right-hand side of (63) is equal to a constant as it was in the case of the nonrelativistic oscillator (see (50)):

$$[A^-,A^+]_{q(x)} = \text{const}. \tag{64}$$

This means that we shall consider (64) as an equation for $\rho(x)$. The remarkable fact is that this problem is solvable, the solution coincides with the nonrelativistic $\rho(x)$ (51) and obviously does not depend on the velocity of light.

The factor $\alpha(x)$ defined from (64) is equal to

$$\alpha(x) = \frac{1}{\cos\,\omega x/2c}. \tag{65}$$

The $a(x)$ and $q(x)$ (see (59)) are now constants:

$$a(x) = -\frac{\omega}{4}, \quad q(x) = \text{const} = q = \exp\left(-\frac{\omega}{4}\right),$$

or in dimensional units

$$q = \exp\left(-\frac{\omega\hbar}{4mc^2}\right), \tag{66}$$

and instead of the $q(x)$-mutator we have the q-mutator (cf. [22–25])

$$[A^-,A^+]_q = q\,A^-A^+ - q^{-1}A^{\div}A^-, \tag{67}$$

and this q-mutator is equal to a constant

$$[A^-,A^+]_q = 2(q^{-1}-q) = 4\sinh\,\frac{\omega}{4}. \tag{68}$$

The operators $A^\pm$ can be calculated now in an explicit form

$$A^\pm = \pm i\sqrt{2}\,\exp\left(\pm\frac{\omega}{8}\right)\left(\sinh\frac{i}{2}\frac{d}{dx} \mp i\tan\frac{\omega x}{2}\cosh\frac{i}{2}\frac{d}{dx}\right) = \\ \mp\frac{1}{\sqrt{2}}\exp\left(\pm\frac{\omega x^2}{2}\right)\hat{\mathcal{D}}\exp\left(-\frac{\omega x^2}{2}\right), \tag{69}$$

where

$$\hat{\mathcal{D}} = -\frac{2i}{\cos\,\omega x/2}\,\sinh\,\frac{i}{2}\frac{d}{dx}. \tag{70}$$

The Hamiltonian of the relativistic oscillator has the form [5]

$$\hat{h} = \frac{1}{2}\{A^-,\, A^+\}_q = \frac{1}{2}\left(q\, A^- A^+ + q^{-1} A^+ A^-\right) = 2\left(\hat{T}^2 - \cosh\frac{\omega}{4}\right), \quad (71)$$

where

$$\hat{T} = \frac{1}{\cos\omega x/2}\cosh\frac{i}{2}\frac{d}{dx} = q^{-N-1/2}, \quad (72)$$

and operator N, the analog of the occupation number, was introduced. In [5], relations (72) between Hamiltonian operator (71), finite-difference operator $\hat{T}$ and the «occupation number operator» N were established. It was also shown (see also [23]) that the eigenvalues of N are equal to $0, 1, 2, \ldots$ The q-mutator of the creation operator A^+ with the Hamiltonian reflects the integrability property of the problem considered

$$[A^+, \hat{h}]_{q^{-1}} = (q^2 - q^{-2})A^+. \quad (73)$$

The solution of the oscillator RSE

$$\hat{h}\Psi(x) = e\Psi(x) \quad (74)$$

corresponding to the nth energy level $e = e_n$ has the form

$$\Psi_n(x) = \exp\left(-\frac{\omega x^2}{2}\right) h_n(x), \quad (75)$$

where $h_n(x)$ are relativistic Hermite polynomials. We refer the reader for the details to [5] and write down here only the necessary relations for the h_n.

• The Rodrigues formula:

$$h_{n+1}(x) = \hat{R}\, h_n(x) = \left(\frac{1}{\sqrt{\omega}}\right) \mathrm{e}^{\omega x^2}\, \hat{\mathcal{D}}\, \mathrm{e}^{-\omega x^2}\, h_n(x). \quad (76)$$

It could be easily seen that operator $\hat{R}$ acting on *arbitrary* polynomial of nth degree gives a polynomial of $n+1$-th degree.

• The connection of the relativistic Hermite polynomials with the q-Hermite polynomials $H_n(z|q)$ (cf. [17–21]) is given by

$$h_n(x) \;=\; \left(\frac{4}{\omega}\right)^{n/2} q^{-\frac{n(n+1)}{4}}\, H_n\left(\sin\frac{\omega x}{2}\,|q\right), \quad q = \mathrm{e}^{-\omega/2}. \quad (77)$$

• The finite-difference equation:

$$\hat{K}\, h_n(x) = \exp\left[\frac{(n+1/2)\omega}{4}\right] h_n(x), \quad (78)$$

where

$$\hat{K} = \exp\left(\frac{\omega x^2}{2}\right)\hat{T}\,\exp\left(-\frac{\omega x^2}{2}\right). \quad (79)$$

4. THE INHOMOGENEOUS LATTICE, DOUBLE PERIODICITY OF RELATIVISTIC HAMILTONIAN AND ORTHOGONALITY CONDITIONS

We defined the factor $\alpha(x)$ as the solution of equation (64), but it is important to stress that it coincides with the characteristic lattice factor coming from the general theory of polynomials of the discrete variables (see [16]). It was shown in the theory of relativistic oscillator [5] that the wave functions, as the solutions of the finite-difference RSE, are expressed in terms of the relativistic Hermite polynomials with characteristic dependence on the lattice argument $\sin \omega x/2$. On the other hand, it was stressed in ([5]) that h_n-s are q-generalizations of the usual Hermite polynomials (see [18–20] and references therein). The explicit form of the natural argument of the relativistic Hermite polynomials is defined by the lattice. In other words, it is the lattice, which indicates the specific change of variables in finite-difference RSE for generalized Hermite polynomials.

In usual differential calculus, changing the argument of the function

$$f(x) = f(s(x)), \tag{80}$$

we have the following formula for the derivative in the new argument:

$$\frac{df}{ds} = \frac{df/dx}{ds/dx}. \tag{81}$$

Analogous formula for finite-difference differentiation has the form:

$$\sinh \frac{i}{2}\frac{d}{dx} f(x) = \frac{\sinh \frac{i}{2}\frac{d}{dx} f(s(x))}{\sinh \frac{i}{2}\frac{d}{dx} s(x)}. \tag{82}$$

Taking into account that

$$\sinh \frac{i}{2}\frac{d}{dx} \sin \frac{\omega x}{2} = i \sinh \frac{\omega}{4} \cos \frac{\omega x}{2}, \tag{83}$$

we convince that (70) is indeed the finite-difference differentiation operator in terms of lattice variable $\sin \omega x/2$.

Another important feature of RSE is that in the theory of relativistic oscillator, the lattice factor $\alpha(x)$ is also connected with the cutting factor of the oscillatory wave function or the ground state wave function $\mathrm{e}^{-(\omega x^2)/2}$. We see

$$\cosh \frac{i}{2}\frac{d}{dx}\, \mathrm{e}^{-(\omega x^2)/2} = \mathrm{e}^{\omega/8} \cos \frac{\omega x}{2}\, \mathrm{e}^{-(\omega x^2)/2} = \frac{1}{\alpha(x)}\, \mathrm{e}^{-\omega x^2/2}. \tag{84}$$

We see now clearly that lattice plays nontrivial role in the finite-difference relativistic quantum mechanics. But this is only the beginning of the story.

Maybe the most difficult problem connected with this change of variables in finite-difference RSE is to understand the role which plays the periodicity of the argument of relativistic Hermite polynomials $\sin\,\omega x/2$. In particular, this problem is reflected in the orthogonality condition for the wave functions of the relativistic oscillator (75).

This orthogonality condition is written as

$$\int\limits_{-\infty}^{\infty} h_n(x)\, h_m(x)\, \mu_1(x) dx = \delta_{nm} J_n, \tag{85}$$

where

$$\begin{gathered}\mu_1(x) = \mathrm{e}^{-\omega x^2}\, \cos\frac{\omega x}{2}, \\ J_n = \left(\frac{\pi}{\omega}\right)^{1/2} \left(\frac{8}{\omega}\right)^n \exp\left[\frac{\omega}{8}\left(n^2+n-\frac{1}{2}\right)\right] \prod_{j=1}^{n} \sinh\,\frac{\omega}{4} j.\end{gathered} \tag{86}$$

In fact, equation (85) is not the orthogonality relation, because the quantity $\cos\,\omega x/2$ is not positively definite and could not be considered as a measure. At the same time, the consideration of the infinite interval is necessary from the physical point of view, because this interval is one-dimensional physical space in which the Schrödinger equation is written. There is obvious contradiction between the probabilistic interpretation of the wave function in full physical space and the explicit form of the exact solutions of the Schrödinger equation. On the other hand, in the theory of the q-polynomials (see [18–20] and references therein) the orthogonality condition is written as integral over the finite interval $(-\pi/\omega,\, \pi/\omega)$ (where $\cos\,\omega x/2$ is positive definite)

$$\int\limits_{-\pi/\omega}^{\pi/\omega} h_n(x)\, h_m(x)\, \mu_2(x)\, dx = \delta_{nm}\, J_n \tag{87}$$

with different well-defined measure:

$$\begin{aligned}\mu_2(x) &= \Theta_4\left(2\pi i x,\, \mathrm{e}^{-4\pi^2/\omega}\right)\, \mathrm{e}^{-\omega x^2}\, \cos\frac{\omega x}{2} = \\ &= \Theta_4\left(2\pi i x,\, \mathrm{e}^{-4\pi^2/\omega}\right) \mu_1(x),\end{aligned} \tag{88}$$

where $\Theta_4(2\pi i x,\, \mathrm{e}^{-4\pi^2/\omega})$ is the *theta-function* [19].

Trying to understand the role of the inhomogeneous lattice $s(x) = \sin\,\omega x/2$, the nature of the change of variables (80), and the related problem of the orthogonality condition for the relativistic Hermite polynomial, we consider the periodicity properties of the Hamiltonian (71) for the relativistic oscillator and the corresponding properties of the oscillator wave functions. The Hamiltonian (71) has two different periodicity properties:

1. *The First Periodicity Property.* The step of finite-difference RSE (38) (as well as the step in the free equation (74)) is equal to $i/2$. This means that any function $\alpha(x)$, periodical with the period $i/2$, i.e.,

$$\alpha\left(x+\frac{i}{2}\right)=\alpha(x) \tag{89}$$

«is a constant in respect to RSE». In other words, if $\psi(x)$ is the solution of RSE, then $\alpha(x)\,\psi(x)$ is also the solution of the same equation. The step of difference operators $\hat{\mathcal{D}}$ (70), $\hat{T}$ (72), and $\hat{K}$ (79) also equals $i/2$.

2. *The Second Periodicity Property.* The Hamiltonian (71) is periodic operator, with the period $2\pi/\omega$:

$$\hat{h}\left(x+\frac{2\pi}{\omega},\,\cosh\,\frac{i}{2}\frac{d}{dx}\right)=\hat{h}\left(x,\,\cosh\,\frac{i}{2}\frac{d}{dx}\right). \tag{90}$$

The operators $\hat{\mathcal{D}}$ (70), $\hat{T}$ (72), and $\hat{K}$ (79) are quasi-periodic in respect to shifts by $2\pi/\omega$:

$$\hat{\mathcal{D}}\left(x+\frac{2\pi}{\omega},\sinh\,\frac{i}{2}\frac{d}{dx}\right)=-\hat{\mathcal{D}}\left(x,\sinh\,\frac{i}{2}\frac{d}{dx}\right), \tag{91}$$

$$\hat{T}\left(x+\frac{2\pi}{\omega},\cosh\,\frac{i}{2}\frac{d}{dx}\right)=-\hat{T}\left(x,\cosh\,\frac{i}{2}\frac{d}{dx}\right), \tag{92}$$

$$\hat{K}\left(x+\frac{2\pi}{\omega},\cosh\,\frac{i}{2}\frac{d}{dx}\right)=\hat{K}\left(x,\cosh\,\frac{i}{2}\frac{d}{dx}\right). \tag{93}$$

These periodicity properties of the oscillator RSE are very important. They actually define the characteristic inhomogeneous lattice $s(x)=\sin\omega x/2$, the specific dependence on this variable of the relativistic Hermite polynomials, the form of orthogonality conditions described earlier and the connection of the theory considered with Θ-functions.

The periodicity properties of the Hamiltonian are reflected in the measures $\mu_1(x)$ and $\mu_2(x)$ and relativistic Hermite polynomials $h_n(x)$:

1.

$$\mu_1\left(x+\frac{2\pi}{\omega}\right)=-\mathrm{e}^{-4\pi x}\,\mathrm{e}^{-4\pi^2/\omega}\,\mu_1(x). \tag{94}$$

2. The *theta-function* is quasi-doubly-periodic function of x [19]

$$\Theta_4\left(2\pi i\left(x+\frac{2\pi}{\omega}\right),\,\mathrm{e}^{-4\pi^2/\omega}\right)=-\mathrm{e}^{4\pi x}\,\mathrm{e}^{4\pi^2/\omega}\,\Theta_4\left(2\pi i x,\,\mathrm{e}^{-4\pi^2/\omega}\right), \tag{95}$$

$$\sinh \frac{i}{2}\frac{d}{dx}\Theta_4\left(2\pi ix,\, \mathrm{e}^{-4\pi^2/\omega}\right) = 0. \tag{96}$$

We write the i-periodicity condition (96) for $\Theta_4(2\pi ix,\, \mathrm{e}^{-4\pi^2/\omega})$ in the form, emphasizing that it is a «constant» in the finite-difference analysis:

$$\mu_2\left(x+\frac{2\pi}{\omega}\right) = \mu_2(x), \tag{97}$$

$$h_n\left(x+\frac{2\pi}{\omega}\right) = (-1)^n h_n(x). \tag{98}$$

The periodicity property (97) is the consequence of (94), (95) and definition (88). It is important that the multiplier in the quasi-periodicity property of cut-off factor $\mathrm{e}^{-\omega x^2/2}$ (75), defining the behavior of the oscillator's wave function at infinity, is just inverse of the quasi-periodicity multiplier of the *theta-function*.

Now we shall derive the orthogonality condition for the relativistic Hermite polynomials (87), using the RSE (74), its periodicity properties and well-known properties of the *theta-function* [19]. Let us rewrite equation (78) in the form

$$\exp\left(\frac{n+1/2}{4}\omega\right)\psi_n(x)\,\cos\frac{\omega x}{2} = \cosh\frac{i}{2}\frac{d}{dx}\,\psi_n(x). \tag{99}$$

We introduce the $i/2$ — periodic functions

$$\lambda(\pm x, \omega) = \prod_{n=1}^{\infty}\left(1-\exp\left[-\frac{4\pi^2}{\omega}(2n-1)\right]\mathrm{e}^{\mp 4\pi x}\right) \tag{100}$$

and transfer to new wave functions

$$\phi_n(x) = \lambda(x,\omega)\,\psi_n(x), \tag{101}$$

$$\widetilde{\phi}_n(x) = \lambda(-x,\omega)\,\psi_n(x). \tag{102}$$

The λ-functions transform under the $x \to x + 2\pi/\omega$ as follows:

$$\lambda\left(\left(x+\frac{2\pi}{\omega}\right),\omega\right) = \frac{-\mathrm{e}^{4\pi^2/\omega}\,\mathrm{e}^{4\pi x}}{\left(1-\mathrm{e}^{4\pi^2/\omega}\,\mathrm{e}^{4\pi x}\right)}\,\lambda(x,\omega), \tag{103}$$

$$\lambda\left(-\left(x+\frac{2\pi}{\omega}\right),\omega\right) = \left(1-\mathrm{e}^{4\pi^2/\omega}\,\mathrm{e}^{4\pi x}\right)\,\lambda(-x,\,\omega). \tag{104}$$

The new wave functions (101) and (102) obey the same equation (99) for initial wave function $\psi(x)$, because the i-periodic functions $\lambda(\pm x,\omega)$ could be inserted under the sign of the finite-difference differentiation in that equation. We have

$$\exp\left(\frac{n+1/2}{4}\omega\right)\phi_n(x)\,\cos\,\frac{\omega x}{2} = \cosh\,\frac{i}{2}\frac{d}{dx}\,\phi_n(x), \tag{105}$$

$$\exp\left(\frac{m+1/2}{4}\omega\right)\widetilde{\phi}_m(x)\,\cos\,\frac{\omega x}{2} = \cosh\,\frac{i}{2}\frac{d}{dx}\,\widetilde{\phi}_m(x). \tag{106}$$

Multiplying the first equation from the left by (102), the second one by (101), subtracting the second expression from the first one and integrating in the limits $(-\pi/\omega,\,\pi/\omega)$, we come to the following relation:

$$\left(\exp\left(\frac{2n+1}{8}\omega\right) - \exp\left(\frac{2m+1}{8}\omega\right)\right)\int\limits_{-\pi/\omega}^{\pi/\omega} h_n(x)\,h_m(x)\,\mu_2(x)\,dx =$$

$$= G^{-1}\int\limits_{-\pi/\omega}^{\pi/\omega}\Theta_4\left(2\pi i x,\,\mathrm{e}^{-4\pi^2/\omega}\right)R_{nm}(x,\omega), \tag{107}$$

where

$$R_{nm}(x,\omega) = \mathrm{e}^{-\omega x^2/2}\left(h_n(x)\,\cosh\,\frac{i}{2}\frac{d}{dx}\,\mathrm{e}^{-\omega x^2/2}\,h_m(x) - \right.$$

$$\left. - h_m(x)\,\cosh\,\frac{i}{2}\frac{d}{dx}\,\mathrm{e}^{-\omega x^2/2}\,h_n(x)\right). \tag{108}$$

The following relation was used [19]:

$$\Theta_4\left(2\pi i x,\,\mathrm{e}^{-4\pi^2/\omega}\right) = G\lambda(x,\omega)\,\lambda(-x,\omega), \tag{109}$$

where

$$G = \prod_{n=1}^{\infty}\left(1 - \exp\left(-\frac{8\pi^2\,n}{\omega}\right)\right). \tag{110}$$

The orthogonality condition for the relativistic Hermite polynomials in the form (87) will be proved, if we show that for $n \neq m$ the right-hand side of (107) vanishes. For $n-m$ odd, it is obvious due to the relation [5]

$$h_n(-x) = (-1)^n h_n(x). \tag{111}$$

For $n-m$ even, the only way is to use the finite-difference analogue of the partial integration, which is valid only for infinite limits of integration [5]. Let us consider the integral

$$J = \int\limits_{-\infty}^{\infty} R_{mn}(x)\,dx. \tag{112}$$

We can represent it as the sum

$$J = \sum_{k=-\infty}^{\infty} \int\limits_{2\pi/\omega(k-1/2)}^{2\pi/\omega(k+1/2)} R_{nm}(x)\,dx. \tag{113}$$

Changing the variable of integration in the kth integral

$$x \to x + \frac{2k\pi}{\omega},$$

we come to the expression

$$J = \sum_{k=-\infty}^{\infty} \int\limits_{-\pi/\omega}^{\pi/\omega} R_{nm}\left(x + \frac{2k\pi}{\omega}\right)\,dx. \tag{114}$$

It is easy to verify that $R_{nm}(x,\omega)$ obeys the following quasi-periodicity condition:

$$R_{nm}\left(x + \frac{2k\pi}{\omega}, \omega\right) = (-1)^k\,\mathrm{e}^{-4\pi xk}\,\exp\left(-\frac{4\pi^2 k^2}{\omega}\right) R_{nm}(x,\omega). \tag{115}$$

Substituting this to (114) and using the relation

$$\Theta_4\left(2\pi i x,\, e^{-4\pi^2/\omega}\right) = \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(-\frac{4\pi^2 k^2}{\omega}\right) \mathrm{e}^{-4\pi xk}, \tag{116}$$

we come to required equality

$$\int\limits_{-\pi/\omega}^{\pi/\omega} \Theta_4\left(2\pi i x,\, \mathrm{e}^{-4\pi^2/\omega}\right) R_{nm}(x,\omega) = \int\limits_{-\infty}^{\infty} R_{nm}(x,\omega)\,dx. \tag{117}$$

As was shown in [5], for $n \neq m$, the integral in right-hand side vanishes. Closing this section, it is worthwhile to stress that the doubly-periodicity properties of the oscillator RSE and the existence of two different orthogonality conditions (85) and (87) can have an interesting physical sense. The relation (117) shows that the essential part of the oscillator wave function is accumulated (confined) in a small region

$$\Delta = \left(-\frac{\pi c}{\omega}, \frac{\pi c}{\omega}\right). \tag{118}$$

Let us call it the confinement area. The wave function normalization integral (=1, the full probability) is saturated on this interval. This means that particle must be confined in this area. The values of wave function outside the confinement area could be obtained by periodicity, but there is no probabilistic interpretation of the relativistic wave function in all physical space ($-\infty < x < \infty$). We stress also that it is easily seen from (118) that in nonrelativistic limit the confinement area coincides with all physical space and all effect disappears.

5. DIFFERENT FACTORIZATIONS

The remarkable property of the nonrelativistic oscillator is the symmetry of its Hamiltonian (41) under the substitution

$$p \rightleftarrows x\omega. \tag{119}$$

In the relativistic case, the existence of this symmetry is not evident. Let us consider together with $A^{\pm}$ another couple of operators $B^{\pm}$ introduced by the following relations (for comparison we write them down together with $A^{\pm}$):

$$\begin{aligned} B^{+} &= \sqrt{2\delta_{-}}q^{N/2}\hat{a}^{+}, \quad A^{+} = \sqrt{2\delta_{-}}q^{-N/2}\hat{a}^{+}, \\ B^{-} &= \sqrt{2\delta_{-}}\hat{a}^{-}q^{N/2}, \quad A^{-} = \sqrt{2\delta_{-}}\hat{a}^{-}q^{-N/2}, \end{aligned} \tag{120}$$

where

$$\delta_{-} = q^{-1} - q. \tag{121}$$

Operator N is given by (72). The q-mutators of $A^{\pm}$ and $B^{\pm}$ are

$$[B^{-}, B^{+}]_{q^{-1}} = q^{-1}\, B^{-}B^{+} - q\, B^{+}B^{-} = 2\delta_{-}, \tag{122}$$

$$[A^{-}, A^{+}]_{q} = q\, A^{-}A^{+} - q^{-1}\, A^{+}A^{-} = 2\delta_{-}. \tag{123}$$

The operators $A^{\pm}$ and $B^{\pm}$ are connected by the relations

$$B^{+} = q^{N}\, A^{+}, \quad B^{-} = A^{-}\, q^{N}, \tag{124}$$

$$\begin{aligned} A^{\pm} &= \pm\frac{i\sqrt{2}}{\cos(\omega x/2)}\,\mathrm{e}^{\pm\omega/8}\left(\cos\frac{\omega x}{2}\sinh\frac{i}{2}\frac{d}{dx} \mp i\sin\frac{\omega x}{2}\cosh\frac{i}{2}\frac{d}{dx}\right), \\ B^{\pm} &= \mp\frac{i\sqrt{2}}{\cosh\dfrac{i}{2}\dfrac{d}{dx}}\,\mathrm{e}^{\mp\omega/8}\left(\cos\frac{\omega x}{2}\sinh\frac{i}{2}\frac{d}{dx} \mp i\sin\frac{\omega x}{2}\cosh\frac{i}{2}\frac{d}{dx}\right). \end{aligned} \tag{125}$$

The lattice factor for the operators B, which are inverse operators with respect to $\cosh(i/2)(d/dx)$, is better understood if we transfer to the conjugated χ-representation. Using (34) we can pass to conjugated variables (rapidities) in which the operators $B^{\pm}$ take the form

$$B^{\pm} = \mp\frac{i\sqrt{2}}{\cosh\chi/2}\,\mathrm{e}^{\pm\chi^2/2\omega}\sinh\frac{i}{2}\frac{d}{d\chi}\,\mathrm{e}^{\mp\chi^2/2\omega}. \tag{126}$$

The Hamiltonian for the creation and annihilation operators $B^{\pm}$ has the form

$$h_B = \frac{1}{2}\{B^{-}, B^{+}\}_{q^{-1}} = \frac{1}{2}(q^{-1}B^{-}B^{+} + qB^{+}B^{-}). \tag{127}$$

It could be easily seen that Hamiltonians (71) and (127) are not symmetric in respect to interchange (119). But now it is easy to find the symmetric Hamiltonian for the relativistic oscillator. It has the form (cf. [24])

$$h_s = \frac{1}{2}\{a^-, a^+\} = \frac{1}{2}(a^- a^+ + a^+ a^-) = \\ = \frac{1}{4\delta_-}(A^- B^+ + A^+ B^-) = \frac{1}{4\delta_-}(B^- A^+ + B^+ A^-). \quad (128)$$

We see that there are different possibilities to obtain factorized relativistic oscillator Hamiltonians. The origin of this fact lies in the finite-difference character of the relativistic momentum operator (37). Indeed, let us return for a moment to the nonrelativistic QM. The free energy operator is the second derivative which can be splitted in two factors by only one way, if we do not consider the fractional degrees of differentiation operators:

$$H_0 = \frac{p^2}{2m} = -\frac{1}{2m}\frac{d^2}{dx^2} = -\frac{1}{2m}\left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right).$$

In the relativistic case, the free Hamiltonian is the finite-difference operator given by (36). Using the definition of the hyperbolic function $\sinh z$:

$$\sinh \frac{i}{2}\frac{d}{dx} = \frac{1}{2}\left(\exp\left(\frac{i}{2}\frac{d}{dx}\right) - \exp\left(-\frac{i}{2}\frac{d}{dx}\right)\right)$$

and the relation

$$\exp\left(\pm\frac{i}{2}\frac{d}{dx}\right)\langle x|p\rangle = \mathrm{e}^{\mp\chi/2}\langle x|p\rangle,$$

we can write the free RSE in different forms:

$$\left(1 - \exp\left(-i\frac{d}{dx}\right)\right)^2 \langle x|p\rangle = 2\,\mathrm{e}^{\chi}\sinh^2\,\frac{\chi}{2}\langle x|p\rangle, \quad (129)$$

$$\left(1 - \exp\left(i\frac{d}{dx}\right)\right)^2 \langle x|p\rangle = 2\,\mathrm{e}^{-\chi}\sinh^2\,\frac{\chi}{2}\langle x|p\rangle, \quad (130)$$

$$\left(1 - \exp\left(i\frac{d}{dx}\right)\right)\left(1 - \exp\left(-i\frac{d}{dx}\right)\right)\langle x|p\rangle = -2\sinh^2\,\frac{\chi}{2}\langle x|p\rangle. \quad (131)$$

It is easily seen that (129)–(131) exhaust all the possibilities because we always must have in the left-hand side a term which does not contain the sift operators $\mathrm{e}^{\pm i(d/dx)}$. Let us consider the case (129). Acting by analogy with Sec. 2, we introduce the operators

$$\begin{aligned}\eta^{+} &= -\frac{i}{\sqrt{2}}\,\mathrm{e}^{\rho_1(x)}\left(1-\exp\left(-i\frac{d}{dx}\right)\right)\mathrm{e}^{-\rho_2(x)},\\ \eta^{-} &= \frac{i}{\sqrt{2}}\,\mathrm{e}^{-\rho_3(x)}\left(1-\exp\left(-i\frac{d}{dx}\right)\right)\mathrm{e}^{\rho_4(x)}\end{aligned} \tag{132}$$

and construct the $q(x)$-mutator, which has no shift operators $\mathrm{e}^{i(d/dx)}$ in its right-hand side. This is achieved by matching the $\rho_i(x)$ and $a(x)$. If we require that the right-hand side of this $q(x)$-mutator is a constant (the oscillator case), we obtain again the q-mutator rather than $q(x)$-mutator and can define $\rho_i(x)$. In fact, the subscripts of the exponents $\rho_i(x)$ stand for presence of imaginary parts of these functions. The linear parts of these exponents play here the role analogous to $\alpha(x)$ of Sec. 2 and are connected with lattice, which in turn is bound to the quadratic real part of $\rho_i(x)$. This function is the same for all i-s and coincides with $\rho(x)$ of (51).

So the function $\mathrm{e}^{\rho_i(x)}$ must be understood as the product of the ground state wave function and analogue of the lattice factor $\alpha(x)$. From the dimensional considerations, the requirement of having correct nonrelativistic limit, and the fact that term $(m\omega x^2)/2\hbar$ does not contain the velocity of light, we can conclude that $\rho_i(x)$ have the form

$$\rho_i(x) = \frac{\omega x^2}{2} + i\zeta_i x, \tag{133}$$

where ζ_i are some constants which we must define.

Again we construct the $q(x)$-mutator (cf. (59))

$$\left[\eta^{-},\eta^{+}\right]_{q(x)} = \eta^{-}q(x)\eta^{+} - \eta^{+}q^{-1}(x)\eta^{-}. \tag{134}$$

This is rather long expression

$$\begin{aligned}[\eta^{-},\eta^{+}]_{q(x)} = \frac{1}{2}\,&\mathrm{e}^{\rho_1+\rho_4-\rho_2-\rho_3}[\mathrm{e}^{-a}-\mathrm{e}^{a}]-\\ &-\left[\mathrm{e}^{-\rho_3}\exp\left(-i\frac{d}{dx}\right)\mathrm{e}^{\rho_1}\,\mathrm{e}^{a} - \mathrm{e}^{\rho_1}\exp\left(-i\frac{d}{dx}\right)\mathrm{e}^{-\rho_3}\,\mathrm{e}^{-a}\right]\mathrm{e}^{\rho_4(x)-\rho_2(x)}-\\ &-\mathrm{e}^{\rho_1(x)-\rho_3(x)}\left[\mathrm{e}^{\rho_4}\exp\left(-i\frac{d}{dx}\right)\mathrm{e}^{-\rho_2}\,\mathrm{e}^{a} - \mathrm{e}^{-\rho_2}\exp\left(-i\frac{d}{dx}\right)\mathrm{e}^{\rho_4}\,\mathrm{e}^{-a}\right]+\\ &+\mathrm{e}^{-\rho_3}\exp\left(-i\frac{d}{dx}\right)\mathrm{e}^{\rho_1+\rho_4}\,\mathrm{e}^{a}\exp\left(-i\frac{d}{dx}\right)\mathrm{e}^{-\rho_2}-\\ &-\mathrm{e}^{\rho_1}\exp\left(-i\frac{d}{dx}\right)\mathrm{e}^{-\rho_2-\rho_3}\,\mathrm{e}^{-a}\exp\left(-i\frac{d}{dx}\right)\mathrm{e}^{\rho_4}.\end{aligned} \tag{135}$$

As we wish to have a constant in the right-hand side of (134), it is clear that the first term in (135) is a constant different from zero, and all other terms (containing the shift operators $e^{-i(d/dx)}$ or $e^{-2i(d/dx)}$) must vanish. We have from the first part of this statement

$$a(x) = \text{const} \tag{136}$$

and

$$\rho_1(x) + \rho_4(x) - \rho_2(x) - \rho_3(x) = 0.$$

or, taking into account (133),

$$\zeta_1 + \zeta_4 - \zeta_2 - \zeta_3 = 0. \tag{137}$$

Terms containing $e^{-i\partial}$ and $e^{-2i\partial}$ must cancel separately. Inserting in these conditions the explicit expressions for $\rho_i(x)$ from (133) we obtain equations connecting a and ζ_i:

$$2a + 2\omega = (\zeta_1 + \zeta_3 - \zeta_2 - \zeta_4), \quad 2a = (\zeta_3 - \zeta_4), \quad 2a = (\zeta_1 - \zeta_2). \tag{138}$$

Equation (138) involves (137). Excluding ζ_1 and ζ_4 and introducing new notations

$$i\omega\xi = -\zeta_2, \quad i\omega\kappa = \zeta_3,$$

we come to the resulting expressions for $\eta^\pm$

$$\begin{aligned}\eta^+ &= \frac{i}{\sqrt{2}} \exp\left[\frac{\omega x^2}{2} - ix\omega(2+\xi)\right]\left[1 - \exp\left(-i\frac{d}{dx}\right)\right]\exp\left(-\frac{\omega x^2}{2} + ix\omega\xi\right),\\ \eta^- &= -\frac{i}{\sqrt{2}} \exp\left[-\frac{\omega x^2}{2} - ix\omega\kappa\right]\left[1 - \exp\left(-i\frac{d}{dx}\right)\right]\exp\left[\frac{\omega x^2}{2} + ix\omega(2+\kappa)\right].\end{aligned} \tag{139}$$

If we require that $(\eta^\pm)^\dagger = \eta^\mp$, then $\xi = \kappa$, and we have

$$\begin{aligned}\eta^+ &= \frac{i}{\sqrt{2}}\, e^{-i\omega x(\xi+2)} \left[1 - \exp\left(i\frac{\omega x}{2}\right)\exp\left(-i\frac{d}{dx}\right)\exp\left(i\frac{\omega x}{2}\right)\right] e^{i\omega x\xi},\\ \eta^- &= -\frac{i}{\sqrt{2}} e^{-i\omega x\xi} \left[1 - \exp\left(-i\frac{\omega x}{2}\right)\exp\left(-i\frac{d}{dx}\right)\exp\left(-i\frac{\omega x}{2}\right)\right] e^{i\omega x(\xi+2)}.\end{aligned} \tag{140}$$

In this case, the value of q is

$$\tilde{q} = e^{\omega}.$$

The q-mutator is

$$[\eta^-, \eta^+]_{\widetilde{q}} = \widetilde{q}\ \eta^-\eta^+ - \widetilde{q}^{-1}\ \eta^+\eta^- =$$
$$= \mathrm{e}^{\omega}\eta^-\eta^+ - \mathrm{e}^{-\omega}\eta^+\eta^- = \frac{1}{2}\left(\widetilde{q} - \widetilde{q}^{-1}\right) = \sinh\omega.$$

Another couple of the creation and annihilation operators $\gamma^{\pm}$ comes out if we start with the free Hamiltonian in the form (131):

$$\begin{aligned}
\gamma^+ &= \frac{i}{\sqrt{2}}\exp\left[-i(2+\zeta)\frac{d}{dx}\right]\times \\
&\quad\times\left[1-\exp\left(-i\frac{\omega x}{2}\right)\exp\left(i\frac{d}{dx}\right)\exp\left(-i\frac{\omega x}{2}\right)\right]\exp\left(i\zeta\frac{d}{dx}\right), \\
\gamma^- &= \frac{i}{\sqrt{2}}\exp\left(i\zeta\frac{d}{dx}\right)\times \\
&\quad\times\left[1-\exp\left(-i\frac{\omega x}{2}\right)\exp\left(-i\frac{d}{dx}\right)\exp\left(-i\frac{\omega x}{2}\right)\right]\exp\left[i(2-\zeta)\frac{d}{dx}\right].
\end{aligned} \tag{141}$$

The value of q is

$$\widetilde{\widetilde{q}} = \widetilde{q}^{-1} = \mathrm{e}^{-\omega},$$

and the q-mutator is

$$\left[\gamma^-, \gamma^+\right]_{\widetilde{q}^{-1}} = \widetilde{q}^{-1}\ \gamma^-\gamma^+ - \widetilde{q}\ \gamma^+\gamma^- =$$
$$= \mathrm{e}^{-\omega}\gamma^-\gamma^+ - \mathrm{e}^{\omega}\gamma^+\gamma^- = \frac{1}{2}\left(\widetilde{q} - \widetilde{q}^{-1}\right) = \sinh\omega.$$

The relation between $\eta^{\pm}$ and γ^+ is similar to the relation between $A^{\pm}$ and $B^{\pm}$ and realizes the symmetry between coordinates and momenta. For $\xi = 0$, operators $\eta^{\pm}$ coincide with finite-difference creation and annihilation operators b and $b^{\dagger}$ introduced in [23] (see there relations (30) and (31)).

6. DIMENSIONAL QUANTITIES

Trying to understand how deformations are related to physical objects, it is very instructive to analyze the dimensional quantities entering into the theory. The deformation parameter q is dimensionless. At the same time, we expect it to have physical meaning, i.e., q must depend on the dimensional parameters of the theory considered. This means that different parameters (at least two) of the same dimensionality must be inherent in the theory, entering into the operators, wave functions, etc. That could be the quantities of any dimensionality. Here, for us it is most natural to consider the lengthes. For example, the q plane structure

$$XY = qYX \tag{142}$$

could be realized by simple operators

$$X = \exp\left(A\frac{d}{dx}\right), \quad Y = \mathrm{e}^{x/B}, \quad q = \mathrm{e}^{A/B}, \tag{143}$$

where A and B are constants of the dimension of length.

In the case of nonrelativistic oscillator, we have only one quantity of the dimension of length:

$$l = \sqrt{\frac{\hbar}{m\omega}}. \tag{144}$$

And there is no possibility for a structure like (142), (143) to arise in this case. Transferring to the relativistic theory we acquire another (second) length, the Compton wave length of the particle:

$$\lambda_0 = \frac{\hbar}{mc} \tag{145}$$

and obviously an infinite number of quantities of the same dimensionality of length which could be constructed from (144) and (145). For us important will be the quantity

$$\lambda = \frac{l^2}{\lambda_0} = \frac{c}{\omega}. \tag{146}$$

As we see, the typical objects of the finite-difference RQM are (37) or

$$\exp\left(\frac{i\hbar}{2mc}\frac{d}{dx}\right)$$

and «lattice objects»:

$$\sin\frac{\omega x}{2c}, \quad \cos\frac{\omega x}{2c}, \quad \text{or} \quad \exp\left(i\frac{\omega x}{2c}\right).$$

We can consider the relations like (142), (143) with

$$A = \frac{i\hbar}{2mc} = \frac{i\lambda_0}{2}, \quad B = -i\frac{2c}{\omega} = -2i\lambda,$$
$$q = \mathrm{e}^{A/B} = \mathrm{e}^{-\lambda_0/4\lambda} = \exp\left(-\frac{\omega\hbar}{4mc^2}\right).$$

So the q-plane structure is inherent in this theory from the beginning.

It is instructive to write down the spectra, corresponding to Hamiltonians h (71), h_B (127) and h_s (128), notations are evident, dimensional quantities were restored:

$$e_n = 2mc^2 \left(\exp\left(\frac{2n+1}{4} \frac{\omega\hbar}{mc^2} \right) - \cosh \frac{\omega\hbar}{4mc^2} \right), \tag{147}$$

$$e_{nB} = 2mc^2 \left(-\exp\left(-\frac{2n+1}{4} \frac{\omega\hbar}{mc^2} \right) + \cosh \frac{\omega\hbar}{4mc^2} \right), \tag{148}$$

$$e_{ns} = 2\omega\hbar \frac{\sinh{(\omega\hbar/4mc^2)(n+1/2)}}{\sinh{(\omega\hbar/8mc^2)}}. \tag{149}$$

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