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TIME-REPARAMETRIZATION-INVARIANT DYNAMICS OF A RELATIVISTIC STRING

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1. Introduction

The problem of the reparametrization-invariant description of relativistic systems (particles, string, n-branes, general relativity) has a long history [1, 2, 3, 4, 5]. There are two opposite solutions of this problem in the generalized Hamiltonian approach [6, 7, 8]: the reduction of the extended phase space by the fixation of gauge which breaks reparametrization-invariance from very beginning [1, 4] and the reduction of an action by the explicit resolving of the first class constraints so that one of variables of the extended phase space (with a negative contribution to the energy constraint) converts into the invariant evolution parameter, and its conjugate momentum becomes the invariant Hamiltonian of evolution [2, 3, 5]. The application of the invariant reduction of extended actions in cosmology and general relativity [3, 5] allows one to formulate the dynamics of reparametrization-invariant systems directly in terms of the proper time (as the time-reparametrization-invariant functional) with the nonzero Hamiltonian of evolution (instead of the non-invariant coordinate time with the generalized zero Hamiltonian of evolution in the gauge-fixing method).

An important element of the invariant reduction is the Levi-Civita - Shanmugadhasan canonical transformation [9, 10] that linearizes the energy constraint as the generator of reparametrizations of the coordinate time.

In the present paper, we apply the method of the invariant Hamiltonian reduction (with resolving the first class constraints and the Levi-Civita - Shanmugadhasan canonical transformations) to express dynamics of a relativistic string [11, 12] in terms of the proper time and to consider the correspondence of this dynamics to the unitary representations of the Poincare group [13].

The content of the paper is the following. Section 2 is devoted to the generalized Hamiltonian formulation of a relativistic string and the statement of the problem. In Section 3, local excitations are separated from the "center of mass" coordinates of the string. In Section 4, the Levi-Civita transformations and the invariant Hamiltonian reduction are performed to resolve the global constraint and to convert the time-like variable of the global motion into the proper time. In Section 5, the dynamics of local excitations is described in terms of the proper time. Section 6 is devoted to the generating functional for the Green functions.

2. The generalized Hamiltonian formulation

2.1. Constrained systems

We begin with the action for a relativistic string in the geometrical form [14]

$$W = -\frac{\gamma}{2}\int d^2u\sqrt{-g}g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x_\mu, \qquad u_\alpha = (u_0, u_1) \tag{1}$$

where the variables x_μ are string coordinates given in a space-time with a dimension D and the metric ($x_\mu x^\mu := x_0^2 - x_i^2$); $g_{\alpha\beta}$ is a second-rank metric tensor given in the two-dimensional Riemannian space $u_\alpha = (u_0, u_1)$.

The Hamiltonian scheme is based on the Dirac-Arnovitt-Deser-Misner parametrization of the two-dimensional metric

$$g_{\alpha,\beta} = \Omega^2 \begin{pmatrix} \lambda_1^2 - \lambda_2^2 & \lambda_2 \\ \lambda_2 & -1 \end{pmatrix}, \qquad \sqrt{-g} = \Omega^2\lambda_1 \tag{2}$$

with the invariant interval [15]

$$ds^2 = g_{\alpha\beta}du^\alpha du^\beta = \Omega^2[\lambda_1^2 d\tau^2 - (d\sigma + \lambda_2 d\tau)^2], \qquad u_\alpha = (u_0 = \tau, u_1 = \sigma) \tag{3}$$

where λ_1 and λ_2 are known in general relativity (GR) as the lapse function and shift " vector", respectively [16, 17]. The action (1) after the substitution (3) does not depend on the conformal factor Ω and takes the form

$$W = -\frac{\gamma}{2}\int\limits_{\tau_1}^{\tau_2} d\tau \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \left[\frac{(D_\tau x)^2}{\lambda_1} - \lambda_1 x'^2\right] \tag{4}$$

where

$$D_\tau x_\mu = \dot{x}_\mu - \lambda_2 x'_\mu \qquad (\dot{x} = \partial_\tau x, \ x' = \partial_\sigma x) \tag{5}$$

is the covariant derivative with respect to the two-dimensional metric (3). The metric (3), the action (4), and the covariant derivative (5) are invariant under the transformations (see Appendix A)

$$\tau \Rightarrow \tilde{\tau} = f_1(\tau), \qquad \sigma \Rightarrow \tilde{\sigma} = f_2(\tau, \sigma). \tag{6}$$

A similar group of transformation in GR is well-known as the "kinemetric" group of diffeomorphisms of the Hamiltonian description [18].

The variation of action (4) with respect to λ_1 and λ_2 leads to the equations

$$\frac{\delta W}{\delta\lambda_2} = \frac{x' D_\tau x}{\lambda_1} = 0 \ \Rightarrow \ \lambda_2 = \frac{\dot{x}x'}{x'^2}; \tag{7}$$

$$\frac{\delta W}{\delta\lambda_1} = \frac{(D_\tau x)^2}{\lambda_1^2} + x'^2 = 0 \ \Rightarrow \ \lambda_1^2 = \frac{(\dot{x}x')^2 - \dot{x}^2 x'^2}{(x'^2)^2}.$$

The solutions of these equations convert the action (4) into the standard Nambu-Gotto action of a relativistic string [12, 19]

$$W = -\gamma \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}.$$

The generalized Hamiltonian form [6] is obtained by the Legendre transformation [8] of the action (4)

$$W = \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \left(-p_\mu D_\tau x^\mu + \lambda_1 \phi_1\right) = \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \left(-p_\mu \dot{x}^\mu + \lambda_1 \phi_1 + \lambda_2 \phi_2\right), \tag{8}$$

where

$$\phi_1 = \frac{1}{2\gamma}[p_\mu^2 + (\gamma x'_\mu)^2], \qquad \phi_2 = x'^\mu p_\mu, \tag{9}$$

and the generalized Hamiltonian

$$\mathcal{H} = \lambda_1 \phi_1 + \lambda_2 \phi_2 \tag{10}$$

is treated as the generator of evolution with respect to the coordinate time τ, and λ_1, λ_2 play the role of variables with the zero momenta

$$P_{\lambda_1} = 0, \qquad P_{\lambda_2} = 0 \tag{11}$$

considered as the first class primary constraints [6, 8]. The equations for λ_1, λ_2

$$\frac{\delta W}{\delta \lambda_1} = \phi_1 = 0; \qquad \frac{\delta W}{\delta \lambda_2} = \phi_2 = 0 \tag{12}$$

are known as the first class secondary constraints [6, 8]. The Hamiltonian equations of motion take the form

$$\frac{\delta W}{\delta x^\mu} = \dot{p}_\mu - \partial_\sigma[\gamma \lambda_1 x'_\mu + \lambda_2 p_\mu] = 0, \qquad \frac{\delta W}{\delta p^\mu} = p_\mu - \gamma \frac{D_\tau x_\mu}{\lambda_1} = 0 \tag{13}$$

The problem is to find solutions of the Hamiltonian equations of motion (13) and constraints (12) which are invariant with respect to the kinemetric transformations (6).

2.2. Gauge-fixing method

The standard method of solving the problem is to fix of the second class constraints (i.e. gauges) that accompany the first class constraints (11), (12). The primary first class constraints (11) are accompanied by the following orthonormal gauge

$$\chi_1^{(1)} = \lambda_1 = 1, \qquad \chi_2^{(1)} = \lambda_2 = 0; \tag{14}$$

it breaks the kinemetric symmetry (6) from the beginning. In this case, the classical equations of motion (13) reduce to the D'Alambert ones for x_μ

$$\partial_\tau^2 x_\mu - \partial_\sigma^2 x_\mu = 0 \tag{15}$$

with definite boundary conditions. We choose here the case of an open string $x'(\tau,\sigma_{1,2}) = 0,\ \sigma_1 = 0, \sigma_2 = \pi$ with the solution [12]

$$x_\mu(\tau,\sigma) = X_\mu(\tau) + \xi_\mu(\tau,\ \sigma) = Q_\mu + \frac{P_\mu}{\pi\gamma}\tau + \frac{1}{2\sqrt{\pi\gamma}}\left[\psi_\mu(\tau+\sigma) + \psi_\mu(\tau-\sigma)\right]) \ , \tag{16}$$

where ψ_μ are periodic functions with a period 2π with the Fourier series

$$\psi_\mu(z) = i\sum_{n\neq 0} \exp(-inz)\frac{\alpha_{n\mu}}{n} \ , \quad \alpha_{-n,\mu} = \alpha^*_{n,\mu} \ . \tag{17}$$

The quantity $X_\mu(\tau) = Q_\mu + \tau P_\mu/\pi\gamma$ represents the "center of mass" coordinate

$$X_\mu(\tau) = \frac{1}{\pi}\int_0^\pi d\sigma x_\mu(\tau,\sigma); \tag{18}$$

whereas ξ_μ are local excitations given in the class of functions with the nonzero Fourier harmonics

$$\int_0^\pi d\sigma\xi_\mu(\tau,\ \sigma) = \frac{1}{2\sqrt{\pi\gamma}}\int_0^\pi d\sigma[\psi_\mu(\tau+\sigma) + \psi_\mu(\tau-\sigma)] = 0 \ . \tag{19}$$

The initial momenta P_μ and the function ψ_μ satisfy the first class secondary constraints (12)

$$\phi_1 \pm \phi_2 = 0 \ \Rightarrow \ [\underline{P}_\mu + \psi'_\mu(\tau\pm\sigma)]^2 = 0, \qquad \underline{P}_\mu = \frac{P_\mu}{\sqrt{\pi\gamma}} = \alpha_{0\mu} \ . \tag{20}$$

The Fourier series of these constraints are written as

$$\phi_1 \pm \phi_2 = -\frac{1}{\pi}\sum_{n=-\infty}^{+\infty} L_n \exp(-iz_\pm n) = 0, \qquad (z_\pm = \tau \pm \sigma) \tag{21}$$

with the Fourier coefficients

$$L_0 = -\frac{\underline{P}^2}{2} + \bar{L}_0 = 0, \qquad L_{n\neq 0} = -\underline{P}_\mu \alpha_n^\mu + \bar{L}_n = 0 \ , \tag{22}$$

where $\bar{L}_n$ are the contributions of the nonzero harmonics

$$\bar{L}_0 = -\frac{1}{2}\sum_{k\neq 0}\alpha_{k\mu}\alpha^\mu_{-k} \ , \quad \bar{L}_{n\neq 0} = -\frac{1}{2}\sum_{k\neq 0,n}\alpha_{k\mu}\alpha^\mu_{n-k} \ . \tag{23}$$

and $\bar{L}_0$, according to (22), is identified with the mass of a string

$$M^2 = -2\pi\gamma\bar{L}_0 = -\frac{\gamma}{2}\int_0^\pi d\sigma\left[(\psi'_\mu(z_+))^2 + (\psi'_\mu(z_-))^2\right] . \tag{24}$$

In quantum theory, the Fourier coefficients L_n (22) form the closed Virasoro algebra with the conformal anomaly [11, 12].

The first class secondary constraints (20) are accompanied, as was mentioned above, by the second class gauge constraints chosen, for example, in the form [12, 7]

$$\chi_1^{(2)} = n_\mu\xi^\mu = 0, \qquad \chi_2^{(2)} = n_\mu\pi^\mu = 0, \tag{25}$$

where n_μ is an arbitrary time-like vector. (The algebra of the secondary constraints $\chi_{1,2}^{(2)}, \phi_{1,2}$ (20), (25) is given in [7].)

There is the problem of the solution (16) of the linearized "gauge-fixing" equation (15) in terms of the evolution parameter τ (as the object reparametrizations in the initial theory) being adequate to the initial kinemetric invariant and relativistic invariant system. In particular, the constraints (20), (25) mix the global motion of the "center of mass" coordinates with local excitations of a string ξ_μ, which contradicts to the relativistic invariance of internal degrees of freedom of a string. In this context, it is worth to clear up a set of questions: Is it possible to introduce the reparametrization-invariant evolution parameter for the string dynamics, instead of the non-invariant coordinate time (τ) used as the evolution parameter in the gauge-fixing method? Is it possible to construct the observable nonzero Hamiltonian of evolution of the "center of mass" coordinates? What is relation of the "center of mass" evolution to the unitary representations of the Poincare group?

2.3. Invariant Hamiltonian reduction

To answer the above-mentioned questions for a relativistic string, we use the time-reparametrization-invariant Hamiltonian reduction [3].

To illustrate this invariant reduction and its difference from the gauge-fixing method, let us consider a reparametrization-invariant form of a classical mechanical system [3]

$$W = \int_{\tau^1}^{\tau_2} d\tau\left(p\dot{q} - \Pi_0\dot{Q}_0 - \lambda[-\Pi_0 + H(p,q)]\right). \tag{26}$$

We show (using nothing but the equations of motion) that this system is completely equivalent to the conventional mechanics in the *reduced phase space*

$$W^M = \int_{T(\tau_1)=T_1}^{T(\tau_2)=T_2} dT\left(p\frac{dq}{dT} - H(p,q)\right). \tag{27}$$

in terms of the proper time T defined as

$$dT := \lambda d\tau, \qquad T = \int\limits_0^\tau d\tau' \lambda(\tau') \tag{28}$$

and invariant with respect to reparametrizations of the *coordinate time*

$$\tau \to \tau' = \tau'(\tau), \qquad \lambda \to \lambda' = \lambda \frac{d\tau}{d\tau'} \,. \tag{29}$$

The problem is to obtain the evolution of the physical variables in the invariant parameter T, or to present the effective action (of the equivalent unconstrained theory) directly in terms of T, the equations of which reproduce this evolution. The solution of this problem will be called the *invariant Hamiltonian reduction.*

The invariant Hamiltonian reduction is the explicit resolving of three equations of the extended system (26) i) for the variable λ

$$\frac{\delta W}{\delta \lambda} = -\Pi_0 + H(p,q) = 0 \tag{30}$$

ii) for the momentum Π_0 with a negative contribution to the constraint (30)

$$\frac{\delta W}{\delta \Pi_0} = 0 \Rightarrow \frac{dQ_0}{d\tau} = \lambda \,, \tag{31}$$

and iii) for its conjugate variable Q_0

$$\frac{\delta W}{\delta Q_0} = \frac{d\Pi_0}{d\tau} = 0 \tag{32}$$

The resolving of the constraint (30) expresses the "ignorable" momentum Π_0 through $H(p,q)$ with a positive value $\Pi_0 = H(p,q) > 0$.

The second equation (31) identifies the *dynamic evolution parameter* Q_0 with the proper time (28) $Q_0 = T$. It is not the gauge but the invariant solution of the equation of motion (31). The third equation (32) is the conservation law.

As a result of the invariant reduction (i.e., a result of the substitution of $\Pi_0 = H$ and $Q_0 = T$ into 26) the initial action (26) is reduced to the action of the conventional mechanics (27) in terms of the proper time T where the role of the nonzero Hamiltonian of evolution in the proper time T is played by the constraint-shell value of the "ignorable" momentum $\Pi_0 = H(p,q)$. In other words, this constraint-shell action $W(\text{constraint}) = W^M$ determines the nonzero Hamiltonian $H(p,q)$ in the proper time T, instead of the zero generalized Hamiltonian in the coordinate time τ in (26) $\lambda(-\Pi_0 + H)$. Thus, the equivalent unconstrained system was constructed without any additional constraint of the type: $\lambda = 1$, $\tau = T$.

Now we can compare the "gauge-fixing" $\lambda = 1$, $\tau = T$ with the gaugeless invariant reduction: first of all, there is the contradiction of the gauge-fixing identification of the coordinate time and proper time $T = \tau$ with the difference of their Hamiltonians $\lambda(-\Pi_0 + H) \neq H(p,q)$. The second difference is more essential, namely, the formulation of the theory in terms of the invariant proper time (28) is achieved by the explicit resolving of the constraint (30) and equation of motion (31), as a result of which "ignorable" variables Π_0, Q_0 are excluded from the initial action (26) and from the phase space.

In the present paper, we apply the invariant Hamiltonian reduction to a relativistic string.

3. The separation of the "center of mass" coordinates

To apply the reparametrization-invariant Hamiltonian reduction discussed before to a relativistic string, one should define the *proper time* in the form of the reparametrization-invariant functional of the lapse function (of type (28)), and to point out, among the variables, a *dynamic evolution parameter*, the equation of which identifies it with the proper time of type (31). We identify this *dynamic evolution parameter* with the time-like variable of the center of mass of a string (18) defined as the total coordinate

$$X_\mu(\tau) = \frac{1}{l(\tau)} \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma x_\mu(\tau,\sigma), \qquad l(\tau) = \sigma_2(\tau) - \sigma_1(\tau). \tag{33}$$

Therefore, the invariant reduction requires to separate the "center of mass" variables before variation of the action. We consider this separation on the level of the action (4) which after the substitution

$$x_\mu(\tau,\ \sigma) = X_\mu(\tau) + \xi_\mu(\tau,\ \sigma), \qquad x'_\mu(\tau,\ \sigma) = \xi'_\mu(\tau,\ \sigma) \tag{34}$$

takes the form

$$W = -\frac{\gamma}{2}\int\limits_{\tau_1}^{\tau_2} d\tau \left\{ \frac{\dot{X}^2 l(\tau)}{N_0(\tau)} + 2\dot{X}_\mu \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \frac{D_\tau \xi^\mu}{\lambda_1} + \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \left(\frac{(D_\tau \xi)^2}{\lambda_1} - \lambda_1 \xi'^2 \right) \right\}, \tag{35}$$

where the global lapse function $N_0(\tau)$ is defined as the functional of $\lambda_1(\tau,\sigma)$

$$\frac{1}{N_0[\lambda_1]} = \frac{1}{l(\tau)} \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \frac{1}{\lambda_1(\tau,\sigma)}. \tag{36}$$

To exclude the superfluous coordinates and momenta, the local variables ξ_μ are given (according to (33) and (34)) in the class of functions (with the nonzero

Fourier harmonics) which satisfy the conditions

$$\int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \xi_\mu = 0. \tag{37}$$

A definition of the conjugate momenta is consistent with (33) and the equation for the momentum p_μ (13) if the local momentum is given in the same class (37)

$$\int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \frac{D_\tau \xi^\mu}{\lambda_1} = 0 \ . \tag{38}$$

Then we get

$$P_\mu = \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma p_\mu(\tau, \sigma) = \frac{\delta W}{\delta \dot{X}^\mu} = -\gamma \frac{\dot{X}_\mu l}{N_0}, \quad \pi_\mu = \frac{\delta W}{\delta \dot{\xi}^\mu} = \gamma \frac{D_\tau \xi^\mu}{\lambda_1}. \tag{39}$$

This separation conserves the group of diffeomorphisms of the Hamiltonian [3] and leads to the Bergmann-Dirac generalized action

$$W = \int_{\tau_1}^{\tau_2} d\tau \left[\left(\int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma [-\pi_\mu D_\tau \xi^\mu - \lambda_1 \mathcal{H}] \right) - P_\mu \dot{X}^\mu + N_0 \frac{P_\mu^2}{2\bar{\gamma}} \right], \quad (\bar{\gamma} = \gamma l(\tau)) \tag{40}$$

where $\mathcal{H}$ is the Hamiltonian of local excitations

$$\mathcal{H} = -\frac{1}{2\gamma}[\pi_\mu^2 + (\gamma \xi_\mu')^2] \ . \tag{41}$$

The variation of the action (40) with respect to λ_1 results in the equation

$$\frac{\delta W}{\delta \lambda_1} = \mathcal{H} - \left(\frac{1}{l \bar{\lambda}_1^2} \right) \frac{P^2}{2\bar{\gamma}} = 0, \tag{42}$$

where

$$\bar{\lambda}_1(\tau, \sigma) = \frac{\lambda_1(\tau, \sigma)}{N_0(\tau)} \tag{43}$$

is the reparametrization-invariant component of the local lapse function. Here we have used the variation of the functional $N_0[\lambda_1]$ (36)

$$\frac{\delta N_0[\lambda_1]}{\delta \lambda_1} = \frac{1}{l(\tau) \bar{\lambda}_1^2}.$$

In accordance with our separation of dynamic variables onto the global and local sectors, the first class constraint (42) has two projections onto the global sector (zero Fourier harmonic) and the local one. The global part of the constraint (42) can be obtained by variation of the action (40) with respect to N_0 (after the substitution of (43) into (40))

$$\frac{\delta W}{\delta N_0} = \frac{P^2}{2\bar{\gamma}} - H^R = 0, \qquad H^R = \int\limits_{\sigma_1}^{\sigma_2} d\sigma \bar{\lambda}_1 \mathcal{H} \ , \tag{44}$$

or, in another way, by the integration of (42) multiplied by λ_1. Then, the local part of the constraint (42) can be obtained by the substitution of (44) into (42)

$$\bar{\lambda}_1 \mathcal{H} - \frac{1}{l\bar{\lambda}_1} \int\limits_{\sigma_1}^{\sigma_2} d\sigma \bar{\lambda}_1 \mathcal{H} = 0. \tag{45}$$

The integration of the local part over σ is equal to zero if we take into account the normalization of the local lapse function

$$\frac{1}{l(\tau)} \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \frac{1}{\bar{\lambda}_1} = \mathcal{N}^{-1} = 1 \ . \tag{46}$$

This follows from the definition of the global lapse function (36). We see that the local part (45) takes the form of an integral operator, orthogonal to the operator of integration over σ.

Finally, we can represent the action (40) in the equivalent form

$$W = \int\limits_{\tau_1}^{\tau_2} d\tau \left[\left(\int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma [-\pi_\mu D_\tau \xi^\mu] \right) - P_\mu \dot{X}^\mu - N_0 \left(-\frac{P_\mu^2}{2\bar{\gamma}} + \mathcal{N} H^R\right) \right] , \tag{47}$$

where the global lapse function N_0 and the local one $\bar{\lambda}_1$ are treated as independent variables, with taking the normalization (46) into account after the variation.

The invariant proper time T measured by the watch of an observer in the "center of mass" frame of a string is given by the expression (according to (6) and (36))

$$\sqrt{\bar{\gamma}} dT := N_0 d\tau = d\bar{\tau}, \quad \bar{\tau} = \int\limits_0^\tau d\tau' \left[\frac{1}{l(\tau')} \int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \frac{1}{\bar{\lambda}_1(\tau', \sigma)} \right]^{-1} . \tag{48}$$

We include the constant $\sqrt{\bar{\gamma}}$ to provide the dimension of the time measured by the watch of an observer.

Now we can see from (47) that the dynamics of the local degrees of freedom π, ξ, in the class of functions of nonzero harmonics (37), is described by the same kinemetric invariant and relativistic covariant equations (13) where x, p

are changed by ξ, π, with the set of the first class (primary and secondary) constraints

$$P_{\lambda_1} = 0, \quad P_{\lambda_2} = 0, \quad \pi_\mu \xi'^\mu = 0, \quad \bar{\lambda}_1 \mathcal{H} - \frac{1}{l\bar{\lambda}_1} \int_{\sigma_1}^{\sigma_2} d\sigma \bar{\lambda}_1 \mathcal{H} = 0. \tag{49}$$

We see that the separation of the "center of mass" (CM) variables on the level of the action removes the interference terms which mix the CM variables with the local degrees of freedom; as a result, the new local constraints (49) do not depend on the total momentum P_μ, in contrast to the standard ones (20). In other words, there is the problem: when can one separate the CM coordinates of a relativistic string; before the variation of the action or after the variation of the action? The relativistic invariance dictates the first one, because an observer in the CM frame (which is the preferred frame for a string) cannot measure the total momentum of the string.

The first class local constraints (49) can be supplemented by the second class constraints (25)

$$\bar{\lambda}_1 - 1 = 0, \qquad \lambda_2 = 0, \qquad n^\mu \xi_\mu = 0, \qquad n^\mu \pi_\mu = 0, \qquad n_\mu = (1,0,0,0) \tag{50}$$

so that the equations of the local constraint-shell action

$$W(\text{loc.constrs.}) = \int_{\tau_1}^{\tau_2} d\tau \left[\left(\int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \pi_i \dot{\xi}_i \right) - P_\mu \dot{X}^\mu - N_0 \left(-\frac{P_\mu^2}{2\bar{\gamma}} + H^R \right) \right] \tag{51}$$

coincide with the complete set of equations and the same constraints (49), (50) of the extended action, i.e., the operations of constraining and variation commute. The substitution of the global constraint (44) with $\bar{\lambda}_1 = 1$ into the action (51) leads to the constraint-shell action

$$W_\pm^D = \int_{X_0(\tau_1)}^{X_0 \tau_2} dX_0 \left[\left(\int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \pi_i \frac{d\xi_i}{dX_0} \right) + P_i \frac{dX_i}{dX_0} \mp \sqrt{P_i^2 + 2\bar{\gamma} H^R} \right]. \tag{52}$$

This action describes the dynamics of a relativistic string with repect to the time measured by an observer in the rest frame with the physical nonzero Hamiltonian of evolution. However, in this system, the local equations become nonlinear. To overcome this difficulty, we pass to the "center of mass" frame.

4. Levi-Civita geometrical reduction

To express the dynamics of a relativistic string in terms of the proper time (48) measured by an observer in the comoving (i.e. "center of mass") frame, we use the Levi-Civita-type canonical transformations [9, 20]

$$(P_\mu, X_\mu) \Rightarrow (\Pi_\mu, Q_\mu);$$

they convert the global part of the constraint (44) into a new momentum Π_0

$$\Pi_0 = \frac{1}{2\bar\gamma}[P_0^2 - P_i^2], \quad \Pi_i = P_i, \quad Q_0 = X_0 \frac{\bar\gamma}{P_0}, \quad Q_i = X_i - X_0 \frac{P_i}{P_0}. \tag{53}$$

The inverted form of these transformations is

$$P_0 = \pm\sqrt{2\bar\gamma\Pi_0 + \Pi_i^2}, \quad P_i = \Pi_i, \quad X_0 = \pm Q_0 \frac{\sqrt{2\bar\gamma\Pi_0 + \Pi_i^2}}{\bar\gamma}, \quad X_i = Q_i + Q_0 \frac{\Pi_i}{\bar\gamma}. \tag{54}$$

As a result of transformations (53), the extended action (47) in terms of the Levi-Civita geometrical variables takes the form (compare with (26))

$$W = \int\limits_{\tau_1}^{\tau_2} d\tau \left[\left(\int\limits_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma [-\pi_\mu D_\tau \xi^\mu] \right) - \Pi_\mu \dot{Q}^\mu - N_0(-\Pi_0 + H^R) - \frac{d}{d\tau}(Q_0 \Pi_0) \right]. \tag{55}$$

The Hamiltonian reduction means to resolve constraint (44) with respect to the momentum Π_0

$$\frac{\delta W}{\delta N_0} = 0 \Rightarrow \Pi_0 = H^R. \tag{56}$$

The equation of motion for the momentum Π_0

$$\frac{\delta W}{\delta \Pi_0} = 0 \Rightarrow \frac{dQ_0}{d\tau} = N_0 \qquad (i.e., dQ_0 = N_0 d\tau := d\bar\tau) \tag{57}$$

identifies (according to our definition (48)) the new variable Q_0 with the proper time $\bar\tau = \sqrt{\gamma}T$, whereas the equation for Q_0

$$\frac{\delta W}{\delta Q_0} = 0 \Rightarrow \frac{d\Pi_0}{d\tau} = 0, \qquad i.e., \frac{dH^R}{dT} = 0, \tag{58}$$

in view of (56), gives us the conservation law.

Thus, resolving the global energy constraint $\Pi_0 = H^R$, we obtain, from (55), the reduced action for a relativistic string in terms of the proper time $Q_0 = \bar\tau$

$$W^G = \int\limits_{\bar\tau_1}^{\bar\tau_2} d\bar\tau \left[\left(\int\limits_{\sigma_1}^{\sigma_2} d\sigma [-\pi_\mu D_{\bar\tau} \xi^\mu] \right) + \Pi_i \frac{dQ_i}{d\bar\tau} - H^R - \frac{d}{d\bar\tau}(\bar\tau H^R) \right], \tag{59}$$

where in analogy with (43) we introduced the factorized "shift-vector" $\lambda_2 = N_0(\tau)\bar\lambda_2(\tau, \sigma)$; in this case, the covariant derivative (5) takes the form

$$D_{\bar\tau}\xi_\mu = \partial_{\bar\tau}\xi_\mu - \bar\lambda_2 \xi'_\mu = \frac{D_\tau \xi_\mu}{N_0} \tag{60}$$

The reduced system (59) has trivial solutions for the global variables Π_i, Q_i

$$\frac{\delta W^R}{\delta \Pi_i} = 0 \Rightarrow \frac{dQ_i}{d\bar\tau} = 0; \; Q_i = \text{const}; \qquad \frac{\delta W^R}{\delta Q_i} = 0 \Rightarrow \frac{d\Pi_i}{d\bar\tau} = 0, \; \Pi_i = \text{const} \tag{61}$$

which have the meaning of initial data.

If the solutions of equations (56), (57), and (61) for the system (59)

$$\Pi_0 = H^R := \frac{M^2}{2\bar{\gamma}}, \qquad \Pi_i = P_i, \qquad Q_0 = \bar{\tau}, \quad Q_i = X_i(0), \tag{62}$$

are substituted into the inverted Levi-Civita canonical transformations (54)

$$P_0 = \pm\sqrt{M^2 + P_i^2}, \qquad X_0(\bar{\tau}) = \bar{\tau}\frac{P_0}{\bar{\gamma}}\,, \qquad X_i(\bar{\tau}) = Q_i + \bar{\tau}\frac{P_i}{\bar{\gamma}}\,, \tag{63}$$

the initial extended action (47) can be described in the rest frame of an observer who measures the energy P_0 and the time X_0 and sees the rest frame evolution of the "center of mass" coordinates

$$X_i(X_0) = Q_i + X_0\frac{P_i}{P_0}\,. \tag{64}$$

The Lorentz scheme of describing a relativistic system in terms of the time and energy (X_0, P_0) in the phase space $P_i, X_i, \pi_\mu, \xi_\mu$ is equivalent to the above-considered the Levi-Civita scheme in terms of the proper time and the evolution Hamiltonian $(\bar{\tau}, H^R)$ in the phase space $\Pi_i, Q_i, \pi_\mu, \xi_\mu$, where the variables Π_i, Q_i are cyclic.

We identify the Levi-Civita scheme with the comoving frame with the energy

$$E_0 = -\frac{dW^R}{d\bar{\tau}} = \frac{M^2}{2\bar{\gamma}} + \frac{dS^{lc}}{d\bar{\tau}} = \frac{M^2}{\bar{\gamma}}; \qquad (S^{lc} = \bar{\tau}\frac{M^2}{2\bar{\gamma}})\,. \tag{65}$$

This energy includes the time-surface S^{lc} term in the action (59). Then, the inverted Levi-Civita canonical transformations (54) (obtained on the level of the extended theory) plays the role of the Lorentz transformation from the comoving frame to the rest frame

$$\bar{\tau}\frac{M^2}{\bar{\gamma}} - X_i^{(0)}P_i^{(0)} = \pm X_0|P_0| - X_iP_i. \tag{66}$$

4.1. Dynamics of the local variables

4.2. Reparametrization-invariant reduction of an open string

We restrict ourselves to an open string with the boundary conditions

$$\sigma_1(\bar{\tau}) = 0, \qquad \sigma_2(\bar{\tau}) = \pi, \qquad l(\bar{\tau}) = \pi\,. \tag{67}$$

In the gauge-fixing method, by using the kinemetric transformation (see Appendix A), we can put

$$\bar{\lambda}_1 = 1, \qquad \bar{\lambda}_2 = 0\,. \tag{68}$$

This requirement does not contradict the normalization of $\bar{\lambda}_1$ (46).

In view of (49), it means that the reduced Hamiltonian H^R (44) coincides with its density (41)

$$\bar{\phi}_1 = \mathcal{H} - \frac{1}{\pi}\int\limits_0^\pi d\sigma \mathcal{H} = 0, \qquad \bar{\phi}_2 = \pi_\mu \xi'^\mu = 0 \tag{69}$$

In this case, the reparametrization-invariant equations for the local variables obtained by varying the action (59)

$$\frac{\delta W_s^R}{\delta \xi^\mu} = 0 \Rightarrow \partial_{\bar{\tau}}\pi_\mu - \partial_\sigma(\bar{\lambda}_2 \pi_\mu) = \gamma \partial_\sigma(\bar{\lambda}_1 \xi'_\mu), \quad \frac{\delta W_s^R}{\delta \pi^\mu} = 0 \Rightarrow \gamma D_{\bar{\tau}} \xi_\mu = \bar{\lambda}_1 \pi_\mu \tag{70}$$

again lead to the D'Alambert equations (15) where τ is changed by the proper time $\bar{\tau}$

$$\partial^2_{\bar{\tau}} \xi_\mu - \partial^2_\sigma \xi_\mu = 0. \tag{71}$$

The general solution of these equations of motion in the class of functions (37) with the boundary conditions (67) is given by the Fourier series (17)

$$\xi_\mu(\bar{\tau}, \sigma) = \frac{1}{2\sqrt{\pi\gamma}}[\psi_\mu(z_+) + \psi_\mu(z_-)], \qquad \psi_\mu(z) = i \sum_{n \neq 0} e^{(-inz)} \frac{\alpha_{n\mu}}{n}, \qquad z_\pm = \bar{\tau} \pm \sigma. \tag{72}$$

$$\xi'_\mu(\bar{\tau}, \sigma) = \frac{1}{2\sqrt{\pi\gamma}}[\psi'_\mu(z_+) - \psi'_\mu(z_-)], \qquad \pi_\mu(\bar{\tau}, \sigma) = \frac{1}{2}\sqrt{\frac{\gamma}{\pi}}[\psi'_\mu(z_+) + \psi'_\mu(z_-)] .$$

The total coordinates $Q^{(0)}_\mu$ and momenta P_μ are determined by the reduced dynamics of the "center of mass" (61), (62), (63), and the string mass M obtained from (44)

$$P_\mu^2 = M^2 = 2\pi\gamma H^R = 2\pi\gamma \int\limits_0^\pi d\sigma \mathcal{H}. \tag{73}$$

The substitution of ξ_μ and π_μ from (72) into (41) leads to the density

$$\mathcal{H} = -\frac{1}{4\pi}\left[\psi'^2_\mu(\bar{\tau} + \sigma) + \psi'^2_\mu(\bar{\tau} - \sigma)\right] ,$$

and from (73) we obtain, for the mass, the expression (24).

The second constraint (69) in terms of the vector ψ'_μ in (72) takes the form

$$\xi'_\mu \pi^\mu = \frac{1}{4\pi}\left[\psi'^2_\mu(\bar{\tau} + \sigma) - \psi'^2_\mu(\bar{\tau} - \sigma)\right] = 0 \Rightarrow \psi'^2_\mu(\bar{\tau} + \sigma) = \psi'^2_\mu(\bar{\tau} - \sigma) = \text{const.} , \tag{74}$$

and the first constraint (69) $\bar{\phi}_1 = 0$ is satisfied identically. After the substitution of the constant value (74) into (24) we obtain that const. $= -M^2/\pi\gamma$; thus, finally

$$M^2 = -\pi\gamma\psi'^2_\mu . \tag{75}$$

It means that ψ'_μ is the modulo-constant space-like vector. The constraint (75) in terms of the Fourier components (72) takes the form

$$\psi'^2_\mu = 2 \sum_{n=-\infty}^{\infty} \bar{L}_n = -\frac{M^2}{\pi\gamma} \ , \tag{76}$$

where $\bar{L}_n$ is given by (23). From (76) we can see that the zero harmonic of this constraint determines the mass of a string

$$M^2 = -2\pi\gamma\bar{L}_0 = -\pi\gamma \sum_{k\neq 0} \alpha_{k\mu}\alpha_{-k\mu} \tag{77}$$

and coincides with the gauge-fixing value. However, the nonzero harmonics of constraint (77)

$$\bar{L}_{n\neq 0} = -\frac{1}{2} \sum_{k\neq 0,n} \alpha_{k\mu}\alpha_{n-k\mu} = 0, \qquad \bar{L}_{-n} = \bar{L}^*_n \tag{78}$$

strongly differ from the gauge-fixing constraints (20), as (78) do not depend on the global motion of the center of mass P_μ and include only the contributions of the nonzero Fourier harmonics $\alpha_{n\neq 0}$.

Thus, instead of the standard constraints (20), the invariant dynamics of a relativistic string leads to other local constraints (78) including only the nonzero Fourier harmonics, in the agreement with the Poincare invariance of the local degrees of freedom. The algebra of these constraints (78) is not closed, as it does not contains the zero Fourier harmonic of the energy constraint (which has been resolved to express the dynamic equations in terms of the proper time). The reparametrization-invariant version of the Virasoro algebra (with all its difficulties, including the $D = 26$ - problem and the negative norm states) appears only in the case of the massless string $-2\pi\gamma\bar{L}_0 = M^2 = 0$.

4.3. The explicit resolving of constraints

The ideology of the invariant reduction (with the explicit resolving of constraints to exclude the superfluous variables of the type of the time-like component of the CM coordinates) can be extended onto the local constraints (69). These constraints in the form (75) can be also used to exclude the time component of the local excitations ξ_0, π_0 (with the negative contribution into energy) from the phase space, to proceed the stability of the system and the positive norm of quantum states.

$$\xi_0 = \frac{1}{2\sqrt{\pi\gamma}}[\psi'_0(z_+) - \psi'_0(z_-)], \qquad \xi_0 = \frac{1}{2}\sqrt{\frac{\gamma}{\pi}}[\psi'_0(z_+) + \psi'_0(z_-)] \tag{79}$$

where

$$\psi'_0(z_\pm) = \pm[(\psi'_i(z_\pm))^2 - \frac{M^2}{\pi\gamma}]^{1/2}. \tag{80}$$

The constraining (79) means that only the spatial components ξ_i, π_i are independent variables.

The choice of gauge $\xi_0 = \pi_0 = 0$ 50 fixes a contribution of the time-like component into the string mass. In this case, as it was mentioned above, the equations for the reduced action coincide with the set of equations and the same constraints of the initial extended action. Finenally, the explicit resolving of the local constraints takes the form

$$(\psi_i'(z))^2 = \frac{M^2}{\pi\gamma} \tag{81}$$

4.4. Operator level: the Röhrlich approach

The reparametrization-invariant dynamics of a relativistic string in the form of the first and second class constraints (49), (50) coincides with the Röhrlich approach to the string theory [21]. This approach is based on two points: i) the choice of the gauge condition

$$G_n = P_\mu \alpha_n^\mu = 0, \qquad n \neq 0$$

and ii) the use of that condition for eliminating the states with negative norm, the physical state vectors being constructed in the "center of mass" (CM) frame (in our scheme, the CM frame appears as a result of the geometric Levi-Civita reduction). This reference frame is the only preferred frame for quantizing such a composite relativistic object as the string, as only in this frame one can quantize the initial data. This is a strong version of the principle of correspondence with classical theory: the classical initial data become the quantum numbers of quantum theory. All previous attempts for quantization of the string fully ignored this meaning of the CM frame.

The Röhrlich approach distinguishes two cases: $M^2 = 0$ and $M^2 \neq 0$.

The first case, in our scheme, the equality $M^2 = 0$ together with the local constraints (78) form the Virasoro algebra.

The second case $M^2 \neq 0$ allows us to exclude the time Fourier components α_{n0}, and it is just these components that after quantization lead to the states with negative norm because of the system being unstable. This means that the state vectors in the CM frame are constructed only by the action on vacuum of the spatial components of the operators $a_{ni}^+ = \alpha_{-ni}/\sqrt{n}, n > 0$

$$|\mathbf{\Phi}_\nu >_{CM} = \prod_{n=1}^{\infty} \frac{(a_{nx}^+)^{\nu_{nx}}}{\sqrt{\nu_{nx}!}} \frac{(a_{ny}^+)^{\nu_{ny}}}{\sqrt{\nu_{ny}!}} \frac{(a_{nz}^+)^{\nu_{nz}}}{\sqrt{\nu_{nz}!}} |0> \ , \tag{82}$$

where the three-dimensional vectors $\nu_n = (\nu_{nx}, \nu_{ny}, \nu_{nz})$ have only nonnegative integers as components. These state vectors automatically satisfy the constraint

$$\alpha_{n0}|\mathbf{\Phi}_\nu >_{CM} = 0, \qquad n > 0 \tag{83}$$

The physical states (82) are subjected to further constraints (78) with $n \geq 0$

$$\bar{L}_n|\Phi_\nu>_{CM}=0, \qquad n>0, \qquad P^2=M_\nu^2=\pi\gamma<\Phi_\nu \sum_{m\neq 0}\alpha_{-m,i}\alpha_{m,i}|\Phi_\nu> \ , \quad (84)$$

where $\bar{L}_n$ can be represented in the normal ordering form

$$\bar{L}_{n>0}=\sum_{k=0}^{\infty}\alpha_{k,i}^{+}\alpha_{n+k,i}+\frac{1}{2}\sum_{k=1}^{n-1}\alpha_{k,i}\alpha_{n-k,i}. \quad (85)$$

Constraints $G_n=\alpha_{n0}, n>0$ (83) and $\bar{L}_m, m\geq 0$ (84), taken together, represent the first class constraints, in accordance with the Dirac classification [6] as they form a closed algebra for $(n,m>0)$

$$[G_n,G_m]=0, \quad [\bar{L}_n,\bar{L}_m]=(n-m)\bar{L}_{n+m}, \quad [G_n,\bar{L}_m]=nG_{m+n} \ . \quad (86)$$

Therefore the conditions (83) eliminating the ghosts and the conditions (84) defining the physical vector states are consistent. Note that the commutator $[\bar{L}_n,\bar{L}_m]$ does not contain a c-number since $n\geq 0$ and $m\geq 0$.

On the operator level, equations determining the resolution of the constraints are fulfilled in a weak sense, as only the " annihilation" part of the constraints is imposed on the state vectors.

5. The causal Green functions

Now we can construct the causal Green function for a relativistic string as the analogy of the causal Green function for a relativistic particle [13]

$$G_c(X)=G_+(X)\theta(X_0)+G_-(X)\theta(-X_0)=i\int\frac{d^4P}{(2\pi)^4}\exp(-iPX)\frac{1}{P^2-m^2-i\epsilon}, \quad (87)$$

where $G_+(X)=G_-(-X)$ is the "commutative" Green function [13]

$$G_+(X)=\int\frac{d^4P}{(2\pi)^3}\exp(-iPX)\delta(P^2-m^2)\theta(P_0). \quad (88)$$

The latter can be represented in the form of the functional integral over the complete phase space P_μ, X_μ [22]

$$G_+(X)=\int\limits_{X(\tau_1)=0}^{X(\tau_2)=X}\frac{dN_0(\tau_2)d^4P(\tau_2)}{(2\pi)^3} \quad (89)$$

$$\prod_{\tau_1\leq\tau<\tau_2}\left\{d\bar{N}_0(\tau)\prod_{\mu}\left(\frac{dP_\mu(\tau)dX_\mu(\tau)}{2\pi}\right)\right\}\exp(iW[P,X,N_0,m])$$

where $\bar{N}_0 = N_0/2\pi\delta(0)$, $\delta(0) = \int dN_0$ is the infinite volume of the group of reparametrization of the coordinate time, and

$$W[P,X,N_0,m] = \int\limits_{\tau_1}^{\tau_2} d\tau[-P_\mu \dot{X}^\mu - \frac{N_0}{2m}(-P_\mu^2 + m^2)] \qquad (90)$$

is the extended action for relativistic mechanics [3, 22] in the Hamiltonian form.

The Veneziano-type causal Green function is the spectral series with the Hermite polynomials $< \xi|\nu >$ over the physical state vectors $|\Phi_\nu >= |\nu >$

$$G_c(X|\xi_1,\xi_2) = G_+(X|\xi_1,\xi_2)\theta(X_0) + G_-(X|\xi_1,\xi_2)\theta(-X_0) = \qquad (91)$$
$$i\int \frac{d^4P}{(2\pi)^4}\exp(-iPX)\sum_\nu \frac{< \xi_1|\nu >< \nu|\xi_2}{P^2 - M_\nu^2 - i\epsilon}.$$

The commutative Green function for a relativistic string $G_+(X|\xi_1,\xi_2)$ can be represented in the form of the Faddeev-Popov functional integral [23] in the gauge (50)

$$G_+(X|\xi_2,\xi_1) = \qquad (92)$$
$$\int\limits_{X(\tau_1)=0}^{X(\tau_2)=X} \frac{dN_0(\tau_2)d^4P(\tau_2)}{(2\pi)^3} \prod_{\tau_1 \le \tau < \tau_2} \left\{ d\bar{N}_0(\tau)\prod_\mu \left(\frac{dP_\mu(\tau)dX_\mu(\tau)}{2\pi}\right)\right\} F_+(\xi_2,\xi_1),$$

where

$$F_+(\xi_2,\xi_1) = \sum_\nu < \xi_2|\nu > \exp\{iW[P,X,N_0,M_\nu]\} < \nu|\xi_1 >= \qquad (93)$$

$$\int\limits_{\xi_1}^{\xi_2} D(\xi,\pi)\Delta_{fp}\exp\{iW_{fp}\}\ ,$$

$W[P,X,N_0,M_\nu]$ is the action (90) with the mass M_ν

$$W_{fp} = \int\limits_0^{\tau(X_0)} d\tau \left[-\left(\int\limits_0^\pi d\sigma \pi_\mu \dot{\xi}^\mu\right) - P_\mu \dot{X}^\mu - N_0\left(-\frac{P^2}{2\pi\gamma} + H^R\right)\right] \qquad (94)$$

is the constraint-shell action (51),

$$D(\xi,\pi) = \prod_{\tau,\sigma}\prod_\mu \frac{d\xi_\mu d\pi_\mu}{2\pi}\ , \qquad (95)$$

and

$$\Delta_{fp} = \prod_{\tau,\sigma}\delta(\phi_1))\delta(\pi_0)\delta(\phi_2))\delta(\xi_0)detB^{-1}, \qquad detB = det\{\phi_1,\phi_2,\pi_0,\xi_0\} \qquad (96)$$

is the FP determinant given in the monograph [7].

6. Conclusion

We have described a relativistic string in terms of the invariant parameter of evolution with the nonzero Hamiltonians of evolution H^R in agreement with the equations of motion of the initial system.

Recall that the fixing of gauge $\lambda_1 = 1$ identifies the proper time $\tau = \sqrt{\gamma}T$ with the noninvariant coordinate time τ that corresponds to the zero generalized Hamiltonian of evolution of the constrained system.

The definition of the proper time consistent with the group of diffeomorphisms of the Hamiltonian description requires to separate the "center of mass" coordinates before varying the action, whereas in the standard gauge-fixing method, the "center of mass" coordinates are separated after varying the action.

As a result, we have got the Virasoro algebra only for the case of a string with a single value of the mass in the spectrum (in classical theory, this value is equal to zero) that corresponds to the light-like branch of the representation of the Poincare group.

Unique admissible gauge for the secondary constraints is the Röhrlich gauge.

The local constraints do not depend on the "center of mass" momenta and mean that the nonzero Hamiltonian of evolution with respect to the proper time coincides with its density. The invariant local constraints differs from the standard ones, as the operations of separation of the "center of mass" coordinates and variation of the action do not commute. We have constructed the causal Green function in the form of the Faddeev-Popov functional integral over the local variables.

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Appendix A: Kinemetric transformations

The kinemetric transformations of the differentials

$$\tilde{\tau} = \dot{f}_1(\tau)d\tau, \qquad d\tilde{\sigma} = \dot{f}_2(\tau,\sigma)d\tau + f_2'(\tau,\sigma)d\sigma$$

correspond to transformations of the string coordinates

$$x_\mu(\tau,\sigma) = \tilde{x}_\mu(\tilde{\tau}, \qquad \tilde{\sigma}), x'_\mu(\tau,\sigma) = \tilde{x}'_\mu(\tilde{\tau},\tilde{\sigma})f_2'(\tau,\sigma),$$

$$\dot{x}_\mu(\tau,\sigma) = \dot{\tilde{x}}_\mu(\tilde{\tau},\tilde{\sigma})\dot{f}_1(\tau) + \tilde{x}'_\mu(\tilde{\tau},\tilde{\sigma})\dot{f}_2(\tau,\sigma),$$

¿From these equations, we can derive the transformation law for λ_1, λ_2 taking into account (7)

$$\lambda_1(\tau,\sigma) = \frac{\sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}}{x'^2(\tau,\sigma)} = \frac{\sqrt{(\dot{\tilde{x}}\tilde{x}')^2 - \dot{\tilde{x}}^2 \tilde{x}'^2}}{\tilde{x}'^2(\tilde{\tau},\tilde{\sigma})}\frac{\dot{f}_1}{f_2'} = \tilde{\lambda}_1 \frac{\dot{f}_1(\tau)}{f_2'(\tau,\sigma)}.$$

$$\lambda_2(\tau,\sigma) = \frac{\dot{x}x'}{x'^2} = \frac{(\dot{\tilde{x}}\tilde{x}')\dot{f}_1 f_2' + \tilde{x}'^2 \dot{f}_2 f_2'}{\tilde{x}'^2 f_2'^2} = \tilde{\lambda}_2 \frac{\dot{f}_1}{f_2'} + \frac{\dot{f}_2}{f_2'}.$$

The kinemetric-invariance of the interval (3) with respect to (6) follows from these transformation laws and the transformation of the conformal factor

$$\Omega(\tau,\sigma) = f_2'(\tau,\sigma)\widetilde{\Omega}(\tilde{\tau},\tilde{\sigma})$$

The covariant derivative (5) is transformed under (6) as

$$D_\tau x_\mu = \dot{x}_\mu - \lambda_2 x'_\mu = \dot{f}_1(\tau)\left[\dot{\tilde{x}}_\mu - \tilde{\lambda}_2 \tilde{x}'_\mu\right] = \dot{f}_1(\tau) D_{\tilde{\tau}}\tilde{x}_\mu \ .$$

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Времени-репараметризационно-инвариантная динамика релятивистской струны

Динамика релятивистской струны изучается в рамках обобщенного гамильтонового формализма путем разрешения связей первого рода. Репараметризационно-инвариантным параметром эволюции системы выбирается временная компонента координаты «центра масс» струны, которая отделяется от локальных степеней свободы преобразованием, сохраняющим группу диффеоморфизмов обобщенного гамильтониана и пуанкаре-ковариантность локальных связей. Для такого отождествления временной компоненты «центра масс» струны с собственным временем, измеряемым наблюдателем в сопутствующей системе координат, мы прибегаем к каноническому преобразованию Леви-Чивита–Шанмугадхасана, которое преобразует глобальную связь (массовая поверхность) в новый импульс, так что для редукции гамильтониана не требуется дополнительного условия калибровки. Разрешение локальных условий связи ведет к редуцированному фазовому пространству с гамильтонианом типа Рорлиха и с параметром эволюции, совпадающим с собственным временем наблюдателя, находящегося в «центре масс» струны.

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Time-Reparametrization-Invariant Dynamics of a Relativistic String

The time-reparametrization-invariant dynamics of a relativistic string is studied in the Dirac generalized Hamiltonian theory by resolving the first class constraints. The reparametrization-invariant evolution parameter is identified with the time-like coordinate of the «center of mass» of a string which is separated from local degrees of freedom by transformation conserving the group of diffeomorphisms of the generalized Hamiltonian formulation and the Poincare covariance of local constraints. To identify the «center of mass» time-like coordinate with the invariant proper time (measured by an observer in the comoving frame of reference), we apply the Levi-Civita–Shanmugadhasan canonical transformations which convert the global (mass-shell) constraint into a new momentum, so that the corresponding gauge is not needed for the Hamiltonian reduction. The resolving of local constraints leads to an equivalent unconstrained system in the reduced phase space with the Röhrlich-type Hamiltonian of evolution with respect to the proper time.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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