ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2-2000-210

N.A.Chernikov

FOUR MODELS OF THE GRAVITATIONAL FIELD OF A STAR

To be published in the Proceedings of XXIII International Colloquium
on Group Theoretical Methods in Physics,
July 31 – August 5, 2000, Dubna, Russia

2000

In this paper four models are being discussed, concerning the gravitational field of a star at rest and the equations of motion of the companion planet.

The first model has been created by Newton and the second model by Lobachevsky. The third model has been initiated by Einstein, further developed Schwarzschild and completed by Fock. The fourth model has been created by the author of this paper.

In the second and in the fourth models the Lobachevsky geometry with the characteristic constant k is introduced in the background space. The constant k is the absolute measure of the length in the background space.

In the third and in the fourth models the Lobachevsky geometry with the characteristic constant c is introduced in the velocity space. The constant c is the absolute measure of the rapidity in the velocity space. It equals the light velocity.

In the first and in the third models the gravitational field of the star obeys the Einstein's equations. In the second and in the fourth models the gravitational field of the star obeys new equations, proposed by the author.

As space co-ordinates x^1, x^2 and x^3 we choose the distance ρ from the star, the polar angle θ and the azimuth ϕ on a sphere $\rho = const$; the notation $x^4 = t$ we will preserve for the time co-ordinate. Co-ordinates ρ, θ, ϕ, t are independent of c and k, therefore operations $c \to \infty$ and $k \to \infty$ have a common sense. The fourth model turns into the third model, if $k \to \infty$. The fourth model turns into the second model, if $c \to \infty$. The fourth model turns into the first model, if $c \to \infty$ and $k \to \infty$.

We must solve the new equations of the gravitational field

$$R_{mn} = \breve{R}_{mn} \ ,$$

where R_{mn} is Ricci tensor for the gravitational connection, $\breve{R}_{mn}$ is Ricci tensor for the background connection. Here the second connection is given, but the first connection is to be found.

I have shown in my works [1], that any physical theory is founded on the concept of velocity space and that the geometry of this space is the Euclidean or the Lobachevsky one. In the first case the theory is named nonrelativistic and in the second case it is named relativistic. It is strange, of course, but it has been named in this way.

It is interesting, that the background connection $\breve{\Gamma}^a_{mn}$ does not depend on the light velocity c. Consequently it refers to the Absolute Geometry of Bolyai in velocity space.

The gravitational connection is defined by the equations of motion for a particle in a gravitational field , when there are no any other forces.

The background connection is defined by the equations of motion for a free particle.

The trivial solution $\Gamma^a_{mn} = \breve{\Gamma}^a_{mn}$ means that the background connection is the gravitational connection in its trivial form. In this case there is no gravitational field.

If the gravitational field is absent, we put

$$g_{mn}dx^m dx^n = \breve{g}_{mn}dx^m dx^n = h_{\mu\nu}dx^\mu dx^\nu - c^2\, dtdt\ ,$$

where $h_{\mu\nu}$ do not depend on $x^4 = t$.

The quadratic form $h_{\mu\nu}dx^\mu dx^\nu$ is either the metrics of the Euclidean space in the Newton's model (the case $k = \infty$), or the metrics of the Lobachevsky space in the Lobachevsky model (the case $k < \infty$).

The components of the Christoffell's connection for the metrics $h_{\mu\nu}dx^\mu dx^\nu$ we shall denote by $h^\alpha_{\mu\nu}$.

The Ricci tensor $r_{\mu\nu}$ for the connection $h^\alpha_{\mu\nu}$ equals $r_{\mu\nu} = -k^{-2}\, h_{\mu\nu}$ in the case $k < 0$ and it equals zero ($r_{\mu\nu} = 0$) in the case $k = \infty$.

Both in the relativistic case and in the nonrelativistic case the background connection equals

$$\breve{\Gamma}^\alpha_{\mu\nu} = h^\alpha_{\mu\nu}\ , \breve{\Gamma}^\alpha_{\mu 4} = 0\ , \breve{\Gamma}^\alpha_{4\nu} = 0\ , \breve{\Gamma}^\alpha_{44} = 0\ , \breve{\Gamma}^4_{mn} = 0\ .$$

Accordingly, both in the relativistic and in the nonrelativistic case the background Ricci tensor equals

$$\breve{R}_{\mu\nu} = r_{\mu\nu} \ , \ \breve{R}_{4n} = 0 \ , \ \breve{R}_{m4} = 0 \ .$$

1. The Newton's Model: the Case $(k = \infty, \ c = \infty)$.

According to Newton, the equations of motion for a planet are:

$$\frac{d^2\rho}{d\tau^2} - \rho \frac{d\theta}{d\tau}\frac{d\theta}{d\tau} - \rho \sin^2\theta \frac{d\phi}{d\tau}\frac{d\phi}{d\tau} + \frac{\gamma M}{\rho^2}\frac{dt}{d\tau}\frac{dt}{d\tau} = 0 \ ,$$

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{\rho}\frac{d\rho}{d\tau}\frac{d\theta}{d\tau} - \sin\theta\cos\theta\frac{d\phi}{d\tau}\frac{d\phi}{d\tau} = 0 \ ,$$

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{\rho}\frac{d\rho}{d\tau}\frac{d\phi}{d\tau} - 2\cot\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau} = 0 \ , \quad \frac{d^2 t}{d\tau^2} = 0 \ .$$

Here γ is the Newton's constant, M is a mass of the star.

From here we find the gravitational connection Γ^a_{mn} in the Newton's case. If $M = 0$, it coincides with the background connection. In this case all components of the tensor

$$P^a_{mn} = \breve{\Gamma}^a_{mn} - \Gamma^a_{mn}$$

equal zero except

$$P^1_{44} = -\frac{\gamma M}{\rho^2} \ ,$$

which equals the force, with which the star attracts the planet's unit mass.

It is remarkable that the Newton's gravitational connection with an arbitrary constant γM is an exact solution of Einstein equations

$$R_{mn} = 0 \ .$$

2. The Lobachevsky Model: the Case ($k < \infty$, $c = \infty$):

In the Lobachevsky geometry the length of a circle of radius ρ equals $2\pi r$, and the area of a sphere of the same radius equals $4\pi r^2$, where $r = k \sinh \frac{\rho}{k}$. Because of it Lobachevsky has shown [2, c. 159] that in the new model the force, with which the star attracts the planet's unit mass should equal

$$P^1_{44} = -\frac{\gamma M}{r^2}.$$

The rest components of tensor P^a_{mn} must be equal zero. The force of attraction in the Lobachevsky model has potential [3], which equals

$$U = \frac{\gamma M}{k}(1 - \coth \frac{\rho}{k}).$$

In order to find the background connection in this model we must write down the equations of motion for a particle in the case when the Lagrangian equals

$$\frac{1}{2}\frac{d\rho}{dt}\frac{d\rho}{dt} + \frac{1}{2}r^2\frac{d\theta}{dt}\frac{d\theta}{dt} + \frac{1}{2}r^2 \sin^2\theta \frac{d\phi}{dt}\frac{d\phi}{dt}.$$

From these equations we receive

$$\begin{aligned}
\breve{\Gamma}^1_{22} &= -k \sinh \tfrac{\rho}{k} \cosh \tfrac{\rho}{k}, & \breve{\Gamma}^1_{33} &= \breve{\Gamma}^1_{22} \sin^2\theta, \\
\breve{\Gamma}^2_{12} &= k^{-1} \coth \tfrac{\rho}{k} = \breve{\Gamma}^2_{21}, & \breve{\Gamma}^2_{33} &= -\sin\theta\cos\theta, \\
\breve{\Gamma}^3_{13} &= k^{-1} \coth \tfrac{\rho}{k} = \breve{\Gamma}^3_{31}, & \breve{\Gamma}^3_{23} &= \cot\theta = \breve{\Gamma}^3_{32},
\end{aligned}$$

the remaining components $\breve{\Gamma}^a_{mn}$ being equal to zero.

The background Ricci tensor in the coordinates ρ, θ, ϕ, t is a diagonal one. Its diagonal elements are

$$\breve{R}_{11} = -2k^{-2}, \ \breve{R}_{22} = -2k^{-2}r^2, \ \breve{R}_{33} = -2k^{-2}r^2 \sin^2\theta, \ \breve{R}_{44} = 0.$$

The Lobachevsky gravitational connection with an arbitrary constant γM is an exact solution of the equations

$$R_{\mu\nu} = -2k^{-2}h_{\mu\nu}, \quad R_{m4} = R_{4m} = 0.$$

3. The Einstein-Schwarzschild-Fock Model: the Case ($k = \infty$, $c < \infty$).

The construction of this model began Einstein, and it was continued by Schwarzschild, and completed by Fock, who insisted on the application of the harmonicity condition. The gravitational metrics in this case equals

$$\Big(\frac{\rho+\alpha}{\rho-\alpha}\Big)d\rho^2 + (\rho+\alpha)^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \Big(\frac{\rho-\alpha}{\rho+\alpha}\Big)c^2dt^2 \ ,$$

where $\alpha = \gamma M c^{-2}$ is the gravitational radius of mass M. This metrics satisfies the equation $R_{mn} = 0$. (See [4, c. 263]).

In the case of the static spherical symmetric metrics

$$g_{mn}dx^m dx^n = F^2 d\rho^2 + H^2(d\theta^2 + \sin^2\theta d\phi^2) - V^2 dt^2$$

the second, the third and the fourth components of anharmonicity vector $\Phi^a = g^{mn}P^a_{mn} = g^{mn}(\check{\Gamma}^a_{mn} - \Gamma^a_{mn})$ equal zero.

In regard to the first (radial) component Φ^1, it depends on the choice of the background connection. In the considered case it equals

$$\Phi^1 = g^{mn}P^1_{mn} = \frac{1}{VFH^2}\,[\frac{d}{d\rho}(F^{-1}VH^2) - 2FV\rho]\ .$$

As a consequence of the Fock harmonicity condition

$$\frac{d}{d\rho}(F^{-1}VH^2) - 2FV\rho = 0\ ,$$

the radial component Φ^1 equals zero. The Fock condition does not follow from the Einstein's equations, but it follows from our equality

$$\frac{d}{d\rho}(F^{-1}VH^2) - FVk\sinh\frac{2\rho}{k} = 0$$

taking in the limit $k \to \infty$.

4. The General Case ($k < \infty,\ c < \infty$).

In the case ($k < \infty,\ c < \infty$) we must solve the following equations of gravity:

$$R_{11} = -\frac{2}{k^2}\,, \qquad R_{22} = -2\sinh^2\frac{\rho}{k}\,, \tag{1}$$

$$R_{33} = -2\sinh^2\frac{\rho}{k}\sin^2\theta\,, \qquad R_{44} = 0,$$

$$R_{mn} = 0\,, \quad if \quad m \neq n\,.$$

The gravitational metric we find as the static spherical symmetric one. In this case we have all nondiagonal components of tensor R_{mn} being equal to zero; as to the diagonal components we have the equality $R_{33} = R_{22}\ \sin^2\theta$.

Consequently, we must solve the following equations:

$$R_{44} = 0\,, \quad R_{11} = -\frac{2}{k^2}\,, \quad R_{22} = -2\sinh^2\frac{\rho}{k}\,. \tag{2}$$

We have

$$R_{44} = \frac{V}{FH^2}\frac{d}{d\rho}\left(\frac{H^2}{F}\frac{dV}{d\rho}\right),$$

$$\frac{H}{2}\left(R_{11} + V^{-2}F^2R_{44}\right) = \frac{dH}{d\rho}\frac{1}{FV}\frac{d(FV)}{d\rho} - \frac{d^2H}{d\rho^2}\,, \tag{3}$$

$$R_{22} = 1 - \frac{1}{FV}\frac{d}{d\rho}\left(\frac{VH}{F}\frac{dH}{d\rho}\right).$$

Therefore we must solve the following equations:

$$\frac{d}{d\rho}\left(\frac{H^2}{F}\frac{dV}{d\rho}\right) = 0\,, \quad \frac{d^2H}{d\rho^2} - \frac{H}{k^2} = \frac{dH}{d\rho}\frac{1}{FV}\frac{d(FV)}{d\rho}\,, \tag{4}$$

$$\frac{d}{d\rho}\left(\frac{VH}{F}\frac{dH}{d\rho}\right) = FV\cosh\frac{2\rho}{k}\,. \tag{5}$$

We shall solve this system provided that $FV = C = const$, when we shall have

$$\frac{d}{d\rho}\left(H^2\frac{dV^2}{d\rho}\right) = 0\,, \quad \frac{d^2H}{d\rho^2} - \frac{H}{k^2} = 0\,, \tag{6}$$

$$\frac{d^2}{d\rho^2}\left(V^2H^2\right) = 2C^2\cosh\frac{2\rho}{k}\,. \tag{7}$$

It follows from (6), that

$$H = Pk\sinh(\xi+\hat{\xi}), \quad V^2 = N - \frac{B^2}{kP^2}\coth(\xi+\hat{\xi}), \tag{8}$$

where $\xi = \rho/k$, $\hat{\xi} = \hat{\rho}/k$, and $B^2, P, \hat{\rho}, N$ are integration constants. Now it follows from (7) and (8), that

$$V^2H^2 = C^2k^2\sinh(\xi+\hat{\xi})\sinh(\xi-\hat{\xi}). \tag{9}$$

Consequently, we have

$$g_{mn}dx^m dx^n = F^2d\rho^2 + H^2(d\theta^2+\sin^2\theta d\phi^2) - V^2dt^2 =$$

$$= P^2k^2\left[\Xi^{-1}d\xi^2 + \sinh^2(\xi+\hat{\xi})(d\theta^2+\sin^2\theta\; d\phi^2\right] - -C^2P^{-2}\Xi dt^2\,, \tag{10}$$

where

$$\Xi = \frac{\sinh(\xi-\hat{\xi})}{\sinh(\xi+\hat{\xi})}\,.$$

On large distances from the star the gravitational metric (10) must approximate the bacground one. It follows from this that

$$C = c, \quad P = \exp(-\hat{\xi}).$$

Taking into consideration the nonrelativistic limit we get

$$\sinh(2\hat{\xi}) = 2\,\frac{\alpha}{k}.$$

In the case under consideration

$$\Phi^1 = \frac{1}{VFH^2}\,[\frac{d}{d\rho}(F^{-1}VH^2) - FVk\sinh\frac{2\rho}{k}] =$$

$$= \frac{k}{H^2}[\frac{d}{d\xi}\sinh(\xi+\hat{\xi})\sinh(\xi-\hat{\xi}) - \sinh(2\xi)] = 0.$$

References

1. N.A. Chernikov. Lobachevsky Geometry and Relativistic Mechanics. Particles & Nuclei. Vol. 4 [3], 772-810. 1973.

2. N.I.Lobachevsky. New Foundations of Geometry with a Complete Theory of Parallels (1835-1838). In Complete Collected Works, Vol. 2 Gostekhizdat, Moscow-Leningrad, 1949, p. 159.

3. N.A. Chernikov. Introduction of Lobachevsky Geometry into the Theory of Gravitation. Particles & Nuclei. Vol. 23 [5], 1155-1191 (September- October 1992).

4. V.A. Fock. The Theory of Space, Time and Gravitation. Gostekhizdat, Moscow, 1955.

Received by Publishing Department
on September 7, 2000.

Черников Н.А. E2-2000-210
Четыре модели гравитационного поля звезды

Рассмотрены четыре модели гравитационного поля покоящейся звезды вместе с уравнениями движения ее спутницы — планеты.

Первая модель построена Ньютоном. Вторая модель построена Лобачевским. Построение третьей модели начато Эйнштейном, продолжено Шварцшильдом и завершено Фоком. Четвертая модель построена автором данной работы.

Во второй и четвертой моделях в фоновое пространство вводится геометрия Лобачевского с характерной константой k, являющейся в фоновом пространстве абсолютной мерой длины.

В третьей и четвертой моделях геометрия Лобачевского с характерной константой c вводится в пространство скоростей. Константа c является абсолютной мерой быстроты в пространстве скоростей. Она равняется скорости света.

В первой и третьей моделях гравитационное поле звезды подчиняется уравнениям Эйнштейна. Во второй и четвертой моделях гравитационное поле звезды подчиняется новым уравнениям, предложенным автором.

Работа выполнена в Лаборатории теоретической физики им. Н.Н.Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 2000

Chernikov N.A. E2-2000-210
Four Models of the Gravitational Field of a Star

Four models are being discussed, concerning the gravitational field of a star at rest and the equations of motion of the companion planet.

The first model has been created by Newton and the second model by Lobachevsky. The third model has been initiated by Einstein, further developed Schwarzschild and completed by Fock. The fourth model has been created by the author of this paper.

In the second and in the fourth models the Lobachevsky geometry with the characteristic constant k is introduced in the background space. The constant k is the absolute measure of the length in the background space.

In the third and in the fourth models the Lobachevsky geometry with the characteristic constant c is introduced in the velocity space. The constant c is the absolute measure of the rapidity in the velocity space. It equals the light velocity.

In the first and in the third models the gravitational field of the star obeys the Einstein's equations. In the second and in the fourth models the gravitational field of the star obeys new equations, proposed by the author.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 2000

Макет Т.Е.Попеко

Подписано в печать 27.09.2000
Формат 60 × 90/16. Офсетная печать. Уч.-изд. листов 0,83
Тираж 425. Заказ 52254. Цена 1 р.

Издательский отдел Объединенного института ядерных исследований
Дубна Московской области