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**QCD COUPLING UP TO THIRD ORDER  
IN STANDARD AND ANALYTIC  
PERTURBATION THEORIES**

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# 1 Introduction

In publications [1, 2] a new approach to perturbative QCD, the renormalization group (RG) invariant analytic approach (IAA) was proposed (for the exhaustive review on this approach we recommend paper [3]). This method consistently takes into account the renormalization invariance and analyticity in perturbation theory (PT).

In work [4], a particular version of IAA has been formulated. According this version, analytic perturbation theory (APT), the observables in the space-like region are represented as a non-power series in the special universal functions  $\mathcal{A}_n(Q^2, f)$  ( $n=1,2,\dots$  and  $f$  denotes number of quark flavor) [5]. Analogical set of functions  $\{\mathfrak{A}_n(s, f)\}$ ,  $s > 0$ , was introduced in the time-like region [6]. Both sets of functions are determined by the QCD running coupling  $\bar{\alpha}_s(Q^2, f)$  and can be calculated in APT analytically or numerically. A systematic mathematical investigation of these functions have been undertaken in works [5-7], in particular the oscillating behavior in the infrared region was established. To calculate these functions beyond the one-loop level the iterative approximation for the coupling was used [6, 8, 9], or RG equation for the coupling has been solved numerically in the complex domain [10].

In papers [11, 12] the RG equation for the QCD running coupling, at the two-loop order, has been solved explicitly as a function of the scale. The solution has been written in terms of the Lambert W function. The three-loop order solution (with Pade transformed  $\beta$ -function) also was expressed in terms of the same function [12]. Using the explicit two-loop solution, the analytical structure of the coupling in the complex  $Q^2$ -plane has been determined [11, 12]. The analytical formulae for the corresponding spectral function was found. Then using the method of APT the analytically improved coupling [13, 14] has been reconstructed.

Afterwards, in paper [15], the running coupling of an arbitrary renormalization scheme, to the  $k$ -th order ( $k \geq 3$ ), was expanded as a power series in the scheme independent explicit two-loop order solution. The new method for reducing the scheme ambiguity for the QCD observables has been proposed in this work. A similar expansion for the single scale dependent observable, motivated differently, has been suggested in [16].

In Sec.2 and Sec.3 we use the explicit solutions for the running coupling to calculate the universal quantities  $\mathcal{A}_n(Q^2, f)$  and  $\mathfrak{A}_n(s, f)$  beyond the one loop order. The results for  $\mathfrak{A}_n(s, f)$ , to second and third orders, are presented in the analytical form.

In Sec.4, the matching conditions for crossing the quark flavor thresholds for the  $\overline{MS}$  scheme running coupling  $\alpha_s(Q, f)$ , to the three loops, are solved analytically. By the way we construct the global (independent on  $f$ ) universal functions,  $\mathcal{A}_n(Q^2)$  and  $\mathfrak{A}_n(s)$  (both introduced in [6]). These functions can be used in the whole momentum space.

In Sec.5 we present numerical estimations of the explicit solutions for the coupling. We consider the cases of standard PT and of APT separately. To third order, we compare numerical results for Pade and standard iterative approximations to the coupling. The scope of validity for the iterative approximation is estimated. We give numerical results for the "analyticized couplings"  $\mathcal{A}_1(Q^2, f)$  and  $\mathfrak{A}_1(s, f)$  calculated to second and third orders. The differences between these quantities and the standard coupling,  $\alpha_s(Q^2, f)$ , are estimated. We also give numerical results for the global functions  $\mathcal{A}_n(Q^2)$  and  $\mathfrak{A}_n(s)$  ( $n = 1, 2, 3$ ), to second and third orders (see Tables 7-12).

## 2 Exact solution for the two-loop coupling in the space-like region

The running coupling of QCD satisfies the differential equation <sup>a</sup>

$$Q^2 \frac{\partial \alpha_s(Q^2, f)}{\partial Q^2} = \beta^f(\alpha_s(Q^2, f)) = - \sum_{N=0}^{\infty} \beta_N^f \alpha_s^{N+2}(Q^2, f), \quad (1)$$

$\bar{\alpha}_s(\mu^2) = \alpha_s = \frac{g^2}{4\pi}$ ,  $g$  is the renormalized coupling constant,  $\mu$  is the renormalization point, and  $f$  denotes the number of quark flavors. In the class of schemes where the beta-function is mass independent  $\beta_0^{(f)}$  and  $\beta_1^{(f)}$  are universal and the result for  $\beta_2^f$  is available in the modified  $\overline{MS}$  ( $\overline{MS}$ ) scheme

$$\beta_0^f = \frac{1}{4\pi} \left( 11 - \frac{2}{3}f \right), \quad \beta_1^f = \frac{1}{(4\pi)^2} \left( 102 - \frac{38}{3}f \right), \quad \beta_2^f = \frac{1}{(4\pi)^3} \left( \frac{2857}{2} - \frac{5033f}{18} + \frac{325f^2}{54} \right). \quad (2)$$

For convenience in what follows we shall omit index  $f$  in the coefficients  $\beta_k^f$ . Exact two-loop solution to Eq. (1) is given by [11, 12]

$$\alpha_s^{(2)}(Q^2, f) = -\frac{\beta_0}{\beta_1} \frac{1}{1 + W_{-1}(\zeta)} : \quad \zeta = -\frac{1}{eb} \left( \frac{Q^2}{\Lambda^2} \right)^{-\frac{1}{b}}, \quad (3)$$

$b = \beta_1/\beta_0^2$ ,  $\Lambda \equiv \Lambda_{\overline{MS}}$  and  $W(\zeta)$  denotes the Lambert W function [17]. This is the multivalued inverse of  $\zeta = W(\zeta) \exp W(\zeta)$ . The branches of W are denoted  $W_k(\zeta)$ ,  $k = 0, \pm 1, \dots$ . A detailed review of properties and applications of this special function can be found in [17]. The three-loop solution (with Pade transformed beta-function) for the coupling is [12]

$$\alpha_{\text{Pade}}^{(3)}(Q^2, f) = -\frac{\beta_0}{\beta_1} \frac{1}{1 - \beta_0\beta_2/\beta_1^2 + W_{-1}(\xi)} : \quad \xi = -\frac{1}{eb} \exp \left( \frac{\beta_0\beta_2}{\beta_1^2} \right) \left( \frac{Q^2}{\Lambda^2} \right)^{-\frac{1}{b}}. \quad (4)$$

Expressions (3) and (4) allows us to perform analytical continuation in the complex  $Q^2$  plane and to calculate discontinuity along the negative  $Q^2$  axis. <sup>b</sup> In this way we construct the corresponding analytically improved expressions for the coupling. The ‘‘analytized n-th power’’ of the coupling, obtained from the solution (4), can be written as

$$A_n^{(3)}(Q^2, f) \equiv \{\alpha_s^{(3)}(Q^2, f)\}_{\text{an.}}^n = \frac{1}{\pi} \int_0^{\infty} \frac{\rho_n^{(3)}(\sigma, f)}{\sigma + Q^2} d\sigma = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^t}{(e^t + Q^2/\Lambda^2)} \tilde{\rho}_n^{(3)}(t, f) dt, \quad (5)$$

where the spectral function,  $\rho_n(\sigma, f) \equiv \tilde{\rho}_n(t, f) = \Im\{\alpha_s(-\sigma - i0)\}^n$ , is given by

$$\tilde{\rho}_n^{(3)}(t, f) = \left( \frac{\beta_0}{\beta_1} \right)^n \Im \left( -\frac{1}{1 - \beta_2\beta_0/\beta_1^2 + W_1(Z(t))} \right)^n, \quad (6)$$

with

$$Z(t) = \frac{1}{be} \exp(\beta_0\beta_2/\beta_1^2 - t/b + i(1/b - 1)\pi). \quad (7)$$

Note that formula (6) is relevant for  $0 \leq f \leq 6$ . Taking the limit  $\beta_2 \rightarrow 0$  in (6) one can reproduce the corresponding formula for the two loop case.

<sup>a</sup>We use the notation  $Q^2 = -q^2$ ,  $Q^2 > 0$  corresponds to a spacelike momentum transfer.

<sup>b</sup>For details of analytical continuation we recommend papers [11-14].

### 3 Continuation of the QCD running coupling to the timelike region

The APT approach allows us to define the QCD coupling in the timelike region in a correct manner [8, 9, 18, 19]. Here, instead of common power series for a timelike observables, there appears asymptotic series over the set of oscillating functions  $\{\mathfrak{A}_n(s, f)\}$  [6, 7]. The functions  $\mathfrak{A}_n(s, f)$  are defined by the elegant formula [20]

$$\mathfrak{A}_n(s, f) = \frac{1}{\pi} \int_s^\infty \frac{d\sigma}{\sigma} \rho_n(\sigma, f), \quad (8)$$

here the spectral function is  $\rho_n(\sigma, f) = \Im\{\alpha_s(-\sigma - i0)\}^n$ . In the one-loop order the set  $\{\mathfrak{A}_n(s)\}$  has been studied in Ref.[6]. In this case, the first four functions of the set are given by

$$\begin{aligned} \mathfrak{A}_1^{(1)}(s, f) &= \frac{1}{\beta_0} \left(0.5 - \frac{1}{\pi} \arctan\left(\frac{\ln \bar{s}}{\pi}\right)\right), & \mathfrak{A}_2^{(1)}(s, f) &= \frac{1}{\beta_0^2} \frac{1}{(\ln^2 \bar{s} + \pi^2)}, \\ \mathfrak{A}_3^{(1)}(s, f) &= \frac{1}{\beta_0^3} \frac{\ln \bar{s}}{(\ln^2 \bar{s} + \pi^2)^2}, & \mathfrak{A}_4^{(1)}(s, f) &= \frac{1}{\beta_0^4} \frac{\ln^2 \bar{s} - \pi^2/3}{(\ln^2 \bar{s} + \pi^2)^3}, \end{aligned} \quad (9)$$

here  $\bar{s} = s/\Lambda^2$ . In this section we will calculate  $\mathfrak{A}_n(s, f)$  to second and third orders. Let us define the auxiliary functions

$$R_n(s, f) = \frac{1}{\pi} \int_{\ln \bar{s}}^\infty dt \tilde{a}^n(t, f). \quad (10)$$

where  $\Im \tilde{a}^n(t, f) = \tilde{\rho}_n(t, f) \equiv \rho_n(\sigma, f)$  with  $\sigma = \exp(t)$ . Then  $\mathfrak{A}_n(s, f) = \Im R_n(s, f)$ . The expressions for  $\tilde{a}$ , at the two and three loop orders, can be read from (6). In the two loop case

$$\tilde{a}^{(2)}(t, f) = -\frac{\beta_0}{\beta_1} \frac{1}{1 + W_1(z(t))} : \quad z(t) = \frac{e^{-1-t/b+i\phi}}{b}, \quad \phi = \pi \left(\frac{1}{b} - 1\right). \quad (11)$$

Integral (10), with (11), can be rewritten as a contour integral in the complex  $z$ -plane

$$R_n^{(2)}(s, f) = p_n \int_{z_\epsilon}^{z_s} \frac{dz}{z} \frac{1}{(1 + W_1(z))^n}, \quad (12)$$

here

$$p_n = \frac{b}{\pi} (-\beta_0/\beta_1)^n, \quad z_s = \frac{1}{e^b} (\bar{s})^{(-\frac{1}{b})} e^{i\phi}, \quad z_\epsilon = \epsilon e^{i\phi}, \quad \phi = \pi \left(\frac{1}{b} - 1\right), \quad (13)$$

and the limit  $\epsilon \rightarrow 0$  is assumed. Let us introduce the new integration variable in (12)

$$\omega = W(z), \quad d\omega = \frac{W(z)}{1 + W(z)} \frac{dz}{z}, \quad (14)$$

then

$$R_n^{(2)}(s, f) = p_n \int_{W_1(z_\epsilon)}^{W_1(z_s)} \frac{1 + \omega}{\omega(1 + \omega)^n} d\omega. \quad (15)$$

For  $n > 2$ , from (15) we find the relation

$$\mathfrak{A}_n^{(2)}(s, f) = -\frac{\beta_0}{\beta_1} \mathfrak{A}_{n-1}^{(2)}(s, f) + \frac{p_n}{(n-2)} \Im(1 + W_1(z_s))^{2-n}. \quad (16)$$

Eq. (16) can be rewritten as the recurrence relation for  $\mathfrak{A}_n(s)$

$$\frac{\partial \mathfrak{A}_{n-2}^{(2)}(s, f)}{\partial \ln s} = -(n-2)(\beta_0 \mathfrak{A}_{n-1}^{(2)}(s, f) + \beta_1 \mathfrak{A}_n^{(2)}(s, f)), \quad (17)$$

formula (17) gives the generalization of the similar one-loop order relation obtained in paper [6]. From (17) with the help of Eq. (8) we find analogical formula for the spectral function

$$\frac{\partial \rho_{n-2}^{(2)}(\sigma, f)}{\partial \ln \sigma} = -(n-2)(\beta_0 \rho_{n-1}^{(2)}(\sigma, f) + \beta_1 \rho_n^{(2)}(\sigma, f)). \quad (18)$$

Let us multiply Eq. (18) by the factor  $(\sigma + Q^2)^{-1}$  and take the integral over the region  $0 < \sigma < \infty$ . Integrating by parts and taking into account the condition  $\rho_{n-2}(\sigma)\sigma/(Q^2 + \sigma)|_{\sigma=0}^{\infty} = 0$  we obtain

$$\frac{\partial \mathcal{A}_{n-2}^{(2)}(Q^2, f)}{\partial \ln Q^2} = -(n-2)(\beta_0 \mathcal{A}_{n-1}^{(2)}(Q^2, f) + \beta_1 \mathcal{A}_n^{(2)}(Q^2, f)). \quad (19)$$

Note that for  $n = 3$  Eqs. (17) and (19) are analogous to the basis Eq. (1) with  $\alpha_s$  replaced by  $\mathfrak{A}_n$  and  $\mathcal{A}_n$  respectively.

It is sufficient to calculate  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$ . Note that  $R_1^{(2)}(s, f)$  is divergent in the limit  $\epsilon \rightarrow 0$ , (see (12)). Nevertheless, it has finite imaginary part<sup>c</sup>. By direct calculation we find

$$\tilde{\alpha}^{(2)}(s, f) = \mathfrak{A}_1^{(2)}(s, f) = \frac{1}{\beta_0} - \frac{1}{\pi \beta_0} \Im \ln W_1(z_s), \quad (20)$$

$$\mathfrak{A}_2^{(2)}(s, f) = \frac{1}{\pi \beta_1} \Im \ln \left( \frac{W_1(z_s)}{1 + W_1(z_s)} \right), \quad (21)$$

with  $z_s$  given by (13). Using the known asymptotic behavior of the W-function [17], in the limit  $s \rightarrow 0$ , we verify [1, 2]

$$\tilde{\alpha}^{(2)}(s, f) = \mathfrak{A}_1^{(2)}(s, f) \rightarrow \frac{1}{\beta_0}, \quad \text{and} \quad \mathfrak{A}_n^{(2)}(s, f) \rightarrow 0 \quad \text{for} \quad n > 1. \quad (22)$$

Analogically in three loop case we find

$$\mathfrak{A}_1^{(3)}(s, f) = \tilde{\alpha}^{(3)}(s, f) = -\frac{1}{\pi \beta_0} \left( \frac{1}{\eta} \Im \ln(W_1(Z_s)) + (1 - \frac{1}{\eta}) \Im \ln(\eta + W_1(Z_s)) - \pi \right), \quad (23)$$

$$\mathfrak{A}_2^{(3)}(s, f) = \frac{1}{\pi \beta_1} \left( \frac{1}{\eta^2} \Im \ln \left( \frac{W_1(Z_s)}{\eta + W_1(Z_s)} \right) - (1 - \frac{1}{\eta}) \Im \left( \frac{1}{\eta + W_1(Z_s)} \right) \right), \quad (24)$$

$$\mathfrak{A}_n^{(3)}(s, f) = -\frac{\beta_0}{\eta \beta_1} \mathfrak{A}_{n-1}^{(3)}(s, f) + \frac{p_n}{\eta(n-2)} \Im(\eta + W_1(Z_s))^{2-n} + \frac{p_n}{n-1} \left( \frac{1}{\eta} - 1 \right) \Im(\eta + W_1(Z_s))^{1-n} \quad (25)$$

where  $n > 2$ ,  $\eta = 1 - \beta_0 \beta_2 / \beta_1^2$  and

$$Z_s = \frac{1}{b} (s/\Lambda^2)^{-1/b} \exp(-\eta + i(1/b - 1)\pi).$$

Note that the ‘‘analytized’’ perturbative expansions for timelike observables (which contain specific functions  $\mathfrak{A}_n$ ) may be rewritten as power series in traditional coupling  $\tilde{\alpha}_s(s)$  with

<sup>c</sup>for the asymptotic behaviour of the W function see paper [17]

modified by  $\pi^2$ -factors coefficients [6, 7]. Previously, these modified power series have been obtained in [21, 22]. Application of the series can be found in papers [23-28]. Thus, “ $\pi^2$ -effects” for various timelike quantities have been estimated in paper [27], in particular, it was found that the  $\pi^2$ -factors give dominating contributions to the coefficients of  $R(s) = \sigma_{tot}(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ . On the other hand, in recent paper [7] various timelike events was analyzed in the  $f=5$  region. Higher-order “ $\pi^2$ -effects” have been taken into account properly. It was found that the extracted values for  $\alpha_s(M_z^2)$  are influenced significantly by these effects.

## 4 Matching procedure and construction of the global space-like and time-like couplings

In MS-like renormalization schemes important issue is how to introduce the matching conditions for the strong coupling constant at the heavy quarks thresholds. In literature few different recipes are known (see for example [6, 7, 29, 30, 31, 32, 33]). Here, we follow works [6, 7]. Let us impose the continuity relations

$$\alpha_s(M_f^2, \Lambda_{f-1}, f-1) = \alpha_s(M_f^2, \Lambda_f, f). \quad (26)$$

Inserting in (26) the three-loop solution (4) we solve equation (26) for  $\Lambda_f$

$$\Lambda_f = M_f \{-b_f F(z_{f-1}) \exp(\eta_f + F(z_{f-1}))\}^{b_f/2} \quad (27)$$

where  $b_f = \beta_1^f/(\beta_0^f)^2$ ,  $\eta^f = 1 - \beta_0^f \beta_2^f/(\beta_1^f)^2$ ,

$$z_{f-1} = -\frac{\exp(-\eta_{f-1})}{b_{f-1}} \left( \frac{\Lambda_{f-1}}{M_f} \right)^{2/b_{f-1}}, \quad (28)$$

$$F(z_{f-1}) = (\eta_{f-1} + W_{-1}(z_{f-1})) \frac{\beta_1^{f-1} \beta_0^f}{\beta_0^{f-1} \beta_1^f} - \eta_f. \quad (29)$$

In paper [6] special model for the spectral functions  $\rho_n(\sigma)$  was proposed

$$\rho_n(\sigma) = \rho_n(\sigma, \Lambda_3, 3) + \sum_{f \geq 4} \Theta(\sigma - M_f^2) (\rho_n(\sigma, \Lambda_f, f) - \rho_n(\sigma, \Lambda_{f-1}, f-1)), \quad (30)$$

here the mass  $M_f$  corresponds to the quark with flavor  $f$ , and  $\Lambda_f$  is determined according formula (27). Inserting (30) in formula (8) we find following expression for the “analytized powers” of the global coupling in the timelike region [6]

$$\mathfrak{A}_n(s) = \mathfrak{A}_n(s, f) + C_n(f) \quad \text{for} \quad M_f \leq \sqrt{s} \leq M_{f+1}, \quad (31)$$

where the shift coefficients  $C_n(f)$  are defined by relation

$$C_n(f) = \mathfrak{A}_n(M_{f+1}^2, f+1) - \mathfrak{A}_n(M_{f+1}^2, f) + C_k(f+1) \quad (32)$$

with  $C_n(6) = 0$ . The analogical formula follows for the corresponding global spacelike functions  $\mathcal{A}_n(Q^2)$

$$\begin{aligned} \mathcal{A}_n(Q^2) &= \frac{1}{\pi} \int_0^{M_4^2} \frac{d\sigma}{\sigma + Q^2} \rho_n(\sigma, \Lambda_3, 3) + \\ & \frac{1}{\pi} \sum_{f=4}^5 \int_{M_f^2}^{M_{f+1}^2} \frac{d\sigma}{\sigma + Q^2} \rho_n(\sigma, \Lambda_f, f) + \frac{1}{\pi} \int_{M_6^2}^{\infty} \frac{d\sigma}{\sigma + Q^2} \rho_n(\sigma, \Lambda_6, 6) \end{aligned} \quad (33)$$

## 5 Numerical estimations

For the quarks masses throughout this paper we assume the values  $M_1 = M_2 = M_3 = 0$ ,  $M_4 = 1.3 \text{ GeV}$ ,  $M_5 = 4.3 \text{ GeV}$  and  $M_6 = 170 \text{ GeV}$ .

In practice usually used the iterative approximation for the coupling [34]

$$\alpha_{it}^{(3)}(Q^2) = \frac{1}{\beta_0 L} - \frac{\beta_1 \ln L}{\beta_0^2 L^2} + \frac{1}{\beta_0^3 L^3} \left( \frac{\beta_1^2}{\beta_0^2} (\ln^2 L - \ln L - 1) + \frac{\beta_2}{\beta_0} \right), \quad (34)$$

where  $L = \ln Q^2 / \Lambda_{\overline{MS}}^2$ . The same formula is used in the timelike region. It is instructive to compare solutions (4) and (34). Numerical results for these functions are summarized in the Table 2. We see, that the difference between these functions becomes noticeable for  $Q < 1.2 \text{ GeV}$ .

In Table 3 we give results for the three-loop Pade approximated coupling  $\alpha^{(3)}(Q^2, 5)$  and the corresponding analytic coupling  $\mathcal{A}_1^{(3)}(Q^2, 5)$ . In Table 4,  $\alpha^{(3)}(s, 5)$  and  $\mathfrak{A}_1^{(3)}(s, 5)$  are compared. The interval  $5 \text{ GeV} < Q$ ,  $\sqrt{s} < 200 \text{ GeV}$  is chosen and it is assumed that  $\Lambda_5 = 264 \text{ MeV}$ . Here, we observe the inequality

$$\alpha^{(k)}(Q^2, 5) > \mathcal{A}_1^{(k)}(Q^2, 5) > \mathfrak{A}_1^{(k)}(Q^2, 5), \quad (35)$$

the relative difference  $\Delta$  (%) between  $\alpha^{(3)}(Q^2, 5)$  and  $\mathcal{A}_1^{(3)}(Q^2, 5)$  decreases from 1.4% at  $Q = 5 \text{ GeV}$  to 0.15% at  $Q = 200 \text{ GeV}$ , whereas the difference between  $\alpha^{(3)}(s, 5)$  and  $\mathfrak{A}_1^{(3)}(s, 5)$  is more appreciable:  $\Delta(\%) = 7.5\%$  at  $\sqrt{s} = 5 \text{ GeV}$  and  $\Delta(\%) = 1.6\%$  at  $\sqrt{s} = 200 \text{ GeV}$ . The ‘‘mirror symmetry’’ [8] between  $\mathcal{A}_1^{(3)}(Q^2, 5)$  and  $\mathfrak{A}_1^{(3)}(s, 5)$  is essentially violated (see Tables 3 and 4). Thus, the relative difference,  $\Delta(\%) = (\mathcal{A}_1^{(3)}(Q^2, 5) - \mathfrak{A}_1^{(3)}(s, 5)) / \mathcal{A}_1^{(3)}(Q^2, 5) * 100$ , monotonically decreases from 5.6% at  $Q = \sqrt{s} = 1 \text{ GeV}$  to 1.5% at  $200 \text{ GeV}$ .

In Table 5 various functions at the two-loop and three-loop orders are compared. We choose  $f = 5$  and  $\Lambda_5 = 215 \text{ MeV}$ . Then, from the matching formula (27), in the two-loop case ( $\eta = 1$ ) we find  $\Lambda_3 = 363 \text{ MeV}$ , while  $\Lambda_3 = 340 \text{ MeV}$  in the three-loop case. The interval  $20 \text{ GeV} < Q$ ,  $\sqrt{s} < 170 \text{ GeV}$  is chosen. Here, we demonstrate the stability of results of PT and of APT, with respect to the higher-loop corrections. In this region, the numerical difference between the three-loop and the corresponding two-loop couplings are of order 0.3% – 0.2%.

In Tables 7-12 we have summarized numerical results for the global three-loop functions,  $\mathcal{A}_n^{(3)}(Q^2)$  and  $\mathfrak{A}_n^{(3)}(s)$  for  $n=1,2,3$ . The values for the parameter  $\Lambda_3$  are chosen 350, 400, and 450  $\text{MeV}$ . The corresponding values for the  $\Lambda_f$  for  $f > 3$  are calculated from the three-loop matching condition (27) (see Table 1).

Let us compare the global three-loop function  $\mathcal{A}_1^{(3)}(Q^2)$  (see Table 9) and  $\alpha^{(3)}(Q^2, 5)$  (Table 3) in the case  $\Lambda_3 = 400 \text{ MeV}$  (the corresponding value for  $\Lambda_5$  is  $264 \text{ MeV}$ ). We see that the global coupling  $\mathcal{A}_1^{(3)}(Q^2)$  does not obey the inequality (35). The difference  $\alpha^{(3)}(Q^2, 5) - \mathcal{A}_1^{(3)}(Q^2)$  becomes negative when  $Q$  increases. For  $\Lambda_3 = 400 \text{ MeV}$ , this takes place at  $Q \sim 17 \text{ GeV}$ . This difference is small but it increases with  $Q$ . It is about 0.6% at  $Q = 200 \text{ GeV}$ . The same effect is occurred for other values of the  $\Lambda_3$ . This enhancement of the global APT coupling can be explained from formula (33). It is easy to verify that (33) contains additional positive non-perturbative contribution, which was not occurred in the case of the local APT coupling (5). This contribution increase the coupling for large values of  $Q$ .

Table 1: The values of  $\Lambda_j$  determined by the threshold matching condition (27).  $\Lambda_3$  is chosen as a basis quantity.

| The three loop case |             |             |             | Exact two loop case |             |             |             |
|---------------------|-------------|-------------|-------------|---------------------|-------------|-------------|-------------|
| $\Lambda_3$         | $\Lambda_4$ | $\Lambda_5$ | $\Lambda_6$ | $\Lambda_3$         | $\Lambda_4$ | $\Lambda_5$ | $\Lambda_6$ |
| <i>MeV</i>          | <i>MeV</i>  | <i>MeV</i>  | <i>MeV</i>  | <i>MeV</i>          | <i>MeV</i>  | <i>MeV</i>  | <i>MeV</i>  |
| 300                 | 258.4       | 183.0       | 76.2        | 300                 | 248.4       | 170.4       | 69.2        |
| 310                 | 268.2       | 190.8       | 79.7        | 310                 | 257.6       | 177.3       | 72.3        |
| 320                 | 278.0       | 198.6       | 83.4        | 320                 | 266.8       | 184.3       | 75.5        |
| 330                 | 287.9       | 206.5       | 87.0        | 330                 | 276.0       | 191.4       | 78.7        |
| 340                 | 297.8       | 214.4       | 90.8        | 340                 | 285.3       | 198.5       | 81.9        |
| 350                 | 307.8       | 222.5       | 94.6        | 350                 | 294.6       | 205.7       | 85.2        |
| 360                 | 318.0       | 230.6       | 98.4        | 360                 | 303.9       | 212.9       | 88.5        |
| 370                 | 328.0       | 238.8       | 102.3       | 370                 | 313.2       | 220.1       | 91.9        |
| 380                 | 338.1       | 247.1       | 106.2       | 380                 | 322.6       | 227.4       | 95.2        |
| 390                 | 348.4       | 255.5       | 110.2       | 390                 | 332.0       | 234.8       | 98.7        |
| 400                 | 358.7       | 263.9       | 114.3       | 400                 | 341.4       | 242.2       | 102.1       |
| 410                 | 369.0       | 272.5       | 118.4       | 410                 | 350.9       | 249.6       | 105.6       |
| 420                 | 379.4       | 281.1       | 122.6       | 420                 | 360.4       | 257.1       | 109.1       |
| 430                 | 389.9       | 289.8       | 126.8       | 430                 | 369.9       | 264.7       | 112.7       |
| 440                 | 400.5       | 298.6       | 131.1       | 440                 | 379.5       | 272.2       | 116.2       |
| 450                 | 411.1       | 307.5       | 135.4       | 450                 | 389.0       | 279.9       | 119.8       |
| 460                 | 421.8       | 316.5       | 139.8       | 460                 | 398.6       | 287.5       | 123.5       |
| 470                 | 432.6       | 325.6       | 144.3       | 470                 | 408.3       | 295.2       | 127.2       |
| 480                 | 443.4       | 334.8       | 148.8       | 480                 | 417.9       | 303.0       | 130.9       |
| 490                 | 454.4       | 344.1       | 153.4       | 490                 | 427.6       | 310.8       | 134.6       |
| 500                 | 465.4       | 353.5       | 158.1       | 500                 | 437.3       | 318.6       | 138.4       |



Table 2: The three loop Pade improved solution (4) versus iterative formula (34). We choose  $\Lambda_3 = 400$  MeV. Then, from formula (27) we have  $\Lambda_4 = 358.7$  MeV,  $\Lambda_5 = 263.9$  MeV and  $\Lambda_6 = 114.3$  MeV.  $\Delta(\%)$  denotes the relative difference of the approximants.

| Q<br>GeV | $\alpha_{pd}^{(3)}(Q^2)$<br>f=3 | $\alpha_{it}^{(3)}(Q^2)$<br>f=3 | $\Delta(\%)$ | Q<br>GeV  | $\alpha_{pd}^{(3)}(Q^2)$<br>f=5 | $\alpha_{it}^{(3)}(Q^2)$<br>f=5 | $\Delta(\%)$ |
|----------|---------------------------------|---------------------------------|--------------|---|---------------------------------|---------------------------------|--------------|
| .8       | .90931                          | .88340                          | 2.8          | 10  | .18849                          | .18847                          | .01          |
| .9       | .68022                          | .69179                          | 1.7          | 11  | .18413                          | .18413                          | .01          |
| 1.0      | .57853                          | .58784                          | 1.6          | 12  | .18033                          | .18033                          | .00          |
| 1.1      | .51548                          | .52195                          | 1.3          | 13  | .17698                          | .17699                          | .01          |
| 1.2      | .47126                          | .47589                          | 1.0          | 14  | .17398                          | .17400                          | .01          |
| 1.3      | .43803                          | .44153                          | .8           | 15  | .17129                          | .17131                          | .01          |
| 1.4      | .41190                          | .41469                          | .7           | 16  | .16884                          | .16887                          | .02          |
| 1.5      | .39068                          | .39301                          | .6           | 17  | .16661                          | .16664                          | .02          |
| 1.6      | .37302                          | .37503                          | .5           | 18  | .16457                          | .16460                          | .02          |
| 1.7      | .35803                          | .35982                          | .5           | 19  | .16268                          | .16271                          | .02          |
| 1.8      | .34511                          | .34674                          | .5           | 20  | .16092                          | .16096                          | .02          |
| 1.9      | .33384                          | .33535                          | .5           | 21  | .15929                          | .15933                          | .03          |
| 2.0      | .32389                          | .32530                          | .4           | 22  | .15777                          | .15781                          | .03          |
| 2.1      | .31502                          | .31637                          | .4           | 23  | .15634                          | .15639                          | .03          |
| 2.2      | .30707                          | .30835                          | .4           | 24  | .15500                          | .15504                          | .03          |
| 2.3      | .29987                          | .30111                          | .4           | 25  | .15374                          | .15378                          | .03          |
| 2.4      | .29333                          | .29452                          | .4           | 26  | .15254                          | .15259                          | .03          |
| 2.5      | .28735                          | .28850                          | .4           | 27  | .15141                          | .15145                          | .03          |
| 2.6      | .28185                          | .28297                          | .4           | 28  | .15033                          | .15038                          | .03          |
| Q<br>GeV | $\alpha_{pd}^{(3)}(Q^2)$<br>f=4 | $\alpha_{it}^{(3)}(Q^2)$<br>f=4 | $\Delta(\%)$ | 29 <td>.14931</td> <td>.14936</td> <td>.03</td> | .14931                          | .14936                          | .03          |
| 2        | .33390                          | .33391                          | .0           | 30  | .14834                          | .14839                          | .03          |
| 3        | .27539                          | .27564                          | .1           | 40  | .14057                          | .14062                          | .04          |
| 4        | .24564                          | .24597                          | .1           | 50  | .13510                          | .13515                          | .04          |
| 5        | .22690                          | .22725                          | .2           | 60  | .13094                          | .13099                          | .04          |
| 6        | .21371                          | .21405                          | .2           | 70  | .12763                          | .12768                          | .04          |
| 7        | .20376                          | .20410                          | .2           | 80  | .12489                          | .12494                          | .04          |
| 8        | .19590                          | .19623                          | .2           | 90  | .12258                          | .12263                          | .04          |
| 9        | .18948                          | .18980                          | .2           | 100   | .12058                          | .12063                          | .04          |
| 10       | .18410                          | .18441                          | .2           | 110   | .11883                          | .11888                          | .04          |
|          |                                 |                                 |              | 120   | .11728                          | .11732                          | .04          |

Table 3: The  $Q^2$  dependence of the three loop Pade improved coupling  $\alpha^{(3)}(Q^2, f)$ , (4), and the corresponding analytic coupling  $\mathcal{A}_1^{(3)}(Q^2, f)$  for  $f = 5$  and  $\Lambda_5 = 264 \text{ MeV}$  ( $\Lambda_3 = 400 \text{ MeV}$ ).  $\Delta(\%)$  denotes the relative difference between the couplings.

| $Q \text{ GeV}$ | $\alpha^{(3)}(Q^2, 5)$ | $\mathcal{A}_1^{(3)}(Q^2, 5)$ | $\Delta(\%)$ | $Q \text{ GeV}$ | $\alpha^{(3)}(Q^2, 5)$ | $\mathcal{A}_1^{(3)}(Q^2, 5)$ | $\Delta(\%)$ |
|-----------------|------------------------|-------------------------------|--------------|-----------------|------------------------|-------------------------------|--------------|
| 5               | .22814                 | .22494                        | 1.4          | 60              | .13094                 | .13075                        | .14          |
| 6               | .21610                 | .21383                        | 1.0          | 62              | .13022                 | .13003                        | .14          |
| 7               | .20692                 | .20521                        | .82          | 64              | .12953                 | .12934                        | .14          |
| 8               | .19959                 | .19825                        | .67          | 66              | .12887                 | .12868                        | .14          |
| 9               | .19357                 | .19247                        | .56          | 68              | .12823                 | .12805                        | .14          |
| 10              | .18849                 | .18757                        | .48          | 70              | .12762                 | .12744                        | .14          |
| 11              | .18413                 | .18334                        | .42          | 72              | .12703                 | .12686                        | .14          |
| 12              | .18033                 | .17964                        | .38          | 74              | .12647                 | .12629                        | .14          |
| 13              | .17697                 | .17636                        | .34          | 76              | .12592                 | .12575                        | .14          |
| 14              | .17398                 | .17343                        | .31          | 78              | .12540                 | .12522                        | .14          |
| 15              | .17128                 | .17078                        | .29          | 80              | .12489                 | .12471                        | .14          |
| 16              | .16884                 | .16838                        | .27          | 82              | .12439                 | .12422                        | .14          |
| 17              | .16661                 | .16618                        | .25          | 84              | .12392                 | .12374                        | .14          |
| 18              | .16456                 | .16416                        | .24          | 86              | .12345                 | .12328                        | .14          |
| 19              | .16267                 | .16230                        | .22          | 88              | .12301                 | .12283                        | .14          |
| 20              | .16092                 | .16057                        | .21          | 90              | .12257                 | .12240                        | .14          |
| 21              | .15929                 | .15895                        | .21          | 92              | .12215                 | .12198                        | .14          |
| 22              | .15777                 | .15745                        | .20          | 94              | .12174                 | .12157                        | .14          |
| 23              | .15634                 | .15603                        | .19          | 96              | .12134                 | .12117                        | .14          |
| 24              | .15500                 | .15470                        | .19          | 98              | .12095                 | .12078                        | .14          |
| 25              | .15373                 | .15345                        | .18          | 100             | .1205                  | .1204                         | .14          |
| 26              | .15254                 | .15226                        | .18          | 105             | .11967                 | .11950                        | .14          |
| 27              | .15140                 | .15114                        | .17          | 110             | .11882                 | .11865                        | .14          |
| 28              | .15033                 | .15007                        | .17          | 115             | .11803                 | .11786                        | .14          |
| 29              | .14931                 | .14905                        | .16          | 120             | .11727                 | .11710                        | .14          |
| 30              | .14833                 | .14808                        | .16          | 125             | .11656                 | .11639                        | .14          |
| 32              | .14651                 | .14628                        | .16          | 130             | .11588                 | .11571                        | .14          |
| 34              | .14485                 | .14462                        | .15          | 135             | .11523                 | .11507                        | .14          |
| 36              | .14331                 | .14309                        | .15          | 140             | .11462                 | .11445                        | .14          |
| 38              | .14189                 | .14167                        | .15          | 145             | .11403                 | .11387                        | .14          |
| 40              | .14056                 | .14035                        | .15          | 150             | .11347                 | .11331                        | .14          |
| 42              | .13933                 | .13912                        | .14          | 155             | .11294                 | .11277                        | .14          |
| 44              | .13817                 | .13797                        | .14          | 160             | .11242                 | .11225                        | .14          |
| 46              | .13709                 | .13689                        | .14          | 165             | .11193                 | .11176                        | .14          |
| 48              | .13606                 | .13586                        | .14          | 170             | .11145                 | .11128                        | .14          |
| 50              | .13509                 | .13490                        | .14          | 175             | .11099                 | .11083                        | .14          |
| 52              | .13418                 | .13398                        | .14          | 180             | .11055                 | .11038                        | .15          |
| 54              | .13331                 | .13312                        | .14          | 185             | .11012                 | .10996                        | .15          |
| 56              | .13248                 | .13229                        | .14          | 190             | .10971                 | .10955                        | .15          |
| 58              | .13169                 | .13150                        | .14          | 195             | .10932                 | .10915                        | .15          |
|                 |                        |                               |              | 200             | .10893                 | .10877                        | .15          |

Table 4: The three loop analytic coupling  $\mathfrak{A}_1^{(3)}(s, 5)$  ( see (23)) versus the ordinary 3-loop Pade approximated coupling  $\alpha^{(3)}(s, 5)$ . We have assumed that  $\Lambda_3 = 400 \text{ MeV}$ , correspondingly  $\Lambda_5 = 264 \text{ MeV}$ .

| $\sqrt{s} \text{ GeV}$ | $\alpha^{(3)}(s, 5)$ | $\mathfrak{A}_1^{(3)}(s, 5)$ | $\Delta(\%)$ | $\sqrt{s} \text{ GeV}$ | $\alpha^{(3)}(s, 5)$ | $\mathfrak{A}_1^{(3)}(s, 5)$ | $\Delta(\%)$ |
|------------------------|----------------------|------------------------------|--------------|------------------------|----------------------|------------------------------|--------------|
| 5                      | .22814               | .21221                       | 7.5          | 60                     | .13094               | .12793                       | 2.3          |
| 6                      | .21610               | .20253                       | 6.7          | 62                     | .13022               | .12726                       | 2.3          |
| 7                      | .20692               | .19498                       | 6.1          | 64                     | .12953               | .12662                       | 2.2          |
| 8                      | .19959               | .18887                       | 5.6          | 66                     | .12887               | .12600                       | 2.2          |
| 9                      | .19357               | .18378                       | 5.3          | 68                     | .12823               | .12541                       | 2.2          |
| 10                     | .18849               | .17945                       | 5.0          | 70                     | .12762               | .12484                       | 2.2          |
| 11                     | .18413               | .17570                       | 4.7          | 72                     | .12703               | .12429                       | 2.2          |
| 12                     | .18033               | .17241                       | 4.5          | 74                     | .12647               | .12376                       | 2.1          |
| 13                     | .17697               | .16949                       | 4.4          | 76                     | .12592               | .12325                       | 2.1          |
| 14                     | .17398               | .16687                       | 4.2          | 78                     | .12540               | .12276                       | 2.1          |
| 15                     | .17128               | .16450                       | 4.1          | 80                     | .12489               | .12228                       | 2.1          |
| 16                     | .16884               | .16234                       | 4.0          | 82                     | .12439               | .12182                       | 2.1          |
| 17                     | .16661               | .16037                       | 3.8          | 84                     | .12392               | .12137                       | 2.0          |
| 18                     | .16456               | .15855                       | 3.7          | 86                     | .12345               | .12094                       | 2.0          |
| 19                     | .16267               | .15687                       | 3.7          | 88                     | .12301               | .12052                       | 2.0          |
| 20                     | .16092               | .15530                       | 3.6          | 90                     | .12257               | .12011                       | 2.0          |
| 21                     | .15929               | .15384                       | 3.5          | 92                     | .12215               | .11972                       | 2.0          |
| 22                     | .15777               | .15247                       | 3.4          | 94                     | .12174               | .11933                       | 2.0          |
| 23                     | .15634               | .15119                       | 3.4          | 96                     | .12134               | .11896                       | 2.0          |
| 24                     | .15500               | .14998                       | 3.3          | 98                     | .12095               | .11859                       | 1.9          |
| 25                     | .15373               | .14884                       | 3.2          | 100                    | .12057               | .11823                       | 1.9          |
| 26                     | .15254               | .14776                       | 3.2          | 105                    | .11967               | .11739                       | 1.9          |
| 27                     | .15140               | .14673                       | 3.1          | 110                    | .11882               | .11659                       | 1.9          |
| 28                     | .15033               | .14576                       | 3.1          | 115                    | .11803               | .11583                       | 1.8          |
| 29                     | .14931               | .14483                       | 3.0          | 120                    | .11727               | .11512                       | 1.8          |
| 30                     | .14833               | .14394                       | 3.0          | 125                    | .11656               | .11445                       | 1.8          |
| 32                     | .14651               | .14228                       | 2.9          | 130                    | .11588               | .11381                       | 1.8          |
| 34                     | .14485               | .14076                       | 2.9          | 135                    | .11523               | .11320                       | 1.8          |
| 36                     | .14331               | .13935                       | 2.8          | 140                    | .11462               | .11261                       | 1.7          |
| 38                     | .14189               | .13805                       | 2.7          | 145                    | .11403               | .11206                       | 1.7          |
| 40                     | .14056               | .13683                       | 2.7          | 150                    | .11347               | .11153                       | 1.7          |
| 42                     | .13933               | .13570                       | 2.6          | 155                    | .11294               | .11102                       | 1.7          |
| 44                     | .13817               | .13463                       | 2.6          | 160                    | .11242               | .11053                       | 1.7          |
| 46                     | .13709               | .13363                       | 2.5          | 165                    | .11193               | .11006                       | 1.6          |
| 48                     | .13606               | .13268                       | 2.5          | 170                    | .11145               | .10961                       | 1.6          |
| 50                     | .13509               | .13179                       | 2.5          | 175                    | .11099               | .10917                       | 1.6          |
| 52                     | .13418               | .13094                       | 2.4          | 180                    | .11055               | .10875                       | 1.6          |
| 54                     | .13331               | .13013                       | 2.4          | 185                    | .11012               | .10835                       | 1.6          |
| 56                     | .13248               | .12936                       | 2.4          | 190                    | .10971               | .10796                       | 1.6          |
| 58                     | .13169               | .12863                       | 2.3          | 195                    | .10932               | .10758                       | 1.6          |
|                        |                      |                              |              | 200                    | .10893               | .10721                       | 1.6          |

Table 5: The two and three loop couplings  $\alpha_s^{(k)}(Q^2, 5)$ ,  $\mathcal{A}_1^{(k)}(Q^2, 5)$  and  $\mathfrak{A}_1^{(k)}(s, 5)$  ( $k = 2, 3$ ) as a functions of variables  $Q$  and  $\sqrt{s}$ . We take  $f=5$  and  $\Lambda_5 = 215$  MeV.

| $Q, \sqrt{s}$<br>GeV | $\alpha_s^{(2)}(Q^2, 5)$ | $\mathcal{A}_1^{(2)}(Q^2, 5)$ | $\mathfrak{A}_1^{(2)}(s, 5)$ | $\alpha_s^{(3)}(Q^2, 5)$ | $\mathcal{A}_1^{(3)}(Q^2, 5)$ | $\mathfrak{A}_1^{(3)}(s, 5)$ |
|----------------------|--------------------------|-------------------------------|------------------------------|--------------------------|-------------------------------|------------------------------|
| 20                   | .15373                   | .15345                        | .14888                       | .15429                   | .15400                        | .14934                       |
| 25                   | .14719                   | .14696                        | .14294                       | .14769                   | .14744                        | .14335                       |
| 30                   | .14226                   | .14205                        | .13843                       | .14271                   | .14249                        | .13880                       |
| 35                   | .13835                   | .13815                        | .13483                       | .13876                   | .13856                        | .13517                       |
| 40                   | .13514                   | .13495                        | .13185                       | .13552                   | .13532                        | .13218                       |
| 45                   | .13243                   | .13224                        | .12934                       | .13279                   | .13260                        | .12965                       |
| 50                   | .13010                   | .12992                        | .12717                       | .13044                   | .13025                        | .12746                       |
| 55                   | .12806                   | .12788                        | .12527                       | .12838                   | .12820                        | .12555                       |
| 60                   | .12626                   | .12608                        | .12358                       | .12656                   | .12639                        | .12385                       |
| 65                   | .12464                   | .12447                        | .12207                       | .12494                   | .12476                        | .12233                       |
| 70                   | .12318                   | .12301                        | .12070                       | .12347                   | .12330                        | .12096                       |
| 75                   | .12186                   | .12169                        | .11946                       | .12213                   | .12196                        | .11970                       |
| 80                   | .12064                   | .12047                        | .11831                       | .12091                   | .12074                        | .11855                       |
| 85                   | .11952                   | .11936                        | .11726                       | .11979                   | .11962                        | .11749                       |
| 90                   | .11849                   | .11832                        | .11628                       | .11874                   | .11857                        | .11651                       |
| 95                   | .11753                   | .11736                        | .11538                       | .11778                   | .11761                        | .11560                       |
| 100                  | .11663                   | .11646                        | .11453                       | .11687                   | .11670                        | .11474                       |
| 105                  | .11579                   | .11562                        | .11373                       | .11603                   | .11586                        | .11394                       |
| 110                  | .11500                   | .11483                        | .11298                       | .11523                   | .11506                        | .11319                       |
| 115                  | .11425                   | .11408                        | .11228                       | .11448                   | .11431                        | .11248                       |
| 120                  | .11355                   | .11338                        | .11161                       | .11377                   | .11360                        | .11181                       |
| 125                  | .11288                   | .11271                        | .11098                       | .11310                   | .11293                        | .11117                       |
| 130                  | .11225                   | .11208                        | .11037                       | .11246                   | .11230                        | .11057                       |
| 135                  | .11164                   | .11148                        | .10980                       | .11186                   | .11169                        | .10999                       |
| 140                  | .11107                   | .11090                        | .10926                       | .11128                   | .11111                        | .10945                       |
| 145                  | .11052                   | .11035                        | .10873                       | .11073                   | .11056                        | .10892                       |
| 150                  | .10999                   | .10983                        | .10823                       | .11020                   | .11003                        | .10842                       |
| 155                  | .10949                   | .10932                        | .10776                       | .10969                   | .10953                        | .10794                       |
| 160                  | .10901                   | .10884                        | .10730                       | .10921                   | .10904                        | .10748                       |
| 165                  | .10854                   | .10838                        | .10685                       | .10874                   | .10857                        | .10703                       |
| 170                  | .10810                   | .10793                        | .10643                       | .10829                   | .10813                        | .10660                       |

Table 6: The shift constants  $C_k(f)$  as a functions of the parameter  $\Lambda_3$ .

| $\Lambda_3$ MeV | 250   | 300   | 350   | 400   | 450   |
|-----------------|-------|-------|-------|-------|-------|
| $C_1(3)$        | .0110 | .0137 | .0169 | .0203 | .024  |
| $C_2(3)$        | .0062 | .0079 | .0099 | .0120 | .014  |
| $C_3(3)$        | .0022 | .0028 | .0035 | .0042 | .0049 |
| $C_1(4)$        | .0026 | .0032 | .0037 | .0043 | .0049 |
| $C_2(4)$        | .0013 | .0016 | .0019 | .0023 | .0027 |
| $C_3(4)$        | .0004 | .0005 | .0007 | .0008 | .0010 |
| $C_1(5)$        | .0003 | .0003 | .0003 | .0083 | .0004 |
| $C_2(5)$        | .0001 | .0001 | .0001 | .0020 | .0001 |
| $C_3(5)$        | .0000 | .0000 | .0000 | .0003 | .0000 |

Table 7: The three-loop Padé approximated global analytic coupling  $\mathcal{A}_1(Q^2)$  and the corresponding global “analyticized powers”  $\mathcal{A}_2(Q^2)$  and  $\mathcal{A}_3(Q^2)$  as a functions of the momentum transfer  $Q$  for  $\Lambda_3 = 350 \text{ MeV}$ . The matching conditions give  $\Lambda_4 = 307.8 \text{ MeV}$ ,  $\Lambda_5 = 222.5 \text{ MeV}$  and  $\Lambda_6 = 94.6 \text{ MeV}$ .

| $Q \text{ GeV}$ | $\mathcal{A}_1(Q^2)$ | $\mathcal{A}_2(Q^2)$ | $\mathcal{A}_3(Q^2)$ | $Q \text{ GeV}$ | $\mathcal{A}_1(Q^2)$ | $\mathcal{A}_2(Q^2)$ | $\mathcal{A}_3(Q^2)$ |
|-----------------|----------------------|----------------------|----------------------|-----------------|----------------------|----------------------|----------------------|
| 1               | .36571               | .09320               | .015540              | 38              | .13774               | .01898               | .00258               |
| 2               | .28689               | .06861               | .012776              | 40              | .13650               | .01864               | .00252               |
| 3               | .25144               | .05641               | .010598              | 42              | .13534               | .01833               | .00246               |
| 4               | .23041               | .04903               | .009107              | 44              | .13442               | .018047              | .00240               |
| 5               | .21615               | .04403               | .008042              | 46              | .13340               | .01777               | .00235               |
| 6               | .20566               | .04039               | .007245              | 48              | .13244               | .01752               | .00230               |
| 7               | .19753               | .037601              | .006626              | 50              | .13153               | .01728               | .00225               |
| 8               | .19098               | .035378              | .006131              | 52              | .13067               | .01706               | .00221               |
| 9               | .18555               | .033557              | .005724              | 54              | .12986               | .01685               | .00217               |
| 10              | .18095               | .032033              | .005384              | 56              | .12908               | .01665               | .00213               |
| 11              | .17698               | .030734              | .005095              | 58              | .12834               | .01646               | .00210               |
| 12              | .17351               | .029610              | .004845              | 60              | .12763               | .01628               | .00206               |
| 13              | .17043               | .028626              | .004628              | 62              | .12696               | .01611               | .00203               |
| 14              | .16768               | .027755              | .004436              | 64              | .12631               | .01595               | .00200               |
| 15              | .16520               | .026977              | .004266              | 66              | .12569               | .01580               | .00197               |
| 16              | .16295               | .026277              | .004113              | 68              | .12510               | .01565               | .00195               |
| 17              | .16089               | .025643              | .003975              | 70              | .12452               | .01551               | .00192               |
| 18              | .15900               | .0250652             | .003851              | 72              | .12397               | .01537               | .00190               |
| 19              | .15725               | .024535              | .003737              | 74              | .12344               | .01524               | .00187               |
| 20              | .15563               | .02405               | .00363               | 76              | .12293               | .01511               | .00185               |
| 21              | .15412               | .02359               | .00353               | 78              | .12244               | .01499               | .00183               |
| 22              | .15270               | .02317               | .00344               | 80              | .12196               | .01488               | .00181               |
| 23              | .15138               | .02278               | .00336               | 82              | .12150               | .01477               | .00179               |
| 24              | .15013               | .02242               | .00328               | 84              | .12105               | .01466               | .00177               |
| 25              | .14895               | .02208               | .00321               | 86              | .12062               | .01455               | .00175               |
| 26              | .14784               | .02176               | .00315               | 88              | .12020               | .01445               | .00173               |
| 27              | .14678               | .02145               | .00308               | 90              | .11979               | .01436               | .00171               |
| 28              | .14578               | .02117               | .00303               | 92              | .11940               | .01426               | .00170               |
| 29              | .14483               | .02090               | .00297               | 94              | .11902               | .01417               | .00168               |
| 30              | .14392               | .02064               | .00292               | 96              | .11864               | .01408               | .00166               |
| 32              | .14222               | .02016               | .00282               | 98              | .11828               | .01400               | .00165               |
| 34              | .14066               | .01973               | .00273               | 100             | .11793               | .01391               | .00163               |
| 36              | .13923               | .01934               | .00265               |                 |                      |                      |                      |

Table 8: The three-loop Padé approximated global analytic coupling  $\mathfrak{A}_1(s)$  and the corresponding global “analytized powers”  $\mathfrak{A}_2(s)$  and  $\mathfrak{A}_3(s)$  as a functions of the energy  $\sqrt{s}$  for  $\Lambda_3 = 350 \text{ MeV}$ .

| $\sqrt{s} \text{ GeV}$ | $\mathfrak{A}_1(s)$ | $\mathfrak{A}_2(s)$ | $\mathfrak{A}_3(s)$ | $\sqrt{s} \text{ GeV}$ | $\mathfrak{A}_1(s)$ | $\mathfrak{A}_2(s)$ | $\mathfrak{A}_3(s)$ |
|------------------------|---------------------|---------------------|---------------------|------------------------|---------------------|---------------------|---------------------|
| 1                      | .34270              | .09196              | .01903              | 58                     | .12550              | .01544              | .00185              |
| 2                      | .26679              | .06266              | .01286              | 60                     | .12484              | .01528              | .00183              |
| 3                      | .23451              | .05024              | .00979              | 62                     | .12420              | .01513              | .00180              |
| 4                      | .21571              | .04338              | .00811              | 64                     | .12359              | .01498              | .00178              |
| 5                      | .20343              | .03896              | .00701              | 66                     | .12301              | .01484              | .00175              |
| 6                      | .19452              | .03582              | .00623              | 68                     | .12245              | .01471              | .00173              |
| 7                      | .18755              | .03344              | .00566              | 70                     | .12191              | .01459              | .00171              |
| 8                      | .18190              | .03156              | .00521              | 72                     | .12139              | .01447              | .00169              |
| 9                      | .17718              | .03002              | .00485              | 74                     | .12089              | .01435              | .00167              |
| 10                     | .17316              | .02874              | .00456              | 76                     | .12040              | .01424              | .00165              |
| 11                     | .16967              | .02764              | .00432              | 78                     | .11993              | .01413              | .00163              |
| 12                     | .16661              | .02670              | .00411              | 80                     | .11948              | .01403              | .00161              |
| 13                     | .16389              | .02587              | .00392              | 82                     | .11904              | .01393              | .00160              |
| 14                     | .16144              | .02513              | .00376              | 84                     | .11862              | .01383              | .00158              |
| 15                     | .15923              | .02447              | .00362              | 86                     | .11821              | .01373              | .00156              |
| 16                     | .15722              | .02388              | .00350              | 88                     | .11781              | .01364              | .00155              |
| 17                     | .15537              | .02335              | .00338              | 90                     | .11742              | .01356              | .00153              |
| 18                     | .15367              | .02286              | .00328              | 92                     | .11704              | .01347              | .00152              |
| 19                     | .15209              | .02241              | .00319              | 94                     | .11668              | .01339              | .00151              |
| 20                     | .15063              | .02199              | .00311              | 96                     | .11632              | .01331              | .00149              |
| 21                     | .14926              | .02161              | .00303              | 98                     | .11597              | .01323              | .00148              |
| 22                     | .14798              | .02125              | .00296              | 100                    | .1156               | .01316              | .00147              |
| 23                     | .14677              | .02092              | .00289              | 105                    | .11483              | .01298              | .00144              |
| 24                     | .14563              | .02061              | .00283              | 110                    | .11407              | .01281              | .00141              |
| 25                     | .14456              | .02031              | .00277              | 115                    | .11335              | .01265              | .00139              |
| 26                     | .14355              | .02004              | .00271              | 120                    | .11267              | .01251              | .00136              |
| 27                     | .14258              | .01978              | .00266              | 125                    | .11203              | .01237              | .00134              |
| 28                     | .14166              | .01953              | .00262              | 130                    | .11142              | .01224              | .00132              |
| 29                     | .14079              | .01930              | .00257              | 135                    | .11084              | .01211              | .00130              |
| 30                     | .13996              | .01908              | .00253              | 140                    | .11028              | .01199              | .00128              |
| 32                     | .13840              | .01867              | .00245              | 145                    | .10975              | .01188              | .00126              |
| 34                     | .13696              | .01829              | .00238              | 150                    | .10925              | .01177              | .00125              |
| 36                     | .13564              | .01795              | .00231              | 155                    | .10876              | .01167              | .00123              |
| 38                     | .13441              | .01764              | .00225              | 160                    | .10830              | .01157              | .00122              |
| 40                     | .13326              | .01735              | .00220              | 165                    | .10785              | .01148              | .00120              |
| 42                     | .13219              | .01708              | .00215              | 170                    | .10742              | .01139              | .00119              |
| 44                     | .13118              | .01682              | .00210              | 175                    | .10704              | .01131              | .00118              |
| 46                     | .13023              | .01659              | .00206              | 180                    | .10667              | .01124              | .00116              |
| 48                     | .12933              | .01637              | .00202              | 185                    | .10632              | .01116              | .00115              |
| 50                     | .12849              | .01616              | .00198              | 190                    | .10598              | .01109              | .00114              |
| 52                     | .12768              | .01596              | .00195              | 195                    | .10565              | .01102              | .00113              |
| 54                     | .12692              | .01578              | .00192              | 200                    | .10533              | .01096              | .00112              |

Table 9: The three-loop Padé approximated global analytic coupling  $\mathcal{A}_1(Q^2)$  and the corresponding “analyticized powers”  $\mathcal{A}_2(Q^2)$  and  $\mathcal{A}_3(Q^2)$  as a functions of the momentum transfer  $Q$  for  $\Lambda_3 = 400 \text{ MeV}$ . The matching conditions give  $\Lambda_4 = 358.7 \text{ MeV}$  and  $\Lambda_5 = 263.9 \text{ MeV}$ .

| $Q \text{ GeV}$ | $\mathcal{A}_1(Q^2)$ | $\mathcal{A}_2(Q^2)$ | $\mathcal{A}_3(Q^2)$ | $Q \text{ GeV}$ | $\mathcal{A}_1(Q^2)$ | $\mathcal{A}_2(Q^2)$ | $\mathcal{A}_3(Q^2)$ |
|-----------------|----------------------|----------------------|----------------------|-----------------|----------------------|----------------------|----------------------|
| 1               | .38656               | .09988               | .01629               | 58              | .13190               | .01743               | .00228               |
| 2               | .30263               | .07438               | .01387               | 60              | .13116               | .01723               | .00224               |
| 3               | .26444               | .06128               | .01164               | 62              | .13044               | .01705               | .00221               |
| 4               | .24171               | .05325               | .01005               | 64              | .12976               | .01687               | .00217               |
| 5               | .22627               | .04777               | .00890               | 66              | .12910               | .01670               | .00214               |
| 6               | .21492               | .04377               | .00803               | 68              | .12847               | .01654               | .00211               |
| 7               | .20613               | .04069               | .00734               | 70              | .12786               | .01639               | .00208               |
| 8               | .19905               | .03824               | .00679               | 72              | .12728               | .01624               | .00206               |
| 9               | .19318               | .03623               | .00634               | 74              | .12672               | .01610               | .00203               |
| 10              | .18822               | .03454               | .00596               | 76              | .12618               | .01596               | .00200               |
| 11              | .18394               | .03311               | .00564               | 78              | .12566               | .01583               | .00198               |
| 12              | .18020               | .03187               | .00536               | 80              | .12515               | .01570               | .00196               |
| 13              | .17689               | .03078               | .00490               | 82              | .12467               | .01558               | .00193               |
| 14              | .17393               | .02982               | .00471               | 84              | .12420               | .01547               | .00191               |
| 15              | .17127               | .02896               | .00454               | 86              | .12374               | .01535               | .00189               |
| 16              | .16885               | .02819               | .00438               | 88              | .12330               | .01524               | .00187               |
| 17              | .16664               | .02749               | .00424               | 90              | .12287               | .01514               | .00185               |
| 18              | .16461               | .02685               | .00411               | 92              | .12245               | .01504               | .00183               |
| 19              | .16273               | .02627               | .00400               | 94              | .12204               | .01494               | .00182               |
| 20              | .16099               | .02573               | .00389               | 96              | .12165               | .01484               | .00180               |
| 21              | .15937               | .02523               | .00379               | 98              | .12127               | .01475               | .00178               |
| 22              | .15786               | .02477               | .00370               | 100             | .12090               | .01466               | .00177               |
| 23              | .15644               | .02434               | .00361               | 105             | .12001               | .01445               | .00173               |
| 24              | .15510               | .02394               | .00353               | 110             | .11917               | .01425               | .00169               |
| 25              | .15385               | .02357               | .00346               | 115             | .11839               | .01406               | .00166               |
| 26              | .15266               | .02321               | .00339               | 120             | .11765               | .01389               | .00163               |
| 27              | .15153               | .02288               | .00332               | 125             | .11695               | .01372               | .00160               |
| 28              | .15046               | .02257               | .00326               | 130             | .11628               | .01357               | .00157               |
| 29              | .14944               | .02227               | .00320               | 135             | .11565               | .01342               | .00155               |
| 30              | .14847               | .02199               | .00309               | 140             | .11505               | .01328               | .00153               |
| 32              | .14666               | .02147               | .00299               | 145             | .11448               | .01315               | .00150               |
| 34              | .14500               | .02100               | .00290               | 150             | .11394               | .01303               | .00148               |
| 36              | .14347               | .02056               | .00282               | 155             | .11341               | .01291               | .00146               |
| 38              | .14205               | .02017               | .00275               | 160             | .11291               | .01280               | .00144               |
| 40              | .14074               | .01980               | .00269               | 165             | .11243               | .01269               | .00142               |
| 42              | .13951               | .01946               | .00262               | 170             | .11197               | .01258               | .00141               |
| 44              | .13835               | .01915               | .00256               | 175             | .11153               | .01248               | .00139               |
| 46              | .13727               | .01885               | .00251               | 180             | .11110               | .01239               | .00137               |
| 48              | .13625               | .01858               | .00245               | 185             | .11069               | .01230               | .00136               |
| 50              | .13529               | .01832               | .00241               | 190             | .11029               | .01221               | .00134               |
| 52              | .13438               | .01808               | .00236               | 195             | .10991               | .01212               | .00133               |
| 54              | .13351               | .01785               | .00232               | 200             | .10954               | .01204               | .00132               |



Table 10: The three-loop Padé approximated global analytic coupling  $\mathfrak{A}_1(s)$  and the corresponding global “analyticized powers”  $\mathfrak{A}_2(s)$  and  $\mathfrak{A}_3(s)$  as a functions of the energy  $\sqrt{s}$  for  $\Lambda_3 = 400 \text{ MeV}$ .

| $\sqrt{s} \text{ GeV}$ | $\mathfrak{A}_1(s)$ | $\mathfrak{A}_2(s)$ | $\mathfrak{A}_3(s)$ | $\sqrt{s} \text{ GeV}$ | $\mathfrak{A}_1(s)$ | $\mathfrak{A}_2(s)$ | $\mathfrak{A}_3(s)$ |
|------------------------|---------------------|---------------------|---------------------|------------------------|---------------------|---------------------|---------------------|
| 1                      | .36427              | .10048              | .02061              | 58                     | .12898              | .01628              | .00201              |
| 2                      | .28168              | .06873              | .01440              | 60                     | .12828              | .01611              | .00198              |
| 3                      | .24639              | .05489              | .01100              | 62                     | .12761              | .01595              | .00195              |
| 4                      | .22592              | .04723              | .00911              | 64                     | .12697              | .01579              | .00192              |
| 5                      | .21256              | .04228              | .00785              | 66                     | .12635              | .01564              | .00189              |
| 6                      | .20288              | .03876              | .00696              | 68                     | .12576              | .01550              | .00187              |
| 7                      | .19533              | .03610              | .00630              | 70                     | .12519              | .01537              | .00184              |
| 8                      | .18922              | .03400              | .00579              | 72                     | .12464              | .01523              | .00182              |
| 9                      | .18413              | .03230              | .00539              | 74                     | .12411              | .01511              | .00180              |
| 10                     | .17980              | .03087              | .00505              | 76                     | .12360              | .01499              | .00178              |
| 11                     | .17605              | .02966              | .00477              | 78                     | .12311              | .01487              | .00176              |
| 12                     | .17276              | .02861              | .00453              | 80                     | .12263              | .01476              | .00174              |
| 13                     | .16984              | .02769              | .00433              | 82                     | .12217              | .01465              | .00172              |
| 14                     | .16722              | .02688              | .00415              | 84                     | .12172              | .01455              | .00170              |
| 15                     | .16485              | .02616              | .00399              | 86                     | .12129              | .01445              | .00168              |
| 16                     | .16269              | .02551              | .00385              | 88                     | .12087              | .01435              | .00167              |
| 17                     | .16072              | .02492              | .00372              | 90                     | .12046              | .01425              | .00165              |
| 18                     | .15890              | .02438              | .00360              | 92                     | .12006              | .01416              | .00164              |
| 19                     | .15721              | .02388              | .00350              | 94                     | .11968              | .01407              | .00162              |
| 20                     | .15565              | .02343              | .00340              | 96                     | .11930              | .01399              | .00161              |
| 21                     | .15419              | .02301              | .00332              | 98                     | .11894              | .01391              | .00159              |
| 22                     | .15282              | .02262              | .00323              | 100                    | .1185               | .01382              | .00158              |
| 23                     | .15154              | .02225              | .00316              | 105                    | .11773              | .01363              | .00155              |
| 24                     | .15033              | .02191              | .00309              | 110                    | .11694              | .01345              | .00152              |
| 25                     | .14919              | .02159              | .00302              | 115                    | .11618              | .01328              | .00149              |
| 26                     | .14811              | .02129              | .00296              | 120                    | .11547              | .01313              | .00146              |
| 27                     | .14708              | .02101              | .00291              | 125                    | .11480              | .01298              | .00144              |
| 28                     | .14610              | .02074              | .00285              | 130                    | .11415              | .01283              | .00142              |
| 29                     | .14518              | .02048              | .00280              | 135                    | .11354              | .01270              | .00139              |
| 30                     | .14429              | .02024              | .00275              | 140                    | .11296              | .01257              | .00138              |
| 32                     | .14263              | .01979              | .00267              | 145                    | .11241              | .01245              | .00136              |
| 34                     | .14111              | .01939              | .00259              | 150                    | .11188              | .01234              | .00134              |
| 36                     | .13970              | .01902              | .00252              | 155                    | .11137              | .01223              | .00132              |
| 38                     | .13840              | .01867              | .00245              | 160                    | .11088              | .01212              | .00130              |
| 40                     | .13718              | .01836              | .00239              | 165                    | .11041              | .01202              | .00129              |
| 42                     | .13605              | .01806              | .00233              | 170                    | .10996              | .01193              | .00127              |
| 44                     | .13498              | .01779              | .00228              | 175                    | .10956              | .01184              | .00126              |
| 46                     | .13398              | .01753              | .00223              | 180                    | .10918              | .01176              | .00125              |
| 48                     | .13303              | .01729              | .00219              | 185                    | .10881              | .01168              | .00123              |
| 50                     | .13213              | .01707              | .00215              | 190                    | .10845              | .01161              | .00122              |
| 52                     | .13129              | .01685              | .00211              | 195                    | .10811              | .01154              | .00121              |
| 54                     | .13048              | .01665              | .00207              | 200                    | .10777              | .01147              | .00120              |

Table 11: The three-loop Padé approximated global analytic coupling  $\mathcal{A}_1(Q^2)$  and the corresponding global “analytized powers”  $\mathcal{A}_2(Q^2)$  and  $\mathcal{A}_3(Q^2)$  as a functions of the momentum transfer  $Q$  for  $\Lambda_3 = 450 \text{ MeV}$ . The matching conditions give:  $\Lambda_4 = 411.1 \text{ MeV}$ ,  $\Lambda_5 = 307.5 \text{ MeV}$  and  $\Lambda_6 = 135.4 \text{ MeV}$ .

| $Q \text{ GeV}$ | $\mathcal{A}_1(Q^2)$ | $\mathcal{A}_2(Q^2)$ | $\mathcal{A}_3(Q^2)$ | $Q \text{ GeV}$ | $\mathcal{A}_1(Q^2)$ | $\mathcal{A}_2(Q^2)$ | $\mathcal{A}_3(Q^2)$ |
|-----------------|----------------------|----------------------|----------------------|-----------------|----------------------|----------------------|----------------------|
| 1               | .40668               | .10611               | .01692               | 38              | .146322              | .02133               | .00306               |
| 2               | .31810               | .07993               | .01488               | 40              | .144927              | .02093               | .00298               |
| 3               | .27716               | .06602               | .01264               | 42              | .143626              | .02057               | .00291               |
| 4               | .25295               | .05739               | .01098               | 44              | .142408              | .02022               | .00284               |
| 5               | .23638               | .05146               | .00975               | 46              | .141263              | .01991               | .00277               |
| 6               | .22419               | .04710               | .00880               | 48              | .140186              | .01961               | .00271               |
| 7               | .21459               | .04374               | .00806               | 50              | .139168              | .01933               | .00266               |
| 8               | .20715               | .04106               | .00746               | 52              | .138205              | .01907               | .00260               |
| 9               | .20086               | .03886               | .00696               | 54              | .137291              | .01882               | .00255               |
| 10              | .19554               | .03702               | .00654               | 56              | .136422              | .01858               | .00251               |
| 11              | .19095               | .03545               | .00618               | 58              | .135595              | .01836               | .00246               |
| 12              | .18695               | .03410               | .00587               | 60              | .134806              | .01815               | .00243               |
| 13              | .18341               | .03291               | .00560               | 62              | .134052              | .01795               | .00238               |
| 14              | .18024               | .03185               | .00537               | 64              | .133330              | .01776               | .00235               |
| 15              | .17740               | .03092               | .00515               | 66              | .132639              | .01758               | .00231               |
| 16              | .17481               | .03007               | .00496               | 68              | .131975              | .01740               | .00228               |
| 17              | .17245               | .02931               | .00479               | 70              | .131338              | .01724               | .00225               |
| 18              | .17029               | .02861               | .00464               | 72              | .130725              | .01708               | .00222               |
| 19              | .16829               | .02797               | .00450               | 74              | .130134              | .01693               | .00219               |
| 20              | .16643               | .02739               | .00437               | 76              | .129565              | .01678               | .00216               |
| 21              | .16471               | .02685               | .00425               | 78              | .12885               | .01664               | .00213               |
| 22              | .16310               | .02634               | .00414               | 80              | .128485              | .01650               | .00211               |
| 23              | .16159               | .02587               | .00403               | 82              | .127972              | .01637               | .00208               |
| 24              | .16017               | .02544               | .00394               | 84              | .127476              | .01625               | .00206               |
| 25              | .15883               | .02503               | .00385               | 86              | .126995              | .01613               | .00204               |
| 26              | .15756               | .02464               | .00377               | 88              | .126530              | .01601               | .00201               |
| 27              | .15637               | .02428               | .00369               | 90              | .126078              | .01590               | .00199               |
| 28              | .15523               | .02394               | .00361               | 92              | .125640              | .01579               | .00197               |
| 29              | .15415               | .02362               | .00354               | 94              | .125215              | .01568               | .00195               |
| 30              | .15312               | .02331               | .00348               | 96              | .124801              | .01558               | .00193               |
| 32              | .15120               | .02274               | .00336               | 98              | .124238              | .01548               | .00192               |
| 34              | .14944               | .02223               | .00325               | 100             | .12384               | .01538               | .00190               |
| 36              | .14782               | .02176               | .00315               |                 |                      |                      |                      |

Table 12: The three-loop Padé approximated global analytic coupling  $\mathfrak{A}_1(s)$  and the corresponding “analyticed powers”  $\mathfrak{A}_2(s)$  and  $\mathfrak{A}_3(s)$  as a functions of the energy  $\sqrt{s}$  for  $\Lambda_3 = 450$  MeV.

| $\sqrt{s}$ GeV | $\mathfrak{A}_1(s)$ | $\mathfrak{A}_2(s)$ | $\mathfrak{A}_3(s)$ | $\sqrt{s}$ GeV | $\mathfrak{A}_1(s)$ | $\mathfrak{A}_2(s)$ | $\mathfrak{A}_3(s)$ |
|----------------|---------------------|---------------------|---------------------|----------------|---------------------|---------------------|---------------------|
| 1              | .38512              | .10857              | .02191              | 58             | .13226              | .01710              | .00216              |
| 2              | .29624              | .07474              | .01590              | 60             | .13152              | .01692              | .00212              |
| 3              | .25798              | .05953              | .01221              | 62             | .13082              | .01674              | .00209              |
| 4              | .23585              | .05108              | .01012              | 64             | .13014              | .01657              | .00206              |
| 5              | .22142              | .04560              | .00871              | 66             | .12950              | .01641              | .00203              |
| 6              | .21096              | .04169              | .00770              | 68             | .12887              | .01626              | .00200              |
| 7              | .20283              | .03875              | .00696              | 70             | .12828              | .01612              | .00198              |
| 8              | .19627              | .03643              | .00638              | 72             | .12770              | .01598              | .00195              |
| 9              | .19081              | .03455              | .00592              | 74             | .12715              | .01584              | .00193              |
| 10             | .18617              | .03298              | .00555              | 76             | .12661              | .01571              | .00191              |
| 11             | .18216              | .03165              | .00523              | 78             | .12609              | .01559              | .00188              |
| 12             | .17865              | .03050              | .00497              | 80             | .12559              | .01547              | .00186              |
| 13             | .17553              | .02949              | .00473              | 82             | .12511              | .01535              | .00184              |
| 14             | .17273              | .02861              | .00453              | 84             | .12464              | .01524              | .00182              |
| 15             | .17021              | .02781              | .00436              | 86             | .12419              | .01513              | .00180              |
| 16             | .16792              | .02710              | .00420              | 88             | .12374              | .01503              | .00179              |
| 17             | .16581              | .02646              | .00405              | 90             | .12332              | .01493              | .00177              |
| 18             | .16388              | .02587              | .00393              | 92             | .12290              | .01483              | .00175              |
| 19             | .16209              | .02533              | .00381              | 94             | .12250              | .01473              | .00174              |
| 20             | .16043              | .02483              | .00370              | 96             | .12211              | .01464              | .00172              |
| 21             | .15888              | .02438              | .00360              | 98             | .12172              | .01455              | .00170              |
| 22             | .15743              | .02395              | .00351              | 100            | .12135              | .01447              | .00169              |
| 23             | .15607              | .02355              | .00343              | 105            | .12046              | .01426              | .00165              |
| 24             | .15479              | .02318              | .00335              | 110            | .11963              | .01407              | .00162              |
| 25             | .15358              | .02284              | .00328              | 115            | .11884              | .01389              | .00159              |
| 26             | .15243              | .02251              | .00321              | 120            | .11809              | .01372              | .00156              |
| 27             | .15135              | .02220              | .00315              | 125            | .11739              | .01356              | .00154              |
| 28             | .15032              | .02191              | .00309              | 130            | .11672              | .01341              | .00151              |
| 29             | .14933              | .02163              | .00303              | 135            | .11608              | .01327              | .00149              |
| 30             | .14840              | .02137              | .00298              | 140            | .11547              | .01313              | .00147              |
| 32             | .14665              | .02089              | .00288              | 145            | .11489              | .01300              | .00145              |
| 34             | .14504              | .02045              | .00280              | 150            | .11434              | .01288              | .00143              |
| 36             | .14355              | .02005              | .00272              | 155            | .11381              | .01276              | .00141              |
| 38             | .14218              | .01968              | .00264              | 160            | .11330              | .01265              | .00139              |
| 40             | .14089              | .01933              | .00258              | 165            | .11281              | .01255              | .00137              |
| 42             | .13970              | .01902              | .00252              | 170            | .11233              | .01244              | .00136              |
| 44             | .13857              | .01872              | .00246              | 175            | .11192              | .01235              | .00134              |
| 46             | .13752              | .01845              | .00241              | 180            | .11152              | .01227              | .00133              |
| 48             | .13652              | .01819              | .00236              | 185            | .11114              | .01218              | .00131              |
| 50             | .13558              | .01794              | .00231              | 190            | .11077              | .01210              | .00130              |
| 52             | .13468              | .01772              | .00227              | 195            | .11041              | .01202              | .00129              |
| 54             | .13383              | .01750              | .00223              | 200            | .11006              | .01195              | .00128              |

Such a behavior does not occur in the case of the global timelike coupling  $\mathfrak{A}_n^{(3)}(s)$  (compare Tables 4 and 10). However, the difference between  $\alpha^{(3)}(s, f = 5)$  and  $\mathfrak{A}_1^{(3)}(s)$  is more appreciable, it is about 7% at  $s = 5 \text{ GeV}$ , 3.4% at  $s = 20 \text{ GeV}$  and 1.75% at  $s = 90 \text{ GeV}$ . It is about 1% at  $s = 200 \text{ GeV}$ . The relative difference between  $\mathcal{A}_1^{(3)}(Q^2)$  and  $\mathfrak{A}_1^{(3)}(s)$  decreases from 6.9% at  $Q = \sqrt{s} = 2 \text{ GeV}$  to 2% at  $100 \text{ GeV}$  (see Tables 9 and 10). The relative differences  $\Delta_n$  (%) between  $\mathcal{A}_n^{(3)}(Q^2)$  and  $\mathfrak{A}_n^{(3)}(s)$ , for  $n=2$  and 3, are even large. Thus,  $\Delta_2$  (%)=6.1% and  $\Delta_3$  (%)=11.1% at  $Q = \sqrt{s} = 100 \text{ GeV}$ .

With the algebraic computer system Maple V release 5 we were able to calculate the quantities  $\{\alpha_s(Q^2, f)\}^n$  and  $\mathfrak{A}_n(s, f)$  (see formulas (3), (4), (23)–(25)) with any arbitrary given accuracy. However, this is no case for formulas (5) and (33). These integrals are singular at  $t \rightarrow \pm\infty$ , therefore, one needs to use a cutoff. With Maple V release 5 the cutoff may be as big as  $10^4$ . This guarantees to obtain 4-5 reliable digits after decimal point. Most of our calculations we have performed with this precision. To obtain more accurate results we suggest following formula

$$\mathcal{A}_n(Q^2, f) = \mathcal{A}_n(Q^2, f, R) + \mathfrak{A}_n(\Lambda^2 e^R, f) + O(e^{-R}), \quad (36)$$

Here  $\mathcal{A}_n(Q^2, f, R)$  denotes the regulated integral (5): the integral is taken over the finite region  $-R \leq t \leq R$ . We remark that formula (36) provides sufficiently high precision even for low values of the cutoff. In addition, for practical calculations, here we suggest the formula

$$\mathcal{A}_n(Q^2, f) = Q^2 \int_0^\infty \frac{ds}{(s + Q^2)^2} \mathfrak{A}_n(s, f), \quad (37)$$

evidently, this relation is also valid for the global quantities,  $\mathcal{A}_n(Q^2)$  and  $\mathfrak{A}_n(s)$ .

## 6 Conclusion

The “analytized powers” of the coupling in the spacelike region  $\mathcal{A}_n^{(k)}(Q^2, f)$ , to second and third orders, are written in terms of the Lambert W function (see integrals (5) and (37)).

The “analytized powers” of the coupling in the timelike region,  $\mathfrak{A}_n^{(k)}(s, f)$  ( $k=2,3$ ), are analytically calculated in terms of the Lambert W function (see formulas (20)–(25)). The recurrence relations for  $\mathfrak{A}_n^{(2)}(s, f)$  and  $\mathcal{A}_n^{(2)}(Q^2, f)$  are derived (see formulas (17) and (19)).

The matching conditions for crossing the quark flavor thresholds (26) are solved explicitly for  $\Lambda_f$  in the cases of the exact 2-loop and Pade improved 3-loop solutions (see formulas (3), (4) and (27)).

The global model for the coupling, of Refs.[6, 7], is considered up to the third order in the context of obtained explicit solutions.

In Sec.5 numerical estimations of the explicit solutions for the powers of the standard and analytical couplings are given. We have compared various solutions in the large region of momentum transfer and energy,  $1 \text{ GeV} \leq \sqrt{s}, Q \leq 200 \text{ GeV}$  (see Tables 1–12). We have confirmed that the differences between the powers of the standard iterative solution (34) and the “analytized timelike powers”  $\mathfrak{A}_n^{(3)}(s)$  are appreciable even for moderate energies.

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#### Note Added

As I was writing this I was informed by D.S. Kourashev that he has also obtained the analytical expressions for the “analytized powers” of coupling in the timelike region in the equivalent form [35] (see formulas (20)-(25)).

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Эффективная константа КХД до третьего порядка включительно в стандартной и аналитической теориях возмущений

Изучены два набора специфических функций  $\{\mathfrak{U}_k(s)\}$  и  $\{\mathfrak{A}_k(Q^2)\}$ , которые образуют базисы нестепенных асимптотических разложений для времени- и пространственноподобных наблюдаемых квантовой хромодинамики в аналитической теории возмущений Ширкова–Соловцова. Эти функции (функция связи и ее аналитизированные степени) выражаются в терминах W-функции Ламберта. Во втором порядке теории возмущений мы используем точное решение уравнения ренормгруппы, а в третьем порядке — решение, которое соответствует паде-преобразованной бета-функции. Для обоих наборов функций получены удобные для численного анализа рекуррентные формулы. Для построения глобальных версий этих функций в схеме минимального вычитания мы используем условия непрерывности эффективного заряда на массовых порогах тяжелых кварков. Полученные выражения исследуются численным методом в широком интервале переданного импульса и энергии от 1 до 170 ГэВ. Составлены подробные численные таблицы для первых трех функций  $\mathfrak{U}_k(s)$  и  $\mathfrak{A}_k(Q^2)$ ,  $k=1,2,3$ , как в пространственной, так и во времениподобной областях. Было установлено, что во времениподобной области они заметно отличаются от соответствующих степеней эффективного заряда, вычисленного итеративным методом для умеренных энергий в 5-ароматной области.

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QCD Coupling up to Third Order in Standard and Analytic Perturbation Theories

We analyse two sets of specific functions,  $\{\mathfrak{U}_k(s)\}$  and  $\{\mathfrak{A}_k(Q^2)\}$  that form the basis of the nonpower asymptotic expansions both in the time-like and space-like regions for single scale dependent QCD observables in the Shirkov–Solovtsov Analytic Perturbation Theory (APT) free of unphysical singularities. These functions (effective running couplings and two higher functions ( $k=2,3$ )) are explicitly derived up to the third order in the closed form in terms of the Lambert W-functions. As an input we used the exact two-loop and the three-loop (corresponding to Padé transformed beta-function) RG solutions for common invariant coupling  $\alpha_s$ . The elegant recurrence formulas, helpful for numerical analysis, are obtained for the both sets of the APT functions. Then we construct the global versions of APT functions using the continuity conditions (at the quark thresholds) on the  $\alpha_s$  in the  $\overline{\text{MS}}$  scheme and give numerical results. For first three of these functions  $\mathfrak{U}_k(s)$  and  $\mathfrak{A}_k(Q^2)$ ;  $k=1,2,3$ , in the large interval of the momentum transfer and energy ( $1 \text{ GeV} < Q, \sqrt{s} < 170 \text{ GeV}$ ), numerical tables are presented. From these we observe that, for the time-like arguments, the differences between functions  $\mathfrak{U}_k(s)$  and the corresponding powers of the standard iteratively approximated coupling  $\alpha_s^k(s)$  are not negligible even for moderate energies in the five-flavor region.

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