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THE DIRAC EQUATION IN THE LOBACHEVSKY SPACE-TIME

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The theory of spinor field in the general case of Riemannian space-time is reviewed in [1]. The metric in four-dimensional space-time for any ortogonal coordinates ξ, η, ζ, τ has the following form:

$$ds^2 = h_1^2\, d\xi^2 + h_2^2\, d\eta^2 + h_3^2\, d\zeta^2 - h_0^2\, d\tau^2\,, \tag{1}$$

where h_0, h_1, h_2, h_3 are some functions depending on ξ, η, ζ and τ.

The Dirac-Fock-Ivanenko equation for the spinor field Ψ in Lame basis

$$f^0 = h_0\, d\tau,\ f^1 = h_1\, d\xi,\ f^2 = h_2\, d\eta,\ f^3 = h_3\, d\zeta \tag{2}$$

we can write as

$$\sum_{\mu=0}^{3} \frac{H^\mu}{\sqrt{h h_\mu}} \frac{\partial}{\partial \xi^\mu} \left[\sqrt{h/h_\mu}\,\Psi\right] = \frac{imc}{\hbar} K \Psi\,, \tag{3}$$

where $h = h_0 h_1 h_2 h_3, \quad \xi_0 = \tau, \quad \xi_1 = \xi, \quad \xi_2 = \eta, \quad \xi_3 = \zeta,$

$$H^0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad H^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$H^2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad H^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

$$K = \begin{pmatrix} -1 & 0 & 0 & \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

These matrixes, H^μ and K, are chosen in accordance to the ones in the book [2] of Elie Cartan, who "is the creator of general spinor theory, which foundations were published in 1913 in his classical research in the theory of representations of simple groups." (The quotation is taken from the preface to the Russian translation of the book [2], which is written by the well-known geometrician P. Shirokov from Kasan university, who recommended this book not only for beginners in mathematics, but for theoretical physicists, wanting to get a deeper knowledge of spinor theory).

The particular case is the Lobachevsky space-time with the metric

$$ds^2 = dl^2 - c^2 dt^2 \ , \tag{6}$$

where dl^2 is a metric in Lobachevsky space, c is the speed of light, t is time. For example, in the spherical coordinates the metric dl^2 is

$$dl^2 = d\rho^2 + r^2 d\theta^2 + r^2 \sin^2\theta \ d\phi^2 \ , \tag{7}$$

where

$$r = k \sinh \frac{\rho}{k} \ , \tag{8}$$

k is characteristic measure of lenghs in Lobachevsky space.

Choosing the Lame basis in the form

$$f^0 = c\ dt, \ f^1 = d\rho, \ f^2 = r\ d\theta, \ f^3 = r \sin\theta \ d\phi \ , \tag{9}$$

we can write the equation of Dirac-Fock-Ivanenko for the spinor field Ψ in the Lobachevsky space-time in the following kind:

$$\frac{H^0}{c}\frac{\partial\Psi}{\partial t} + \frac{H^1}{r}\frac{\partial(r\Psi)}{\partial\rho} + \frac{H^2}{r\sqrt{\sin\theta}}\frac{\partial(\sqrt{\sin\theta}\Psi)}{\partial\theta} + \frac{H^3}{r\sin\theta}\frac{\partial\Psi}{\partial\phi} = \frac{imc}{\hbar}K\Psi \ . \tag{10}$$

Let us transform this equation to the special form like the one, which Dirac has found in the paper [3] in 1935. In the paper [4], the procedure of such transformation have been worked out in the case of De Sitter spherical space-time. The procedure is based on the transformation from basis f to basis dx in enveloping pseudo-Euclidean space.

In conformity with the afore-mentioned procedure let us put

$$x = R \sinh\frac{\rho}{k} \sin\theta \cos\phi \ , \quad y = R \sinh\frac{\rho}{k} \sin\theta \sin\phi \ , \tag{11}$$

$$z = R \sinh\frac{\rho}{k} \cos\theta \ , \quad u = R \cosh\frac{\rho}{k} \ .$$

So far as

$$dx^2 + dy^2 + dz^2 - du^2 = \left(\frac{R}{k}\right)^2 dl^2 - dR^2 \ , \tag{12}$$

then in the four-dimensional pseudo-Euclidean space-time with Cartesian coordinates

$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = u \tag{13}$$

the inner geometry of the three-dimensional surface, which is set by equations (11) with $R = k$, is has coinsided with the Lobachevsky geometry.

Let us consider the differential forms (9) together with the form $f^4 = dR$. Accodding to (12) on the surface $R = k$ have fulfilled the following equation:

$$\eta_{ab}\, dx^a dx^b = \eta_{ab}\, f^a f^b , \tag{14}$$

where is meant summation up from 1 to 4 by the indexes a ad b. The numerals η_{ab} are equal 0, if $a \neq b$, and $\eta_{11} = \eta_{22} = \eta_{33} = 1$, $\eta_{44} = -1$. The numerals η_{ab} form metrical tensor of the four-dimensional pseudo-Euclidean space-time in the basis dx.

In view of the equation (14) the transition from the basis f to the fasis dx is achieved by the Lorenz transformation

$$f^a = L^a_b\, dx^b . \tag{15}$$

Differentiating functions (11), we can get constructively the inverse transformation

$$dx^a = \tilde{L}^a_b\, f^b . \tag{16}$$

The mutually inverse matrixes L and $\tilde{L}$ are linked by the orthogonality condition

$$\eta_{as}\, \tilde{L}^s_b = \eta_{bs}\, L^s_a . \tag{17}$$

Then it is convinient to introduce matrixes

$$H_1 = H^1, \;\; H_2 = H^2, \;\; H_3 = H^3, \;\; H_4 = -iK, \;\; H^4 = iK . \tag{18}$$

They fulfill the following requirements

$$H_a = \eta_{ab} H^b , \tag{19}$$

$$H_a H_b + H_b H_a = 2\,\eta_{ab} . \tag{20}$$

Moreover

$$H_a H^0 + H^0 H_a = 0 , \quad H^0 H^0 = -1 . \tag{21}$$

A matrix S can be selected so, that the following equations are accomplished:

$$SH^aS^{-1} = L^a_b H^b \ , \quad S^{-1}H^aS = \tilde{L}^a_b H^b \ ,$$
$$S^{-1}H_aS = L^b_a H_b \ , \quad SH_aS^{-1} = \tilde{L}^b_a H_b \ . \tag{22}$$

If it is designated as

$$dX = H_a dx^a \ , \quad F = H_a F^a \ , \tag{23}$$

the terms will come out as

$$dX = SFS^{-1} \ , \quad F = S^{-1} dX S \ . \tag{24}$$

When making a transition from basis f to basis dX, the spinor Ψ is transrorms with the help of the substitution

$$\Xi = S\Psi \ , \tag{25}$$

so that

$$dX\Xi = SF\Psi \ . \tag{26}$$

Differentiating (11), we get the transformation

$$dx = \cosh\frac{\rho}{k} \ \sin\theta \ \cos\phi \ f^1 + \cos\theta \ \cos\phi \ f^2 - \sin\phi \ f^3 + \frac{x}{k} f^4 \ ,$$
$$dy = \cosh\frac{\rho}{k} \ \sin\theta \ \sin\phi \ f^1 + \cos\theta \ \sin\phi \ f^2 + \cos\phi \ f^3 + \frac{y}{k} \ f^4 \ ,$$
$$dz = \cosh\frac{\rho}{k} \ \cos\theta \ f^1 - \sin\theta \ f^2 + \frac{z}{k} \ f^4 \ ,$$
$$du = \sinh\frac{\rho}{k} \ f^1 + \frac{u}{k} \ f^4 \ . \tag{27}$$

It follows from (27) that

$$k \ \check{L}^a_4 = x^a \ , \tag{28}$$

and so, according to the last of the formulas (22), we get

$$k \ SH_4S^{-1} = X \ , \tag{29}$$

where

$$X = H_a x^a \ . \tag{30}$$

In a detailed form we get

$$SH_4S^{-1} = S(-iK)S^{-1} = \tag{31}$$

$$= \sinh\frac{\rho}{k}\,[\sin\theta(H^1\cos\phi + H^2\sin\phi) + H^3\cos\theta] - iK\cosh\frac{\rho}{k}\,.$$

In the present case the matrix

$$\breve{L} = \begin{pmatrix} \breve{L}^1_1 & \breve{L}^1_2 & \breve{L}^1_3 & \breve{L}^1_4 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \breve{L}^4_1 & \breve{L}^4_2 & \breve{L}^4_3 & \breve{L}^4_4 \end{pmatrix} \tag{32}$$

can be factorized into the product

$$\breve{L} = \breve{L}_1\breve{L}_2\breve{L}_3\breve{L}_4$$

of four matrixes

$$\breve{L}_1 = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \breve{L}_2 = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\breve{L}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\frac{\rho}{k} & \sinh\frac{\rho}{k} \\ 0 & 0 & \sinh\frac{\rho}{k} & \cosh\frac{\rho}{k} \end{pmatrix}, \quad \breve{L}_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{33}$$

Accordingly matrixes S and S^{-1} can be factorized into the products

$$S = S_1\, S_2\, S_3\, S_4 \quad \textit{and} \quad S^{-1} = (S_4)^{-1}(S_3)^{-1}(S_2)^{-1}(S_1)^{-1}$$

of following matrixes:

$$S_1 = \cos\frac{\phi}{2} + H_2H_1\sin\frac{\phi}{2}, \quad S_2 = \cos\frac{\theta}{2} + H_1H_3\sin\frac{\theta}{2}, \tag{34}$$

$$S_3 = \cosh\frac{\rho}{2k} + H_4H_3\sinh\frac{\rho}{2k}, \quad S_4 = \frac{1}{2}\,(H_1 - H_3)\,(H_2 - H_3)$$

and

$$(S_1)^{-1} = \cos\frac{\phi}{2} + H_1H_2\sin\frac{\phi}{2}$$
$$(S_2)^{-1} = \cos\frac{\theta}{2} + H_3H_1\sin\frac{\theta}{2}$$
$$(S_3)^{-1} = \cosh\frac{\rho}{2k} + H_3H_4\sinh\frac{\rho}{2k}$$
$$(S_4)^{-1} = \frac{1}{2}\,(H_2 - H_3)\,(H_1 - H_3).$$

Now let's make up the following table:

$$(S_1)^{-1}H^1S_1 = H^1\cos\phi - H^2\sin\phi$$
$$(S_1)^{-1}H^2S_1 = H^1\sin\phi + H^2\cos\phi$$
$$(S_1)^{-1}H^3S_1 = H^3$$
$$(S_1)^{-1}H^4S_1 = H^4$$
$$(S_2)^{-1}H^1S_2 = H^1\cos\theta + H^3\sin\theta$$
$$(S_2)^{-1}H^2S_2 = H^2$$
$$(S_2)^{-1}H^3S_2 = -H^1\sin\theta + H^3\cos\theta$$
$$(S_2)^{-1}H^4S_2 = H^4$$
$$(S_3)^{-1}H^1S_2 = H^1$$
$$(S_3)^{-1}H^2S_2 = H^1$$
$$(S_3)^{-1}H^3S_3 = H^3\cosh\frac{\rho}{k} + H^3\sinh\frac{\rho}{k}$$
$$(S_3)^{-1}H^4S_3 = H^3\sinh\frac{\rho}{k} + H^3\cosh\frac{\rho}{k}$$
$$(S_4)^{-1}H^1S_4 = H^2$$
$$(S_4)^{-1}H^2S_4 = H^3$$
$$(S_4)^{-1}H^3S_4 = H^1$$
$$(S_4)^{-1}H^4S_4 = H^4. \tag{35}$$

Using this table we can find the following representation of transformation (27):

$$S^{-1}H^1S = H^1 \cosh \frac{\rho}{k} \sin\theta\cos\phi + H^2\cos\theta\cos\phi - H^3\sin\phi + H^4\,\frac{x}{k},$$

$$S^{-1}H^2S = H^1 \cosh \frac{\rho}{k} \sin\theta\sin\phi + H^2\cos\theta\sin\phi + H^3\cos\phi + H^4\,\frac{y}{k},$$

$$S^{-1}H^3S = H^1 \cosh \frac{\rho}{k} - H^2\sin\theta + H^4\,\frac{z}{k},$$

$$S^{-1}H^4S = H^1 \sinh \frac{\rho}{k} + H^4\,\frac{u}{k}. \tag{36}$$

Multiplaying equations (36) from the left by S and from the right by S^{-1}, we shall get a system of linear equations. It follows from the last system that

$$SH^1S^{-1} = [(H^1\cos\phi + H^2\sin\phi)\sin\theta + H^3\cos\theta]\cosh\frac{\rho}{k} - H^4\sinh\frac{\rho}{k}\,,$$

$$SH^2S^{-1} = (H^1\cos\phi + H^2\sin\phi)\cos\theta - H^3\sin\theta\,,$$

$$SH^3S^{-1} = -H^1\sin\phi + H^2\cos\phi\,,$$

$$SH^4S^{-1} = \frac{X}{k}. \tag{37}$$

Now let's introduce the following operators:

$$e_1 = \frac{\partial}{\partial\rho}, \quad e_2 = \frac{1}{r}\frac{\partial}{\partial\theta}\,, \quad e_3 = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\,. \tag{38}$$

With a help of the table (35) it is not difficult to prove that

$$S^{-1}e_1S = e_1 + \frac{1}{2k}H_1H^4,$$

$$S^{-1}e_2S = e_2 + \frac{1}{2k}H_2H^4 + \frac{1}{2k}H_2H^1\coth\frac{\rho}{k}, \tag{39}$$

$$S^{-1}e_3S = e_2 + \frac{1}{2k}H_3H^4 + \frac{1}{2k}H_3H^1\coth\frac{\rho}{k} + \frac{\cot\theta}{2r}H_3H^2,$$

and so

$$\sum_{\nu=1}^{3} H^\nu S^{-1}e_\nu S = \sum_{\nu=1}^{3} H^\nu e_\nu + \frac{H^1}{r}\frac{dr}{d\rho} + \frac{H^2}{r\sqrt{\sin\theta}}\frac{d\sqrt{\sin\theta}}{d\theta} + \frac{3H^4}{2k}\,. \tag{40}$$

Taking into account the substitution (25), we get

$$\frac{H^1}{r}\frac{\partial(r\Psi)}{\partial\rho}+\frac{H^2}{r\sqrt{\sin\theta}}\frac{\partial(\sqrt{\sin\theta}\Psi)}{\partial\theta}+\frac{H^3}{r\sin\theta}\frac{\partial\Psi}{\partial\phi}= \tag{41}$$

$$=\sum_{\nu=1}^{3}H^\nu S^{-1}e_\nu\Xi+\frac{3}{2k}H_4S^{-1}\Xi\,.$$

Substituting this equality into the equation DFI (10), multiplying the obtained result from the left by kS and taking into account (29), we trasform (10) to the form

$$H^0\,\frac{k}{c}\frac{\partial\Xi}{\partial t}+\sum_{\nu=1}^{3}kSH^\nu S^{-1}e_\nu\Xi+(\mathrm{m}+\frac{3}{2})\frac{X}{k}\Xi=0\,, \tag{42}$$

where

$$\mathrm{m}=\frac{mck}{\hbar}\,. \tag{43}$$

In order to count incoming in (42) the sum $\sum$ together with the vectorial fields (38) let's introduce the field

$$e_4=\frac{\partial}{\partial R}=k^{-1}x^b\frac{\partial}{\partial x^b} \tag{44}$$

and the vectorial fields $\partial/\partial x^a$, which are taken with $R=k$,

Since $L^a_b e_a=\partial/\partial x^b$ and $kSH^4S^{-1}=-x_aH^a$ (see Eq. (37)), the following equality is correct:

$$\sum_{\nu=1}^{3}kSH^\nu S^{-1}e_\nu=kSH^aS^{-1}e_a-kSH^4S^{-1}e_4=H^am_a\,, \tag{45}$$

where

$$m_1=k\frac{\partial}{\partial x}+x\frac{\partial}{\partial R}\,,\quad m_2=k\frac{\partial}{\partial y}+y\frac{\partial}{\partial R}\,,\quad m_3=k\frac{\partial}{\partial z}+z\frac{\partial}{\partial R}\,, \tag{46}$$

$$m_4=k\frac{\partial}{\partial u}-u\frac{\partial}{\partial R}$$

are the generators of the conformal transformations of Lobachevsky space.

As a result, equation (10) transforms to the following equation:

$$H^0 \; \frac{k}{c}\frac{\partial \Xi}{\partial t} + H^a m_a \Xi + (\mathrm{m} + \frac{3}{2})\frac{X}{k}\Xi = 0\,. \qquad (47)$$

In fact in the paper [3] Dirac has offered not one, but two special forms of the spinor equation (in the spherical space-time). We already have come to the one of them in the examinated case (in the Lobachevsky space-time, see Eq. 47). To transform equation (47) to the another special form, let's introduce generators m_{ab} of the isometric transformations of Lobachevsky space, which are equal

$$m_{ab} = x_a \frac{\partial}{\partial x^b} - x_b \frac{\partial}{\partial x^a}\,, \qquad (48)$$

and the operator, which is equal

$$\frac{1}{2}\; H^a H^b m_{ab} = \frac{1}{2}\; (X H^a - H^a X) \frac{\partial}{\partial x^a}\,. \qquad (49)$$

Noticing

$$H^a X + X H^a = 2 x^a\,, \qquad (50)$$

we write following equations:

$$(x^a - H^a X)\frac{\partial}{\partial x^a} \;=\; \frac{1}{2}\; (X H^a - H^a X)\frac{\partial}{\partial x^a} \;= (X H^a - x^a)\frac{\partial}{\partial x^a}\,.$$

It is not difficult to prove that right one of them gives

$$\frac{1}{2}\; H^a H^b m_{ab} = \frac{X}{k}\; H^a m_a \qquad (51)$$

and the left one of them gives

$$\frac{1}{2}\; H^a H^b m_{ab} = 3 - H^a m_a\; \frac{X}{k}\,. \qquad (52)$$

Consequently, the operator

$$N = H^a\, m_a + \frac{3}{2}\,\frac{X}{k} \qquad (53)$$

is anticommutating with the operator X:

$$NX + XN = 0.\,. \tag{54}$$

Equally, the operator

$$M = \frac{1}{2}\, H^a H^b m_{ab} - \frac{3}{2} \tag{55}$$

is anticommutating with the operator X:

$$MX + XM = 0.\,. \tag{56}$$

The equation (47) can be written down through the operator N in one special form, which is:

$$H^0\, \frac{k}{c}\frac{\partial \Xi}{\partial t} + N\Xi + \mathrm{m}\frac{X}{k}\Xi = 0\;, \tag{57}$$

and through the operator M in the other special form, which is:

$$\frac{XH^0}{c}\frac{\partial \Xi}{\partial t} + M\Xi = \mathrm{m}\Xi\;. \tag{58}$$

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Парамонов Д.В., Парамонова Н.Н., Шавохина Н.С. E2-2000-79
Уравнение Дирака в пространстве-времени Лобачевского

Пространство-время Лобачевского определяется как прямое произведение пространства Лобачевского на ось времени. Пространство Лобачевского рассматривается как полость гиперболоида в четырехмерном псевдоевклидовом пространстве. Уравнение Дирака–Фока–Иваненко приводится к уравнению Дирака в двух специальных видах путем перехода от базиса Ламе в пространстве Лобачевского к декартову базису в объемлющем псевдоевклидовом пространстве.

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The Dirac Equation in the Lobachevsky Space-Time

The product of the Lobachevsky space and the time axis is termed the Lobachevsky space-time. The Lobachevsky space is considered as a hyperboloid's sheet in the four-dimensional pseudo-Euclidean space. The Dirac–Fock–Ivanenko equation is reduced to the Dirac equation in two special forms by passing from Lame basis in the Lobachevsky space to the Cartesian basis in the enveloping pseudo-Euclidean space.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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