E2-2001-243

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ON THE CANONICAL TREATMENT OF LAGRANGIAN CONSTRAINTS

Submitted to the International Conference «New Trends in High-Energy Physics», 8 September, 2001, Yalta, Ukraine

The conventional canonical treatment of constrained systems [1] deals with the constraints which follow only from the initial singular Lagrangian. However, there are problems where the Lagrange constraints are introduced 'by hand' in addition to the initial Lagrangian or when from the very beginning of the Hamiltonization procedure, some of the constraints that follow from the Lagrange function, are taken into account manifestly. For example, the Lorentz gauge in electrodynamics cannot be canonically implemented [2]. The purpose of this note is to show that such "noncanonical" constraints can be implemented by the Berezin algorithm [3]. The algorithm provides a unified consideration of the singular Lagrangians and nonsingular ones with constraints that depend on velocities and time:

$$\varphi_a(q,\dot{q},t)=0,\ \ q=(q_1,q_2,\ldots,q_n),\ a=1,2,\ldots,m,\ m\leq n. \quad (1)$$

Let us consider the Lagrangian $L(q,\dot{q},t)$ and the set of the Lagrangian constraints (1). The relevant extended (generalized) Lagrangian reads

$$\mathcal{L}(q,\dot{q},t)=L(q,\dot{q},t)+\sum_{a=1}^{m}\lambda_a(t)\varphi_a(q,\dot{q},t) \quad (2)$$

where λ_a are the Lagrange multipliers. All the constraints to be considered depend explicitly on velocities $\dot{q}_i$. When among them, there exists the equation $\varphi(q,t)=0$, we replace it, after differentiating with respect to time, by the equivalent equation

$$\sum_{i=1}^{n}\frac{\partial\varphi}{\partial q_i}\dot{q}_i+\frac{\partial\varphi}{\partial t}=0. \tag{3}$$

Now, let us introduce the extended momenta for the Lagrangian function (2)

$$\tilde{p}_i=\frac{\partial\mathcal{L}}{\partial\dot{q}_i}=\frac{\partial L}{\partial\dot{q}_i}+\sum_{a=1}^{m}\lambda_a\frac{\partial\varphi_a}{\partial\dot{q}_i},\quad i=1,\ldots,n. \tag{4}$$

Berezin [3] has assumed that the velocities $\dot{q}_i$ and the Lagrange multipliers $\lambda_a(t)$ can be expressed uniquely in terms of q_i and $\tilde{p}_i$ by resolving the constraints (1) together with Eqs. (4). In this case, the variational problem is said to be a nondegenerate (nonsingular) one. On the contrary, the requirement of the initial Lagrangian being nonsingular

$$\det\left\|\frac{\partial^2 L}{\partial\dot{q}_i\partial\dot{q}_j}\right\|\neq 0 \tag{5}$$

becomes superfluous. In the following, only the dynamic systems that satisfy the Berezin assumption will be considered. The method does not lead to the reduction of degrees of freedom of the systems in the phase space. However, the transition to the canonical momenta p, corresponing to the initial Lagrangian L takes place if

$$\lambda_a(\tilde{p},q,t)|_{\tilde{p}=p}=0\,. \tag{6}$$

It will lead to the primary Hamiltonian constraints in this approach. As an illustration, we apply the Berezin method to a number of constrained Lagrangian systems.

1. The Lagrangian linear in velocities [3, 4, 5]

$$L = \sum_{i=1}^{n} f_i(q)\dot{q}_i - V(q), \quad q = (q_1, q_2, \ldots, q_n). \tag{7}$$

Since the Lagrangian (7) is singular, and all the equations of motion

$$\sum_{j=1}^{n} f_{ij}\dot{q}_j + \frac{\partial V}{\partial q_i} = 0, \quad f_{ij} = \frac{\partial f_i}{\partial q_j} - \frac{\partial f_j}{\partial q_i}, \quad \det \parallel f_{ij} \parallel \neq 0 \tag{8}$$

become first-order equations, the extended Lagrangian acquires the form

$$\mathcal{L} = \sum_{i=1}^{n} f_i(q)\dot{q}_i - V(q) + \sum_{i,j=1}^{n} \lambda_i \left(f_{ij}\dot{q}_j + \frac{\partial V}{\partial \dot{q}_i} \right) \tag{9}$$

and the extended momenta read

$$\tilde{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} = f_i(q) + \sum_{j=1}^{n} \lambda_j f_{ji}(q). \tag{10}$$

It is possible to resolve Eqs. (8) with respect to $\dot{q}_i$, because there exists the inverse matrix f_{ij}^{-1} such that

$$\dot{q}_i = -\sum_{j=1}^{n} f_{ij}^{-1} \frac{\partial V}{\partial q_j}. \tag{11}$$

Also, resolution of (10) with respect to λ_i gives us

$$\lambda_i = \sum_{j=1}^{n} f_{ij}^{-1}(\tilde{p}_j - f_j). \tag{12}$$

Taking into account that $\mathcal{L}$ in (9) on the surface of constraints has the form

$$\mathcal{L} = \sum_{i,j=1}^{n} f_i f_{ij} \frac{\partial V}{\partial q_j} - V(q),$$

we find that

$$\mathcal{H} = \sum_{i=1}^{n} \tilde{p}_i \dot{q}_i - \mathcal{L} = \sum_{i,j=1}^{n} (f_i - \tilde{p}_i) f_{ij}^{-1} \frac{\partial V}{\partial q_j} + V(q). \qquad (13)$$

Going over to the canonical momenta p, from (6) and (12), we obtain the primary Hamiltonian constraints (invariant relations)

$$\lambda_i|_{\tilde{p}=p} = 0 \Longrightarrow p_i = f_i(q). \qquad (14)$$

The kinetic term in (13) is linear in $\tilde{p}$, thus, $\mathcal{H}$ is singular. Therefore, again there is no Legendre transformation from $\mathcal{H}$ to $\mathcal{L}$ because the relations

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = -\sum_{j=1}^{n} f_{ij} \frac{\partial V}{\partial q_j} \qquad (15)$$

do not contain p. However, with the help of the Berezin algorithm, the system (13), (14) can be transformed into the initial Lagrangian system. Indeed, we derive the extended Hamiltonian in the form

$$\mathcal{H}_{ext} = \mathcal{H} + \sum_{i=1}^{n} \mu_i (p_i - f_i(q)) \Longrightarrow \tilde{\dot{q}} = -\sum_{j=1}^{n} f_{ij}^{-1} \frac{\partial V}{\partial q_j} + \mu_i.$$

Thus, we arrive at the system of equations

$$\mu_i = \tilde{\dot{q}}_i + \sum_{j=1}^{n} f_{ij}^{-1} \frac{\partial V}{\partial q_j}, \quad p_i = f_i(q) \qquad (16)$$

and can construct the Lagrangian

$$L = \sum_{i=1}^{n} p_i \tilde{\dot{q}}_i - \mathcal{H}_{ext} = \sum_{i=1}^{n} \tilde{\dot{q}}_i f_i(q) - V(q).$$

Going over to the generalized velocities $\tilde{\dot{q}} \to \dot{q}$ via the equation $\mu_i|_{\tilde{\dot{q}}=\dot{q}} = 0$, from (13) we once again obtain the Lagrangian constraints (11). **2.** Relativistic point particle $L = -m\int\sqrt{\dot{x}^2(\tau)}d\tau$ [6].

If the parameter τ is chosen as the proper time, the Lagrangian constraint is $\dot{x}^2 = c^2 = \text{constant}$. The density of the extended Lagrangian for this system takes the form

$$\mathcal{L} = -m\sqrt{\dot{x}^2(\tau)} - \lambda\frac{m}{2}[\dot{x}^2(\tau) - c^2], \tag{17}$$

$$\tilde{p}_\mu = -\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = m\left[\frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} + \lambda\dot{x}_\mu\right]. \tag{18}$$

As before, we use equations (18) and the above constraint to find λ and $\dot{x}_\mu$

$$\lambda = \frac{\sqrt{\tilde{p}^2} - m}{cm}, \quad \dot{x}_\mu = c\frac{\tilde{p}_\mu}{\sqrt{\tilde{p}^2}}. \tag{19}$$

As a result, we arrive at the Hamiltonian in the following form (taking into account that on the constraint shell, $\mathcal{L} = -mc$):

$$\mathcal{H} = -\tilde{p}^\mu\dot{x}_\mu - \mathcal{L} = c(m - \sqrt{\tilde{p}^2}). \tag{20}$$

Turning to the canonical momenta $\lambda|_{\tilde{p}=p} = 0$, from (19), we get the Hamiltonian constraint

$$\sqrt{p^2} = m. \tag{21}$$

The Hamiltonian equations for (20) coincide with the Lagrangian ones:

$$\dot{x}_\mu = -\frac{\partial\mathcal{H}}{\partial p^\mu} = c\frac{p_\mu}{\sqrt{p^2}} = c\frac{p_\mu}{m}; \quad \dot{p}_\mu = \frac{\partial\mathcal{H}}{\partial x^\mu} = 0.$$

The obtained Hamiltonian (20), as well as the Lagrangian, is singular, but applying the same algorithm, it is possible to restore the initial Lagrangian system:

$$\mathcal{H} = c(m - \sqrt{p^2}) + \mu(m - \sqrt{p^2}) \implies \tilde{\dot{x}}_\mu = (c + \mu)\frac{p_\mu}{\sqrt{p^2}}.$$

Adding the constraint (21) to this system, we obtain

$$\mu = \sqrt{\tilde{\dot{x}}^2} - c, \quad p_\mu = m\frac{\tilde{\dot{x}}_\mu}{\sqrt{\tilde{\dot{x}}^2}}.$$

Finally,

$$\mathcal{L} = -p_\mu \tilde{\dot{x}}^\mu - \mathcal{H} = -m\sqrt{\tilde{\dot{x}}^2}, \quad \mu|_{\tilde{\dot{x}}=\dot{x}} = 0 \implies \sqrt{\dot{x}^2} = c.$$

3. Relativistic particle with the gauge $x^0 = \mathcal{P}\tau/m$.

Differentiating this gauge with respect to time $\dot{x}^0 = \mathcal{P}/m$ and substituting it into the extended Lagrangian, we obtain

$$\mathcal{L} = -m\sqrt{(\dot{x}^0)^2 - \dot{\mathbf{x}}^2} - \lambda m(\dot{x}^0 - \mathcal{P}/m), \tag{22}$$

$$\tilde{p}_0 = -\frac{\partial \mathcal{L}}{\partial \dot{x}^0} = m\left(\frac{\dot{x}_0}{\sqrt{\dot{x}^2}} + \lambda\right); \quad \mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} = m\frac{\dot{\mathbf{x}}}{\sqrt{\dot{x}^2}}. \tag{23}$$

Applying once more the condition $\dot{x}^0 = \mathcal{P}/m$, we have

$$\tilde{p}_0 = \frac{\mathcal{P}}{\sqrt{(\mathcal{P}/m)^2 - \dot{\mathbf{x}}^2}} + \lambda m, \quad \mathbf{p} = \frac{m\dot{\mathbf{x}}}{\sqrt{(\mathcal{P}/m)^2 - \dot{\mathbf{x}}^2}}.$$

Hence, it follows that

$$\lambda = \frac{\tilde{p}_0 - \sqrt{\mathbf{p}^2 + m^2}}{m}, \quad \dot{\mathbf{x}} = \frac{\mathcal{P}}{m}\frac{\mathbf{p}}{\sqrt{\mathbf{p}^2 + m^2}}. \tag{24}$$

The Hamiltonian reads

$$\mathcal{H} = -p_0\dot{x}^0 + \mathbf{p}\dot{\mathbf{x}} - \mathcal{L} = \frac{\mathcal{P}}{m}(\sqrt{\mathbf{p}^2 + m^2} - \tilde{p}_0). \tag{25}$$

Going over to the canonical momentum p_0 by means of $\lambda|_{\tilde{p}_0 = p_0} = 0$, we derive the Hamiltonian constraint $p_0 = \sqrt{\mathbf{p}^2 + m^2}$. Then, the Hamiltonian equations are as follows:

$$\dot{x}^0 = -\frac{\partial \mathcal{H}}{\partial p_0} = \frac{\mathcal{P}}{m}; \quad \dot{\mathbf{x}} = \frac{\partial \mathcal{H}}{\partial p^0} = \frac{\mathcal{P}}{m}\frac{\mathbf{p}}{\sqrt{\mathbf{p}^2 + m^2}},$$

$$\dot{p}_0 = \frac{\partial \mathcal{H}}{\partial x^0} = 0, \quad \dot{\mathbf{p}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}} = 0.$$

4. Relativistic string in the orthonormal gauge [7]:

$$L = -\gamma\sqrt{(\dot{x}x')^2 - \dot{x}^2x'^2}, \quad \dot{x}^2 + x'^2 = 0, \quad (\dot{x}x') = 0. \tag{26}$$

The extended Lagrangian in this case reads

$$\mathcal{L} = -\gamma\sqrt{(\dot{x}x')^2 - \dot{x}^2x'^2} - \frac{\lambda_1}{2}(\dot{x}^2 + x'^2) - \lambda_2(\dot{x}x'),$$

and taking account of the constraints we get for extended momenta

$$\tilde{p}_\mu = -\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = (\gamma + \lambda_1)\dot{x}_\mu + \lambda_2 x'_\mu. \tag{27}$$

Projecting this onto x'_μ and using the constraints, we find that $(\tilde{p}x') = \lambda_2 x'^2$, and then,

$$\lambda_2 = \frac{(\tilde{p}x')}{x'^2}. \tag{28}$$

Squaring (27), we obtain

$$\tilde{p}^2 = (\gamma+\lambda_1)^2(-x'^2)+\frac{(\tilde{p}x')^2}{x'^2} \implies \lambda_1+\gamma = \frac{\sqrt{(\tilde{p}x')^2 - \tilde{p}^2x'^2}}{-x'^2}. \tag{29}$$

Given λ_1 and λ_2, we can express $\dot{x}_\mu$ in terms of x'_μ and $\tilde{p}_\mu$ as follows:

$$\dot{x}_\mu = \frac{(\tilde{p}x')x'_\mu - x'^2\tilde{p}_\mu}{\sqrt{(\tilde{p}x')^2 - \tilde{p}^2x'^2}},$$

which satisfies the constraints identically. As a result, the Hamiltonian for the string assumes the form

$$\mathcal{H} = -\tilde{p}^\mu\dot{x}_\mu - \mathcal{L} = -\sqrt{(\tilde{p}x')^2 - \tilde{p}^2x'^2} - \gamma\, x'^2. \tag{30}$$

Going over to the canonical momenta p_μ according to formula (6), we arrive at the Hamiltonian constraints

$$\lambda_i|_{\tilde{p}=p} = 0 \implies (px') = 0, \quad p^2 + \gamma^2x'^2 = 0. \tag{31}$$

On the surface of these Hamiltonian constraints, $\mathcal{H} = 0$, the canonical equations are as follows:

$$\dot{x}_\mu = -\frac{\partial \mathcal{H}}{\partial p^\mu} = \frac{(px')x'_\mu - x'^2 p_\mu}{\sqrt{(\tilde{p}x')^2 - \tilde{p}^2 x'^2}},$$

$$\dot{p}_\mu = -\frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{H}}{\partial x'^\mu}\right) = \frac{\partial}{\partial \sigma}\left[\frac{(px')p_\mu - p^2 x'_\mu}{\sqrt{(\tilde{p}x')^2 - \tilde{p}^2 x'^2}} + 2\gamma x'_\mu\right].$$

Taking account of constraints (31), we can rewrite these equations in the form

$$\dot{x}_\mu = \frac{1}{\gamma} p_\mu, \quad \dot{p}_\mu = \gamma x''_\mu \Longrightarrow \ddot{x}_\mu - x''_\mu = 0.$$

The Hamiltonian (30) is singular $\det \left|\left| \frac{\partial^2 \mathcal{H}}{\partial p^\mu \partial p^\nu} \right|\right| = 0$. But using the Berezin algorithm, we can pass to the initial Lagrangian L and constraints (26). Indeed, the extended Hamiltonian is of the form

$$\mathcal{H}_{ext} = -\sqrt{(px')^2 - p^2 x'^2} - \gamma x'^2 - \frac{\mu_1}{2\gamma}(p^2 + \gamma^2 x'^2) - \mu_2 (px'),$$

from which and the subsidiary conditions we find

$$\dot{\tilde{x}}_\mu = \frac{\partial \mathcal{H}_{ext}}{\partial p^\mu} = \gamma^{-1}(1 + \mu_1)p_\mu + \mu_2 x'_\mu, \Longrightarrow (\dot{\tilde{x}}x') = \mu_2 x'^2, \quad (32)$$

$$\dot{\tilde{x}}^2 = -(1 + \mu_1)^2 x'^2 + \frac{(\dot{\tilde{x}}x')^2}{x'^2} \Longrightarrow 1 + \mu_1 = \frac{\sqrt{(x'\tilde{\dot{x}})^2 - x'^2 \tilde{\dot{x}}^2}}{-x'^2}$$

and therefore, from (32), we get

$$p_\mu = \gamma \frac{(x'\tilde{\dot{x}})x'_\mu - x'^2 \tilde{\dot{x}}^2}{\sqrt{(x'\tilde{\dot{x}})^2 - x'^2 \tilde{\dot{x}}^2}}.$$

Finally, we obtain

$$L = -p_\mu \tilde{\dot{x}}^\mu - \mathcal{H}_{ext} = -\gamma \sqrt{(x'\tilde{\dot{x}})^2 - x'^2 \tilde{\dot{x}}^2}$$

$$\mu_i|_{\tilde{\dot{x}}=\dot{x}} = 0 \Longrightarrow (\dot{x}x') = 0, \quad \dot{x}^2 + x'^2 = 0.$$

5. Electromagnetic field with the Lorentz gauge $\partial_\mu A^\mu = 0$ [8].

The extended Lagrangian with an external source j^μ is of the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu - \lambda(\partial_\mu A^\mu),$$
$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad \partial_\mu j^\mu = 0.$$

This gives the time component of the extended momenta $\tilde{\pi}_0$ as follows

$$\tilde{\pi}_0 = -\frac{\partial \mathcal{L}}{\partial \dot{A}^0} = \lambda. \tag{33}$$

For the space component $\boldsymbol{\pi}$, we derive the canonical expression

$$\tilde{\boldsymbol{\pi}} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{A}}} = \dot{\mathbf{A}} + \boldsymbol{\nabla} A^0. \tag{34}$$

According to the Berezin algorithm, adding the Lorentz gauge

$$\dot{A}^0 = -(\boldsymbol{\nabla}\mathbf{A}) \tag{35}$$

to these equations, resolving the velocities $\dot{A}^\mu$ and the multiplier λ in terms of A^μ and π^μ, we obtain

$$\lambda = \tilde{\pi}^0, \quad \mathbf{A} = \boldsymbol{\pi} - \boldsymbol{\nabla} A^0, \quad \dot{A}^0 = -(\boldsymbol{\nabla}\mathbf{A}). \tag{36}$$

Now, one can construct the Hamiltonian

$$\mathcal{H} = \tilde{\pi}^0(\boldsymbol{\nabla}\mathbf{A}) + \frac{1}{2}(\boldsymbol{\pi}^2 + (\mathrm{rot}\,\mathbf{A})^2) - (\boldsymbol{\pi}\boldsymbol{\nabla} A^0) + j_\mu A^\mu.$$

It gives the canonical equations

$$\dot{A}^0 = -\frac{\partial \mathcal{H}}{\partial \tilde{\pi}_0} = -(\boldsymbol{\nabla}\mathbf{A}), \quad \dot{\mathbf{A}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{\pi}} = \boldsymbol{\pi} - \boldsymbol{\nabla} A^0$$

coinciding with (34) and (35). Also, for the momenta, we get

$$\dot{\tilde{\pi}}_0 = \frac{\partial \mathcal{H}}{\partial A^0} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \frac{\partial \mathcal{H}}{\partial(\partial A^0/\partial x_j)} = j_0 + (\boldsymbol{\nabla}\boldsymbol{\pi}),$$

$$\dot{\boldsymbol{\pi}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{A}} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \frac{\partial \mathcal{H}}{\partial(\partial \mathbf{A}/\partial x_j)} = \mathbf{j} + \boldsymbol{\nabla}\tilde{\pi}^0 - \text{rot rot } \mathbf{A}. \quad (37)$$

And for the components of π^μ and A^μ, we have

$$\left.\begin{aligned} \ddot{A}^0 &= \Delta A^0 + j^0 - \dot{\tilde{\pi}}^0 \\ \ddot{\mathbf{A}} &= \Delta \mathbf{A} + \mathbf{j} + \boldsymbol{\nabla}\tilde{\pi}^0 \end{aligned}\right\} \longrightarrow \Box A^\mu = j^\mu - \partial^\mu \tilde{\pi}^0;$$

$$\Box \boldsymbol{\pi} = \dot{\mathbf{j}} + \boldsymbol{\nabla} j^0; \quad \Box \tilde{\pi}^0 = 0. \quad (38)$$

After the transition $\lambda|_{\pi^0=\pi^0} = 0$, from (36), we obtain $\pi^0 = 0$, and all the equations (37) and (38) are cast into the correct equations of electrodynamics in the Lorentz gauge.

To conclude, we note that contrary to the Dirac approach, the suggested algorithm allows unique construction of the Hamilton formalism for constrained Lagrangian systems with constraints that depend on velocities and, in the general case, do not depend on the Lagrangian form.

Acknowledgments. A am grateful to Professor L.D. Fadeev and Dr. V.V. Nesterenko for critical discussions of this work.

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Received by Publishing Department
on November 8, 2001.

Барбашов Б. М. E2-2001-243
Каноническая трактовка лагранжевых связей

Каноническая трактовка динамических систем с явно заданными зависящими от скоростей лагранжевыми связями, предложенная Березиным, применяется к ряду конкретных систем: лагранжиану, линейному по скоростям, релятивистской частице в калибровке собственного времени, релятивистской струне в ортонормальной калибровке и к электромагнитному полю в калибровке Лоренца.

Работа выполнена в Лаборатории теоретической физики им. Н. Н. Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 2001

Barbashov B. M. E2-2001-243
On the Canonical Treatment of Lagrangian Constraints

The canonical treatment of dynamic systems with manifest Lagrangian constraints proposed by Berezin is applied to concrete examples: a special Lagrangian linear in velocities, relativistic particles in proper time gauge, a relativistic string in orthonormal gauge, and the Maxwell field in the Lorentz gauge.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 2001

Макет Т. Е. Попеко

Подписано в печать 05.12.2001
Формат 60 × 90/16. Офсетная печать. Уч.-изд. л. 1,05
Тираж 425. Заказ 52993. Цена 1 р. 30 к.

Издательский отдел Объединенного института ядерных исследований
Дубна Московской области