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BIFURCATIONS OF THE SOLUTIONS OF MODIFIED GINZBURG–LANDAU EQUATION FOR JOSEPHSON JUNCTIONS

1 Introduction

The physics of Josephson junctions is based on a known sinusoidal relation

$$j_s(\Delta\phi) = j_c \sin \Delta\phi \tag{1}$$

where j_s is the Josephson supercurrent, the amplitude j_c represents the critical Josephson current, and $\Delta\varphi$ is the difference between phases on each side of the barrier.

On the other hand, it is well known that apart from an insulating tunnel structure, any sufficiently short localized weak link, such as a very short constriction in the cross-section of a superconductor, a point contact between two superconductors, as well as two superconductors separated by a thin layer of normal metal, could be used as a Josephson junction, obeying the current-phase relations, usually different from (1). This fact forced Licharev [1] and Waldram [2] to propose a generalized definition: a weak link is supposed to show a Josephson behavior if the supercurrent-phase relation is a single-valued and odd analytical function which can be represented as a Fourier series

$$j_s(\Delta\phi) = \sum_{n=1}^{\infty} j_n \sin(n\Delta\phi). \tag{2}$$

The crossover between an ideal Josephson behavior and a uniform superconducting flow was studied in detail by solving exactly the Ginzburg-Landau (GL) equation for a 1-D superconductor in the presence of an effective δ-function potential of arbitrary strength [3]. Recently, a modified GL type model has been formulated [4]. This model could be equally well applied to a boundary between different superconductors, superconductor-insulator, and superconductor-normal metal. The purpose of our paper is to apply this modified GL model for calculating the supercurrent-phase relation and the crossover between a Josephson behavior and a uniform superconducting flow.

We would like to mention that the Josephson contacts in principle are discussed in recent investigations [5] - [8].

This work represents an extensive explanation of the numerical results from [9].

2 Formulation of the Problem

We focus our attention only on current-conserving solutions of the modified GL equations [4], in which a nonzero current across the boundary is associated with a linearly varying asymptotic phase. Let us accept a one-dimensional approximation (the dependence of all relevant quantities is only on the coordinate z across the boundary). Then the magnetic field of our uniform current density will depend at least on one of the transverse coordinates, and in one-dimensional approximation this field could be neglected.

A simple sketch of a SNS (SIS) Josephson contact is shown in Fig. 1. Here $2\tilde{\Delta}$ is the full length of the contact and $2\tilde{\delta}$ is the thickness of layer. All the sizes in Fig. 1 are in a dimensionless form (see below). We shall consider SNS contacts of finite sizes, thus $0 < \delta < \Delta < \infty$.

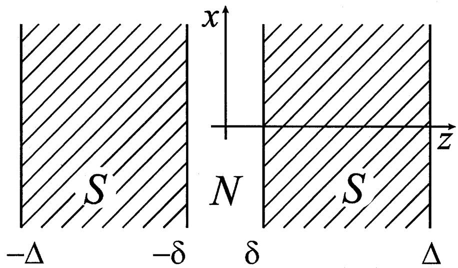

Figure 1: Simple sketch of the Josephson contact.

Let us denote $\tilde{S} \equiv (-\tilde{\Delta}, -\tilde{\delta}) \cup (\tilde{\delta}, \tilde{\Delta})$, $\tilde{N} \equiv (-\tilde{\delta}, \tilde{\delta})$, $\tilde{D} \equiv [-\tilde{\Delta}, \tilde{\Delta}]$, and $\tilde{d} \equiv [-\tilde{\delta}, \tilde{\delta}]$. Then, the order parameter $\widetilde{\Phi}(\tilde{z})$ satisfies the following equations:

$$-\frac{\hbar^2}{2m_s}\widetilde{\Phi}'' + a_s\widetilde{\Phi} + b_s|\widetilde{\Phi}|^2\widetilde{\Phi} = 0, \tag{3}$$

for the superconducting domain $\tilde{z} \in \tilde{S}$, and

$$-\frac{\hbar^2}{2m_n}\widetilde{\Phi}'' + a_n\widetilde{\Phi} + b_n|\widetilde{\Phi}|^2\widetilde{\Phi} = 0, \tag{4}$$

for the normal domain $\tilde{z} \in \tilde{N}$. The necessary boundary conditions (BCs) at $\tilde{z} = \pm\tilde{\Delta}$ as well as the conditions at the interfaces $\tilde{z} = \pm\,\tilde{\delta}$ will be discussed below. The GL phenomenological coefficients satisfy the conditions $a_s = -|a_s|$, $a_n > 0$, $b_s > 0$, $b_n \geq 0$. We are free to choose the value of one of the effective masses m_s and m_n. The effective mass of superconducting electrons m_s is usually chosen to be twice the electron mass. This makes the mass m_n as a parameter depending on the layer's material.

Let us define the coherence length $\xi = \hbar/\sqrt{2m_s|a_s|}$, as well as the dimensionless distances $z = \tilde{z}/\xi$, and the order parameter $\Phi(z) = \widetilde{\Phi}(z\xi)\sqrt{b_s/|a_s|}$. Thereby, $\Delta = \tilde{\Delta}/\xi$, $\delta = \tilde{\delta}/\xi$, so $S \equiv (-\Delta, -\delta) \cup (\delta, \Delta)$, $N \equiv (-\delta, \delta)$, $D \equiv [-\Delta, \Delta]$, and $d \equiv [-\delta, \delta]$. In a dimensionless form the equations (3) and (4) can be rewritten as follows:

$$-\Phi'' - \Phi + |\Phi|^2\Phi = 0, \quad z \in S, \tag{5}$$

$$-\frac{1}{m_0}\Phi'' + a_0\Phi + b_0|\Phi|^2\Phi = 0, \quad z \in N. \tag{6}$$

Here we merely substitute $m_0 = m_n/m_s \geq 1$, $a_0 = a_n/|a_s| \geq -1$, and $b_0 = b_n/b_s \geq 1$. The case $a_0 = -1$, $b_0 = 1$, and $m_0 = 1$ corresponds to a uniform superconductor occupying the whole interval D.

It is advisable to write the above equations in a general form using the step functions:

$$m(z) \equiv \begin{cases} 1, & z \in S; \\ m_0, & z \in d, \end{cases} \quad a(z) \equiv \begin{cases} -1, & z \in S; \\ a_0, & z \in d, \end{cases} \quad b(z) \equiv \begin{cases} 1, & z \in S; \\ b_0, & z \in d. \end{cases}$$

In this way we formally obtain

$$-\frac{1}{m(z)}\Phi'' + a(z)\Phi + b(z)|\Phi|^2\Phi = 0, \quad z \in S \cup N. \tag{8}$$

We note that Eq. (8) has the known first integral

$$J = \frac{i}{2m(z)}\left(\Phi\Phi^{*\,\prime} - \Phi^*\Phi\,'\right), \tag{9}$$

where the constant J physically represents the supercurrent density, so from now on we substitute $j_s \equiv J$.

Under this assumption we can achieve our goal to investigate the full crossover from the Josephson effect to that of a bulk superconducting flow.

3 Variational Formulation

As usual [3] we can factorize the order parameter $\Phi(z) = R(z)\exp\left[i\varphi(z)\right]$. In that way we can rewrite the free energy functional within norm factor $2/\Delta$ in the form:

$$F[R] = \int_{-\Delta}^{\Delta} \mathcal{F}(x, R, R')\, dx, \tag{10}$$

where the energy density $\mathcal{F}$ is given by:

$$\mathcal{F} = \frac{1}{2}\left[\frac{1}{m(z)}R'^2 + a(z)R^2 + \frac{1}{2}\,b(z)R^4 - \frac{m(z)\,J^2}{R^2}\right].$$

The necessary extremum conditions [17] for (10) within the class of sectionally smooth for $z \in D$ functions $R(z)$ result in the following boundary value problem (BVP):
– the Euler-Lagrange equation for the amplitude $R(z)$

$$-\frac{1}{m(z)}R'' + a(z)R + b(z)R^3 + \frac{m(z)J^2}{R^3} = 0, \tag{11}$$

– the boundary conditions at $z = \pm\Delta$

$$R'(-\Delta) = 0, \quad R'(\Delta) = 0, \tag{12}$$

– the Weierstrass conditions at the points $z = \pm\delta$

$$R'\left(-\delta - 0\right) = \frac{1}{m_0}R'\left(-\delta + 0\right), \quad \frac{1}{m_0}R'\left(\delta - 0\right) = R'\left(\delta + 0\right). \tag{13}$$

When $m_0 = 1$, the derivative $R'(z)$ is continuous in whole D.

In terms of amplitude $R(z)$ and phase $\varphi(z)$ the first integral (9) can be converted to the form:

$$\frac{1}{m(z)}\,R^2\varphi\,' = J, \tag{14}$$

and we assume that the function $\varphi(z)$ satisfies the continuity conditions

$$\frac{1}{m_0}\varphi'(-\delta+0)-\varphi'(-\delta-0)=\frac{J}{R^2(-\delta)},\quad \varphi'(\delta+0)-\frac{1}{m_0}\varphi'(\delta-0)=\frac{J}{R^2(\delta)}.$$

Let us introduce the vector of physical parameters $p=\{J,a_0,b_0,m_0\}$. It is clear that apart from the space coordinate z, the virtual solutions $R(z,p)$ of the formulated nonlinear BVP also depend on the parameters p. We suppose that $R(z,p)$ as well as its space-derivatives $R'(z,p)$ are continuous functions of all the parameters p on some given domain $\mathcal{P}\in\mathbb{R}^4$. Then the functional (10) is a differentiable function $F(p)$ of p and it is easy to prove that:

$$\frac{\partial F(p)}{\partial p_k}=\int_{-\Delta}^{\Delta}\frac{\partial\mathcal{F}}{\partial p_k}\,dz,\quad k=1,\ldots,4. \tag{15}$$

We notice that the formulated nonlinear BVP (11) - (13) has more than one solution (see below). In the infinite case $\Delta\to\infty$ and $\delta\to 0$ this fact is pointed out in [3, 9], where analytical solutions of the above problem have been derived. The existence of many solutions of BVP in conjunction with their dependence upon parameters p forces us to study the stability of these solutions.

The varying of each of parameters p causes a variation of the distribution $R(z,p)$. This means that the primary stable solution $R(z,p)$ can lose its stability by changing p. In order to examine the stability of some concrete solution $R(z,p)$, we introduce the Sturm-Liouville problem (SLP):

$$-\frac{1}{m(z)}\psi''+q(z,p)\,\psi=\lambda\,\psi\,, \tag{16}$$

$$\psi'(-\Delta)=0,\quad \psi'(\Delta)=0, \tag{17}$$

$$\psi'(-\delta-0)=\frac{1}{m_0}\psi'(-\delta+0)\,,\quad \frac{1}{m_0}\psi'(\delta-0)=\psi'(\delta+0)\,. \tag{18}$$

Here

$$q(z,p)=a(z)+3\,b(z)\,R^2(z,p)-3\frac{m(z)J^2}{R^4(z,p)}$$

is the potential of SLP, originated from the mentioned solution $R(z,p)$. Mathematically SLP represents a sufficient extremum condition for the functional (10) - when $\lambda=0$ then Eq. (16) is just Jacobi's equation [17].

In accordance with the physical sense we only shall consider the solutions satisfying the condition $0<R(z,J)\leq 1$, $z\in D$, for fixed $J\neq 0$, and assume that $R(z)=0$ onto some point set can be reached only in the case $J=0$. Under these assumptions, the potential $q(z,J)$ is bounded on the interval D, so SLP (16) has (see for details the classical book [18]) counted lower bounded number of different eigenvalues $\lambda_{min}\equiv\lambda_0<\lambda_1<\lambda_2<\ \ldots\ \lambda_n<\ \ldots$ Every eigenvalue λ_n, $n=0,1,2,\ldots$ conforms unique eigenfunction $\psi_n(z,p)$ which satisfies some norm condition, for example,

$$\int_{-\Delta}^{\Delta}\psi_n^2(z)\,d\,z=1\,. \tag{19}$$

The stability of the solution $R(z,p)$ is determined by the sign of the minimal eigenvalue $\lambda_{min}(p)$. If the condition $\lambda_{min}(p) > 0$ is satisfied, then the solution is stable. Conversely, if $\lambda_{min}(p) < 0$, the solution is unstable. In the parametric space $\mathcal{P}$ the equation

$$\lambda_{min}(p) = 0 \tag{20}$$

determines a hypersurface, whose points appear to be bifurcation points corresponding to the solution under consideration. The cross-sections of the bifurcation hypersurface (20), when there are chosen pairs of the parameters p for fixed values of others, are called bifurcation curves and the respective values of the parameters - bifurcation (critical) parameters. From a physical viewpoint the most interesting seem to be the bifurcation curves of kind "critical current - barrier strength":

$$\lambda_{min}(j_c, g_1) = 0, \tag{21}$$

for given g_2 and g_3.

4 Numerical Modelling

The Generalized Continuous Analogue of Newton's Method (see the survey by I.V. Puzynin et al. [10]) for solving the nonlinear differential equation (11) on the interval $z \in S \cup N$ with zero Neumann conditions (12) at boundaries $z = \pm\Delta$ and continuity conditions (13) at the interfaces $z = \pm\delta$, is applied.

At each iteration the corresponding linear boundary value problem is solved numerically using the finite elements method on a nonuniform grid, condensed to the boundaries $z = \pm\delta$ of the layer. Then the corresponding phase $\varphi(z, J)$ is calculated by means of the integral (14).

We calculate the first three eigenvalues of SLP using the sub-space iterations method [11].

Obviously, the trivial method to calculate the bifurcation curves is through the point by point construct them using BVP (11) - (13). In order to calculate the bifurcation curves directly we consider the system of equations (11) - (13), together with SLP (16) - (18) and the norm condition (19) as well, as a closed nonlinear eigenvalue problem with respect to the amplitude $R(z)$, eigenfunction $\psi(z)$ and one of the parameters p, for example, J, while the other parameters, and the eigenvalue λ as well, supposed to be given. Fixing $\lambda > 0$ to be small enough (for example $\lambda = 0.001 \div 0.01$), we guarantee a stable solution to be obtained. We note that the derivative $\partial\lambda/\partial J \to \infty$ when J approaches its critical (bifurcation) value j_c, so, the solutions of the above system with a priori prescribed accuracy belong to the small vicinity of the sought bifurcation curve.

This method has been applied to solve various physical problems [12] - [16].

In order to make further comparison with paper [3], we introduce the parameters [9]

$$g_1 \equiv 2\,(1 + m_0 a_0)\,\delta, \quad g_2 \equiv 2\,(1 - m_0 b_0)\,\delta, \quad g_3 \equiv 2\left(1 - m_0^2\right)\delta.$$

Clearly for very thin normal layers, when $\delta \to +0$, the parameters g_i, $(i = 1,2,3)$, scaling like δ should be small too. This limiting case corresponds to a uniform superconducting flow and to small deviations from this state. However, if the value of δ is fixed for a relatively thin layer (for example, $\delta = 0.2$ in our numerical study), the situation is quite different. For example, for normal materials possessing very low electrical conductivity, the effective mass m_n can take very large values as compared to m_s, so that we can have $m_0 \gg 1$. Thus, for a finite size layer the parameter's domains of definition are

$$g_1 \in [0, \infty), \quad g_2 \in (-\infty, \delta], \quad g_3 \in (-\infty, 0].$$

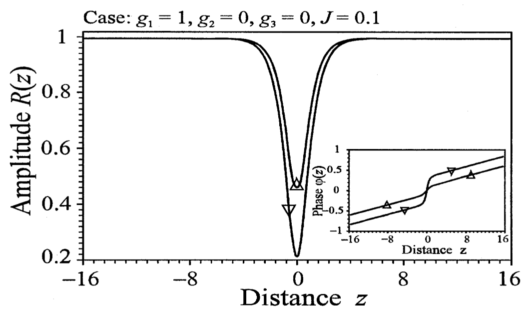

Figure 2: Two basic solutions for $J = 0.1, g_1 = 1, g_2 = 0$ and $g_3 = 0$.

All numerical results from now on were obtained for $\Delta = 16$ and the width of the layer $\delta = 0.2$.

Fig. 2 demonstrates the two basic solutions $R(z, J)$ we found numerically (in this case $J = 0.2, g_1 = 1, g_2 = 0, g_3 = 0$). The first solution (marked by Δ, $\lambda_{min} \approx 1.14$ and full energy $F \approx -1.05$) is a smooth function of J in the closed interval $[-j_c, j_c]$, where j_c is the critical current (see below). For the second solution (marked by ∇, $\lambda_{min} \approx -3.09$, and $F \approx -1.02$) the definition domain is $J \in [-j_c, 0) \cup (0, j_c]$.

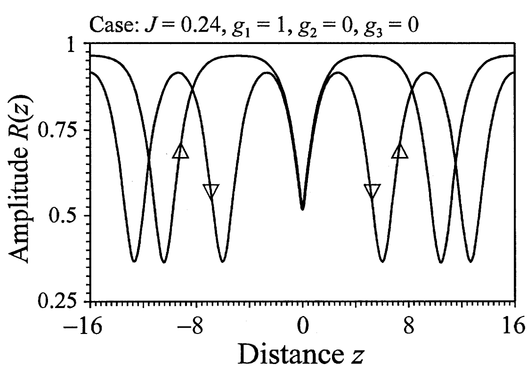

Figure 3: Chains of solitons: Δ: $N = 3, F \approx -0.51$; $\nabla : N = 5, F \approx -0.47$.

The inset displays the corresponding phase differences $\varphi(z)$. Since $R(z, J) \to$ const ≤ 1 when $z \to \pm\Delta$, the phase asymptotically is a linear function of z.

It is necessary to note that except the mentioned basic solutions, there exists a number of multi-soliton solutions, some of which are demonstrated in Fig. 3. These solutions can be considered as chains of $N > 1$ solitons and the corresponding energy increases when the number N increases.

The graphics in Fig. 4 represent the $J(\Delta\varphi)$ curves obtained numerically for four different values of g_1 at $g_2 = 0$ and $g_3 = 0$.

For each curve in this figure we denote $j_c = \max J(\Delta\varphi)$ when $\Delta\varphi/\pi \in [0, 1)$, so $J'(\Delta\varphi_c) = 0$. If the effective barrier strength $g_1 = 0$ (the corresponding curve is marked by $\square$), the extreme points $B = (\pm j_c, \pm\Delta\varphi_c/\pi)$ are achieved at $j_{dep} = \pm 2/3\sqrt{3} \approx \pm 0.385$. The value j_{dep} is the depairing current density in a uniform superconductor, so if $g_1 > 0$ then $j_c < j_{dep}$. For large values of g_1 ($a_0 \gg -1$) we found results close to the ideal Josephson relation (1). We note that the numerical results displayed in Fig. 5 are in a good agreement with Fig. 2 in paper [3].

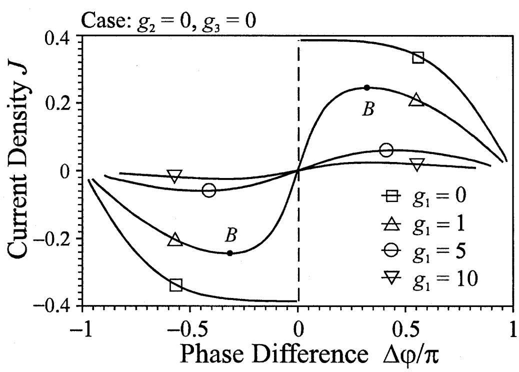

Figure 4: Some typical curves $J(\Delta\varphi/\pi)$ for $g_2 = 0$ and $g_3 = 0$.

The dependence of the free energy $F(J)$ on the current density J for these two solutions is represented graphically in Fig. 5 for different values of g_1. Such a behaviour of curves $F(J)$ can be qualitatively explained by means of formula (15). In fact,

$$F'(J) = -J \int_{-\Delta}^{\Delta} \frac{m(z)}{R^2(z)}\, dz = -\chi,$$

where $\chi \equiv \varphi(\Delta) - \varphi(-\Delta)$ is the total phase difference [3]. It follows that if $J > 0$ then $F'(J) < 0$, so the energy decreases, and vice versa - if $J < 0$ then $F'(J) < 0$, and function $F(J)$ increases. For $J = 0$ the energy connected with the solution under consideration has a maximum.

We note that Fig. 5 gives a typical bifurcation diagram: at the points B where $J = j_c$, the two branches, which correspond to the basic solutions with different energies, coalesce and acquire a common cusp.

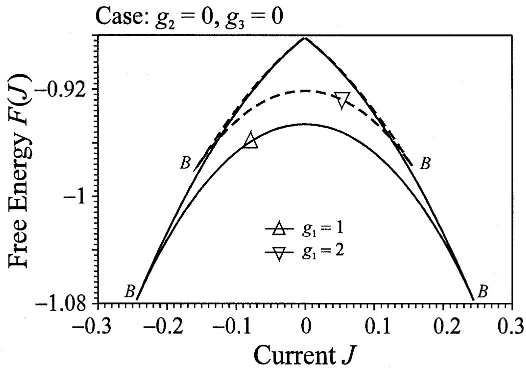

Figure 5: Bifurcation: the full energy $F(J)$ has a cusp for $J = j_c$.

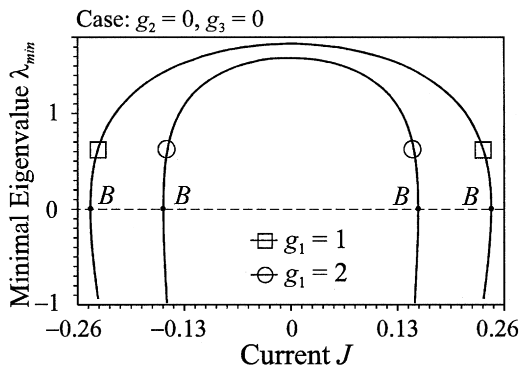

Figure 6: Bifurcation: the minimal eigenvalue $\lambda_{min}(j_c) = 0$.

This fact is corroborated in Fig. 6 which displays the relations $\lambda_{min}(J)$ for $g_1 = 1$ and $g_1 = 2$. For the solutions under consideration the corresponding curve has two zeros (marked by "B"), that conform to the critical current $\pm j_c$. This means that the critical Josephson current mathematically represents a bifurcation point for the basic solutions. Because of $F(J) < 0$, both solutions $R(z, J)$ guarantee the minimum of the functional (10). Hence, the solution marked by " Δ " in Fig. 2 is still stable, whereas the second one, marked by "∇" appears to be quasi-stable. The lifetime of the quasi-stable solution can be estimated as $(\sqrt{|\lambda_{min}|})^{-1}$.

For completeness the dependence of full energy $F(\Delta\varphi/\pi)$ on normalized phase difference is demonstrated in Fig. 7 for different values of g_1, $g_2 = 0$, and $g_3 = 0$. The minima B of these curves correspond to the critical current density $\pm j_c$ (see Fig. 5). In fact, we have $F'(\Delta\varphi) = F'(J)J'(\Delta\varphi)$, so $F'(\Delta\varphi_c) = 0$ as $J'(\Delta\varphi_c) = 0$ (see Fig. 4), and $F'(\Delta\varphi = 0) = 0$ in view of $F'(J = 0) = 0$ (see Fig. 5).

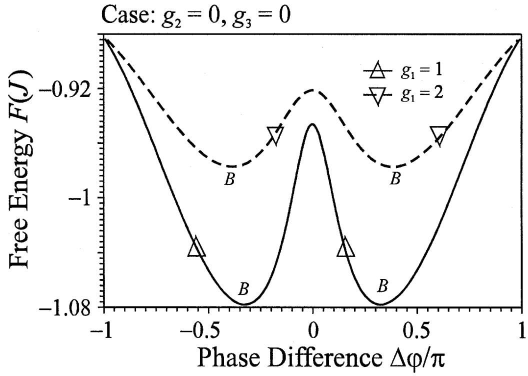

Figure 7: The critical current j_c corresponds to minimums of full energy.

Thus, we can generally conclude that each curve $J(\Delta\varphi)$ consists of three smoothly joined branches $(-\pi, -\Delta\varphi_c)$, $(-\Delta\varphi_c, \Delta\varphi_c)$, and $(\Delta\varphi_c, \pi)$, corresponding to stable and unstable states of the order parameter's amplitude $R(z)$. The critical Josephson current j_c appears to be a bifurcation value for these states and conforms to the points of confluence of the separated branches. The stable solution can be originated from the stable uniform solution $R(z) = 1$, $\varphi(z) = 0$, $\lambda_{min} = 2$, and $F = -1$ (existing for $J = 0$, $g_1 = 0$, $g_2 = 0$, and $g_1 = 0$) via continuous change of the barrier strength g_1 and/or the current J (see, for example, Fig. 8). The unstable state can not be obtained by a continuous alternation of J from the second trivial solution $R(z) = 0$, $\varphi(z) = 0$, $\lambda_{min} = -1$, and $F = 0$ which exists if $J = 0$.

A simple qualitative criterion for stability of some solution $R(z, J)$ by changing the current J can be derived in the following way. Let us denote $\rho(z, J) = \partial R/\partial J$. By means of differentiation of Eqs. (11) - (13) we obtain:

$$-\frac{1}{m(z)}\rho'' + q(z)\rho = -\frac{2m(z)J}{R^3}, \tag{22}$$

$$\rho'(-\Delta) = 0, \quad \rho'(\Delta) = 0, \tag{23}$$

$$\rho'(-\delta - 0) = \frac{1}{m_0}\rho'(-\delta + 0), \quad \frac{1}{m_0}\rho'(\delta - 0) = \rho'(\delta + 0). \tag{24}$$

On the other hand, it is well known [18] that the eigenfunction $\psi_0(z, J)$ corresponding to the minimal eigenvalue $\lambda_{min}(J)$ is of constant signs on the whole interval D. Let us multiply Eq. (22) by $\psi_0(z)$ and integrate over D. Then using the conditions (23), (24) together with SLP (16) - (18), it is easy to derive

$$\lambda_0(J) \int_{-\Delta}^{\Delta} \varrho(z, J)\psi_0(z, J)\, dz = -2J \int_{-\Delta}^{\Delta} \frac{m(z)\psi_0(z, J)}{R^3(z, J)}\, dz\,.$$

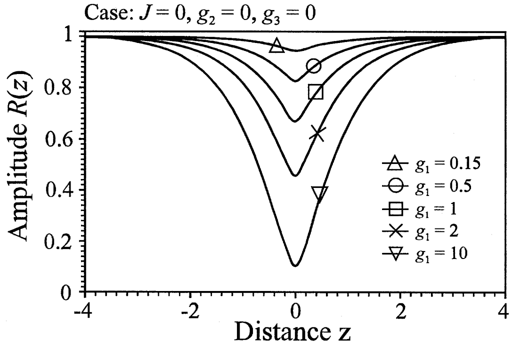

Figure 8: The amplitude $R(0)$ reduces significantly when g_1 increase.

The last formula achieves that for a fixed value of $J \neq 0$ the necessary and sufficient condition for the stability of some solution $R(z, J)$ can be rewritten in the form $\partial R/\partial J(z, J) < 0$. If $\partial R/\partial J(z, J) > 0$, then $R(z, J)$ is unstable.

This conclusion can be graphically observed in Fig. 9. The curves demonstrated in this figure are cross-sections of the surface $R = R(z, J)$ with fixed plains $z = const.$ In view of symmetry only the case $J \geq 0$ is displayed. The dotted lines conform to the unstable branches $(-1, -\Delta\varphi_c/\pi) \cup (\Delta\varphi_c/\pi, 1)$ on $J(\Delta\varphi)$ curve (see Fig. 4), the solid ones correspond to the stable branch $(-\Delta\varphi_c/\pi, -\Delta\varphi_c/\pi)$. The first curve at the left (marked by " $\triangleleft$ ") represents the trajectory of the point $R(0, J)$ at change the current J, while the curve marked by " $\triangleright$ " represents the point $R(\Delta, J)$. It is clearly seen that for every dotted branch the derivative $\partial R/\partial J > 0$ and $\partial R/\partial J < 0$ for solid branches.

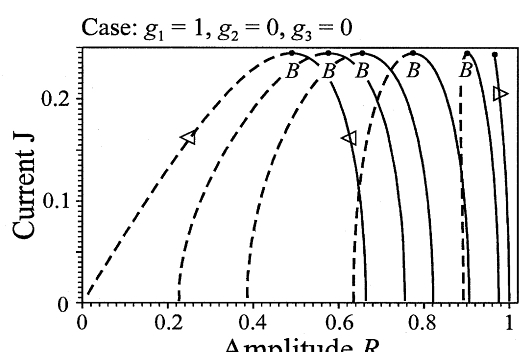

Figure 9.

An analogous approach is used, for example, in the theory of filtration [19].

In order to understand the critical case $J = j_c$, we point out that in accordance with the Schwartz inequality and the norm condition (19) it is easy to derive

$$|\lambda_{\min}(J)| \int_{-\Delta}^{\Delta} \rho^2(z,J)\,dz \leqslant \sqrt{2|J|} \int_{-\Delta}^{\Delta} \frac{m(z)}{R^3(z,J)}\,\psi(z,J)\,dz\,.$$

Therefore, when $J \to j_c$ and, consequently, $|\lambda_{min}(J)| \to 0$ the condition

$$\int_{-\Delta}^{\Delta} \left(\frac{\partial R}{\partial J}\right)^2 dz \to \infty.$$

is necessary to be fulfil. Graphically the last requirement can be observed in Fig. 9: every bifurcation point B is a maximum of the corresponding curve $J(R(z))$ for fixed z, and, therefore, the envelope $\partial R/\partial J \to \infty$ when $J \to j_c$.

In case $J = 0$ the comparison of the problem (22) - (24) with SLP (16) - (18) leads to two possibilities. If the minimal eigenvalue of SLP originated from the solution $R(z,0)$, satisfies $\lambda_{min}(0) \neq 0$, then the problem (22) - (24) has only a trivial solution, so $\partial R/\partial J = 0$ for $z \in D$. As it can be seen from Fig. 8, the last condition is realized for stable (solid) branches and is not reached for unstable ones especially in the vicinity of the center $R(0,J)$. Formally non-trivial solutions are possible if and only if $\lambda_{min}(0) = 0$. It means that $R(z)$ is a bifurcation solution, and $J = 0$ appears to be the corresponding bifurcation point.

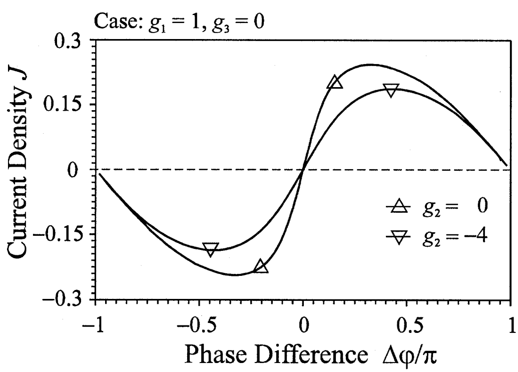

Figure 10: The influence of the parameter g_2.

Figs. 10 and 11 show the influence of the parameters g_2 and g_3, respectively, on the $J(\Delta\varphi)$ curve. The comparison between Fig. 10 and Fig. 4 at $g_1 = 1$ clearly indicates the influence of the parameter g_2 on the current density J - with enlargement of $|g_2|$ we have a more definitely expressed Josephson behaviour of the curve $J(\Delta\varphi)$. For large enough values of g_1 the influence of g_2 is insignificant (see below Fig. 12, 14 and 15).

The comparison between Fig. 12 and Fig. 4 for $g_1 = 1$ shows that the variation of the parameter g_3 ($m_0 > 1$) between 0 and -4 leads to a significant reduction of the maximum current density (approximately twice).

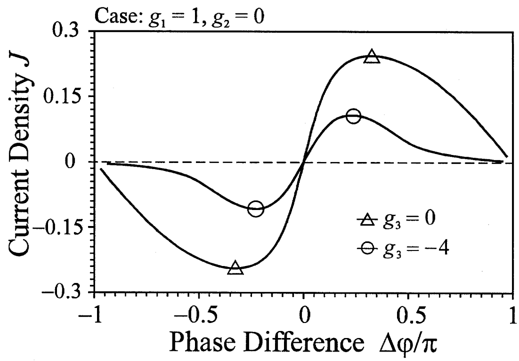

Figure 11: The influence of the parameter g_3.

These quantitative conclusions can be coupled with the Fourier decomposition of $J(\Delta\varphi)$ curves as given by Eq. (2). We are restricted mainly to the analysis of the ratio j_2/j_1 of the first two Fourier coefficients. When $j_2/j_1 \ll 1$, we have approximately a pronounced Josephson behavior $J \simeq j_c \sin\Delta\varphi$, so $j_c \approx j_1$.

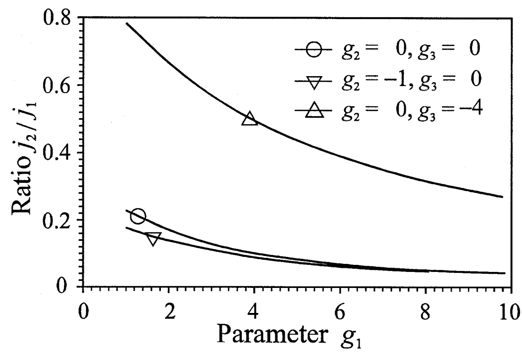

Figure 12: The influence of the parameter g_1.

The ratio j_2/j_1 as a function of the effective barrier strength g_1 is shown in Fig. 12. It is seen that for large values of the parameter g_1 ($g_1 > 8$), when the parameter $g_3 = 0$, the amplitude j_2 of the second harmonic is less than 5% of j_1. On the contrary, for small values of g_1 we have a substantial weight of higher harmonics (for example, if $g_1 = 1$, then the ratio $j_2/j_1 \approx 0.23$).

As can be expected (see the curve marked by ∇), the influence of the parameter g_2 (the ratio b_0) on the Fourier coefficients is essential for small enough values only of the parameter g_1 (for example, if $g_1 = 1$, $g_2 = -1$, and $g_3 = 0$, then $j_2/j_1 \approx 0.17$). For large values of g_1 the influence of the parameter b_0 can be neglected.

On the other hand, taking into account the coefficient $g_3 < 0$ (the ratio $m > 1$)

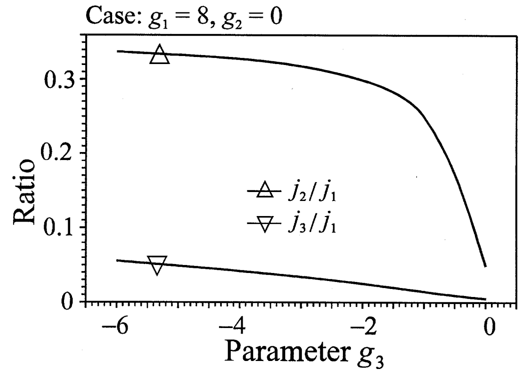

Figure 13: The influence of the parameter g_3.

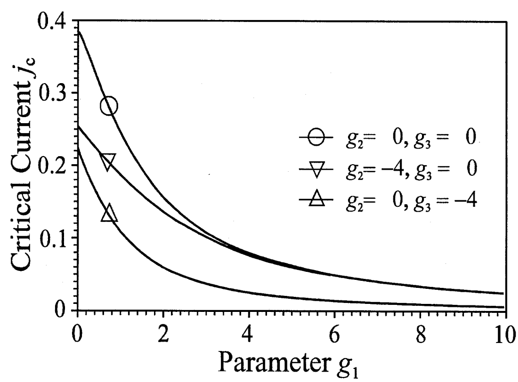

Figure 14: The biffurcation curves $\lambda_{min}(j_c, g_1) = 0$.

leads to a significant increase of the second term in Eq. (2) even at great values of the parameter g_1 (the corresponding curve j_2/j_1 is marked by $\triangle$). This statement is graphically confirmed in Fig. 13, where the ratios j_2/j_1 and j_3/j_1 as functions of g_3 for considerable barrier strength $g_1 = 8$ and $g_2 = 0$ are displayed. Even small differences between the effective masses caused a rapid increase of the second harmonic in 2. In the same time the influence of the third harmonic can be neglected. Mathematically such behavior is clear enough - the existence of discontinuity of the first derivative $R'(z)$ at the boundaries $z = \pm\delta$ increases the number of higher harmonics.

These facts can be considered as arguments for applying the double SG equation (see, for example, [20, 21]) to some Josephson contacts.

As a case in point, in Fig. 14 we display some numerically obtained bifurcation curves of kind (21) for different values of the parameters g_2 and g_3. The curve marked by $\bigcirc$ conforms to Fig. 2 in [3]. Regardless of the fact that the asymptotic behaviour of

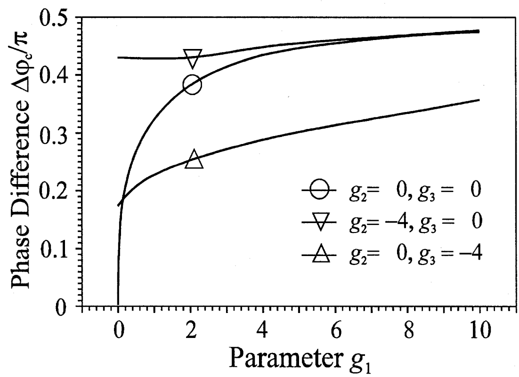

Figure 15: The biffurcation curves $\lambda_{min}(\Delta\varphi_c, g_1) = 0$.

the bifurcation curves for large g_1 is similar when the anisotropy of the effective masses is turned off ($g_3 = 0$), the critical current is essentially larger ($\sim$ 4 times when $g_1 = 8$) as compared with the case $g_3 = -4$.

Based on dependencies $J(\Delta\varphi)$ and (21) it is easy to construct the critical curves of kind (see Fig. 14)

$$\lambda_{min}(\Delta\varphi_c, g_1) = 0.$$

The importance of such curves is due to the fact that the quantity $\Delta\varphi_c$ is a measure of a deviation of dependence $J(\Delta\varphi)$ from ideal Josephson behaviour ($\Delta\varphi_c = \pm 1/2$). As it follows from Fig. 15, the influence of the non-linearity g_2 can be neglected, while the presence of different effective masses leads to a significant error even for large enough barrier strength g_1.

Concluding Remarks

In the present paper we show that by taking into account different phenomenological coefficients in the normal and superconducting regions in SNS sandwich, many harmonics exist and the dependence $J(\Delta\varphi)$ of the current as a function of the phase offset is not sinusoidal. The usually accepted sinusoidal dependence (1) is justified only for a restricted domain of values of the parameters m_0, a_0, and b_0.

We show that each curve $J(\Delta\varphi)$ consists of three continuously joined pieces, corresponding to stable and unstable states of the order parameter's amplitude. The critical Josephson current j_c appears to be a bifurcation value for these states.

We prove numerically that the essential deviation from sinusoidal relation is caused by the possible anisotropy of masses ($m_0 > 1$). The numerical investigation indicates nontrivial specific modifications of the $J(\Delta\varphi)$ curve by the introduced new parameters g_2 and g_3. When the anisotropy of masses is absent ($m_0 = 1$), $b_0 = 1$ and the thickness δ is very small, we recover the results given in [3].

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Бояджиев Т. Л. E11-2001-248
Бифуркации решений модифицированного уравнения Гинзбурга–Ландау для джозефсоновских контактов

При помощи модифицированного уравнения Гинзбурга–Ландау (ГЛ) рассматривается задача численного построения кривых вида «сверхток–разность фаз» для джозефсоновского контакта. Показано, что каждая кривая такого вида состоит из трех ветвей, соответствующих устойчивому и неустойчивому состояниям амплитуды параметра порядка. Критический ток в переходе соответствует точке бифуркации решений (точке «склейки» ветвей). Построены примеры бифуркационных зависимостей, связывающих критический ток с феноменологическими коэффициентами ГЛ-уравнения. При помощи численного фурье-разложения изучено влияние ГЛ-коэффициентов на форму указанных кривых. В частности, в случае неравных эффективных масс-носителей в слоях контакта нельзя пренебрегать амплитудой второй гармоники.

Работа выполнена в Лаборатории информационных технологий ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна, 2001

Boyadjiev T. L. E11-2001-248
Bifurcations of the Solutions
of Modified Ginzburg–Landau Equation for Josephson Junctions

We investigate numerically a class of superconducting/normal/superconducting (SNS) contacts with plane boundaries on the basis of modified Ginzburg–Landau type equations.

The resulted current density-phase offset curves are constructed numerically for different values of the phenomenological coefficients. We prove that each curve consists of three smoothly joined branches corresponding to stable and unstable states of the order parameter's amplitude. The critical Josephson current appears to be a bifurcation value for these states and conforms to the points of confluence of the separated branches.

The investigation has been performed at the Laboratory of Information Technologies, JINR.

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