

Quaternionic Extension of the Double Taub-NUT Metric

Pierre-Yves Casteill^{a,1}, Evgeny Ivanov^{b,a,2}, Galliano Valent^{a,3}

^a *Laboratoire de Physique Théorique et des Hautes Energies,
Unité associée au CNRS URA 280, Université Paris 7
2 Place Jussieu, 75251 Paris Cedex 05, France*

^b *Bogoliubov Laboratory of Theoretical Physics, JINR,
Dubna, 141 980 Moscow region, Russia*

Abstract

Starting from the generic harmonic superspace action of the quaternion-kähler (QK) sigma models and using the appropriate adaptation of the quotient method we present a QK extension of the double Taub-NUT metric in an explicit form. It possesses $U(1) \times U(1)$ isometry and supplies a new example of non-homogeneous QK metric.

p. n. 12, 001, 2001

E-Mail:

- 1) casteill@lpthe.jussieu.fr
- 2) eivanov@thsun1.jinr.ru, eivanov@lpthe.jussieu.fr
- 3) valent@lpthe.jussieu.fr

1. Introduction. In view of the distinguished role of hyper-Kähler (HK) and quaternion-Kähler (QK) manifolds in string theory (see, e.g., [1]), it is important to know the explicit form of the corresponding metrics. One of the approaches to computing them proceeds from the generic actions of bosonic nonlinear sigma models with the HK or QK targets.

A generic action for the bosonic QK sigma models was constructed in [5], based upon the well-known one-to-one correspondence [6] between the QK manifolds and local $N = 2, d = 4$ supersymmetry. This relationship was made manifest in [2], where the most general off-shell action for the hypermultiplet $N = 2$ sigma models coupled to $N = 2$ supergravity was constructed in the framework of $N = 2$ harmonic superspace (HSS) [7]. As was proved in [4], the corresponding analytic superfield Lagrangian of interaction is the QK potential, the fundamental object of QK geometry codifying the entire information about the local properties of the relevant QK metric. Then, based on the one-to-one correspondence mentioned above, the generic QK sigma model bosonic action can be derived [5] by discarding the fermionic fields and part of the bosonic ones in the general HSS sigma model action. The action of physical bosons parametrizing the QK manifold associated with the given HSS potential arises, like in the HK case [1], after elimination of infinite sets of auxiliary fields present in the off-shell hypermultiplet superfields. This amounts to solving some differential equations on the internal sphere S^2 of the $SU(2)$ harmonic variables. It is a difficult problem in general to solve such equations. As was shown in [5], in the case of metrics with isometries the computations can be greatly simplified by using the HSS version of the QK [9] quotient constructions. An attractive feature of the HSS quotient is that it manifests the isometries of the corresponding metric which come out as internal symmetries of the HSS sigma model action.

In [5], using these techniques, we constructed QK extensions of the text-book [10] Taub-NUT and Eguchi-Hanson HK metrics. In this note we apply the HSS quotient approach to construct a QK extension of the 4-dimensional “double Taub-NUT” HK metric. The latter was derived from the HSS approach in [13] by directly solving the corresponding harmonic differential equations. It turns out that the HSS quotient allows one to reproduce the same answer much easier, and it remarkably works in the QK case as well. We gauge two commuting $U(1)$ symmetries of the system of three “free” hypermultiplets and, after solving two algebraic constraints and fully fixing gauges, are left with a 4-dimensional QK metric having two $U(1)$ isometries and going onto the double Taub-NUT in the HK limit. It is an essentially new explicit example of inhomogeneous QK metrics.

2. The gauged HSS action of the QK double Taub-NUT. Details of the general construction can be found in [5]. Here we apply the HSS quotient approach to explicitly construct a sigma model giving rise to a QK generalization of the “double Taub-NUT” HK metric. The latter belongs to the class of two-center ALF metrics with the $U(1) \times U(1)$ isometry (one $U(1)$ is triholomorphic) and was treated in the HSS approach in [13].

We start with the action of three hypermultiplet superfields,

$$Q_A^{+a}(\zeta), \quad g^{+r}(\zeta), \quad a = 1, 2; \quad r = 1, 2; \quad A = 1, 2, \quad (1)$$

possessing no any self-interaction. So, by reasoning of [3, 5], this action corresponds to the “flat” QK manifold $\mathbb{H}H^3 \sim Sp(1, 3)/Sp(1) \times Sp(3)$. In (1), the indices a and r are the doublet indices of two (initially independent) Pauli-Gürsey-type $SU(2)$ s realized on Q_A^{+a} and g^{+r} , the index A is an extra $SO(2)$ index. These superfields are given on the

harmonic analytic $N = 2$ superspace

$$(\zeta) = (x^m, \bar{\theta}^{+\mu}, \bar{\theta}^{+\dot{\mu}}, u^{+i}, u^{-k}), \quad (2)$$

the coordinates u^{+i}, u^{-k} , $u^{+i}u_i^- = 1$, $i, k = 1, 2$, being the $SU(2)/U(1)$ harmonic variables. They satisfy the pseudo-reality conditions

$$(a) Q_A^{+a} \equiv (\widetilde{Q_{aA}^+}) = \epsilon^{ab} Q_{bA}^+, \quad (b) g^{+r} \equiv (\widetilde{g_r^+}) = \epsilon^{rs} g_s^+, \quad (3)$$

where $\epsilon^{ab}\epsilon_{bc} = \delta_c^a$, $\epsilon^{12} = -1$. The generalized conjugation $\widetilde{}$ is the product of the ordinary complex conjugation and a Weyl reflection of the sphere $S^2 \sim SU(2)/U(1)$ parametrized by $u^{\pm i}$. In the QH sigma model action below we shall need only the bosonic components in the θ -expansion of the above superfields:

$$\begin{aligned} Q_A^{+a}(\zeta) &= F_A^{+a}(x, u) + i(\theta^+ \sigma^m \bar{\theta}^+) B_{mA}^{-a}(x, u) + (\theta^+)^2 (\bar{\theta}^+)^2 G_A^{(-3r)}(x, u), \\ g^{+r}(\zeta) &= g_0^{+r}(x, u) + i(\theta^+ \sigma^m \bar{\theta}^+) g_m^{-r}(x, u) + (\theta^+)^2 (\bar{\theta}^+)^2 g^{(-3r)}(x, u) \end{aligned} \quad (4)$$

(possible terms $\sim (\theta^+)^2$ or $\sim (\bar{\theta}^+)^2$ can be shown not to contribute to the final action). The component fields still have a general harmonic expansion off shell. The physical bosonic components $F_A^{ai}(x), g^{ri}(x)$ are defined as the first components in the harmonic expansions of $F_A^{+a}(x, u), g_0^{+r}(x, u)$

$$\begin{aligned} F_A^{+a}(x, u) &= F_A^{ai}(x) u_i^+ + \dots, \quad g_0^{+r}(x, u) = g^{ri}(x) u_i^+ + \dots, \\ (\widetilde{F_A^{ai}(x)}) &= \epsilon_{ab} \epsilon_{ik} F_A^{bk}(x), \quad (\widetilde{g^{ri}(x)}) = \epsilon_{rs} \epsilon_{ik} g^{sk}(x). \end{aligned} \quad (5)$$

The selection of two commuting $U(1)$ symmetries to be gauged and the form of the final gauge-invariant HSS action are uniquely determined by the natural requirement that the resulting action have two different limits corresponding to the earlier considered HSS quotient actions of the QK extensions of Taub-NUT and Eguchi-Hanson metrics [5]. The full action S_{dTN} has the following form

$$\begin{aligned} S_{dTN} &= \frac{1}{2} \int d\zeta^{(-4)} \mathcal{L}_{dTN}^+ - \frac{1}{2\kappa^2} \int d^4x [D(x) + V^{m(ij)}(x) V_{m(ij)}(x)], \\ \mathcal{L}_{dTN}^+ &= -q_a^+ \mathcal{D}^{++} q^{+a} + \kappa^2 (u^- \cdot q^+)^2 [Q_{rA}^+ \mathcal{D}^{++} q_r^+ + g_r^+ \mathcal{D}^{++} g^{+r} \\ &\quad + W^{++} (Q_A^{+a} Q_{aB}^+ \epsilon_{AB} - \kappa^2 c^{(ij)} g_i^+ g_j^+ + c^{(ij)} v_i^+ v_j^+) \\ &\quad + V^{++} (2(v^+ \cdot g^+) - a^{(rj)} Q_{rA}^+ Q_{fA}^+)]. \end{aligned} \quad (7)$$

Here, $d\zeta^{(-4)} = d^4x d^2\theta^+ d^2\bar{\theta}^+ du$ is the measure of integration over (2), the covariant harmonic derivative \mathcal{D}^{++} is defined by

$$\mathcal{D}^{++} = D^{++} + (\theta^+)^2 (\bar{\theta}^+)^2 \{ D(x) \partial^{--} + 6 V^{m(ij)}(x) u_i^- \bar{u}_j^- \partial_m \}, \quad (8)$$

with $D^{++} = \partial^{++} - 2i\theta^+ \sigma^m \bar{\theta}^+ \partial_m$, $\partial^{\pm\pm} = u^{\pm i} / \partial u^{\mp i}$, the non-propagating fields $D, V^{m(ij)}$ are inherited from the $N = 2$ supergravity Weyl multiplet, κ^2 is the Einstein constant (or, from the geometric point of view, the parameter of contraction to the HK case) and the new harmonic v^{+i} is defined by

$$v^{+a} = \frac{q^{+a}}{u_i^- q^{+i}} = u^{+a} - \frac{u_i^+ q^{+i}}{u_i^- q^{+i}} u^{-a}.$$

The superfield $q^{+a} = f^{+a}(x, u) + \dots = f^{ai}(x)u_i^+ + \dots$ is an extra compensating hypermultiplet, with the θ expansion and reality properties entirely analogous to (3), (4). Like in [5], we fully fix the local $SU(2)_c$ symmetry of (6) (which is present in any QK sigma model action) by the gauge condition

$$f_a^i(x) = \delta_a^i \omega(x) . \quad (9)$$

The objects defined so far are necessary ingredients of the generic QK sigma model action. The specificity of the given case is manifested in the particular form of \mathcal{L}^{+4} in (7). It includes two analytic gauge abelian superfields $V^{++}(\zeta)$ and $W^{++}(\zeta)$ and two sets of $SU(2)$ breaking parameters $c^{(ij)}$ and $a^{(rm)}$ satisfying the pseudo-reality condition

$$\overline{c^{(ij)}} = \epsilon_{ik}\epsilon_{jl}c^{(kl)} \quad (10)$$

(and the same for $a^{(rm)}$). The Lagrangian (7) can be checked to be invariant under the following two commuting gauge $U(1)$ transformations, with the parameters $\varepsilon(\zeta)$ and $\varphi(\zeta)$:

$$\begin{aligned} \delta Q_A^{+r} &= \varepsilon [\epsilon_{AB}Q_B^{+r} - \kappa^2 c^{+-} Q_A^{+r}] , & \delta g^{+r} &= \varepsilon \kappa^2 [c^{(rm)}g_r^+ - c^{+-} g^{+r}] , \\ \delta q^{+a} &= \varepsilon \kappa^2 c^{(ab)}q_b^+ , & \delta W^{++} &= \mathcal{D}^{++}\varepsilon , \end{aligned} \quad (11)$$

$$\begin{aligned} \delta Q_A^{+r} &= \varphi a^{(rb)}Q_{bA}^+ - \varphi \kappa^2 (u^- \cdot g^+) Q_A^{+r} , & \delta g^{+r} &= \varphi [v^{+r} - \kappa^2 (u^- \cdot g^+) g^{+r}] , \\ \delta q^{+a} &= \varphi \kappa^2 (u^- \cdot g^+) q^{+a} , & \delta V^{++} &= \mathcal{D}^{++}\varphi . \end{aligned} \quad (12)$$

This gauge freedom will be fully fixed at the end. The only surviving global symmetries are two commuting $U(1)$ s. One of them comes from the Pauli-Gürsey $SU(2)$ acting on Q_A^{+a} and broken by the constant triplet $a^{(rm)}$. Another $U(1)$ is the result of breaking, by the constants $c^{(ij)}$, of the $SU(2)$ which uniformly rotates the doublet indices of harmonics and those of q^{+a} and g^{+r} (it does not commute with supersymmetry and forms the diagonal subgroup in the product of three independent $SU(2)$ s realized on these quantities in the “free” case; this product gets broken down to the diagonal $SU(2)$ and further to $U(1)$ due to the presence of explicit harmonics and $c^{(ik)}$ in the interaction terms in (7)). These two surviving $U(1)$ symmetries will be isometries of the final QK metric, the first one becoming triholomorphic in the HK limit. The SG fields $D(x)$ and $V^{(ik)}$ are inert under any isometry and so are \mathcal{D}^{++} and the S_{SG} part of (6).

It can be shown that the action (6), (7) is a generalization of both the HSS quotient actions describing the QK extensions of the EH and Taub-NUT sigma models: putting $g^{+r} = a^{(rm)} = 0$ yields the EH action as it was given in [3, 5], putting $Q_2^{+a}(Q_1^{+a}) = c^{(ik)} = 0$ yields the Taub-NUT action [5]. Also, fixing the gauge with respect to the λ transformations by the condition $(u^- \cdot g^+) = 0$, varying with respect to the non-propagating superfield V^{++} and eliminating altogether $(v^+ \cdot g^+)$ by the resulting algebraic constraint, we arrive at the form of the action which in the HK limit $\kappa^2 \rightarrow 0$ exactly coincides with the HSS action describing the “double Taub-NUT” manifold [8, 13]. Thus (6), (7) is the natural QK generalization of the action of [8, 13] and therefore the relevant metric is expected to be a QK generalization of the double Taub-NUT HK metric.

3. Towards the target metric. We are going to profit from the opportunity to choose a WZ gauge for W^{++} and V^{++} , in which harmonic differential equations for the components $f^{+a}(x, u)$, $F_A^{+r}(x, u)$ and $g^{+r}(x, u)$ of $q^{+a}(\zeta)$, $Q_A^{+r}(\zeta)$ and $g^{+r}(\zeta)$ are drastically simplified.

In this gauge W^{++} and V^{++} , has the following short expansion

$$\begin{aligned} W^{++} &= i\theta^+ \sigma^m \bar{\theta}^+ W_m(x) + (\theta^+)^2 (\bar{\theta}^+)^2 P^{(ik)}(x) u_i^- u_k^-, \\ V^{++} &= i\theta^+ \sigma^m \bar{\theta}^+ V_m(x) + (\theta^+)^2 (\bar{\theta}^+)^2 T^{(ik)}(x) u_i^- u_k^-, \end{aligned} \quad (13)$$

(once again, possible terms proportional to $(\theta^+)^2$ and $(\bar{\theta}^+)^2$ can be omitted). The hypermultiplet superfields have the same expansions as in (4). At the intermediate step it is convenient to redefine these superfields as follows

$$(Q_A^{+r}, g^{+r}) = \kappa(u^- \cdot q^+) \left(\widehat{Q}_A^{+r}, \widehat{g}^{+r} \right). \quad (14)$$

Due to the structure of the WZ-gauge superfields (13) the highest components in the θ expansions of the redefined HM superfields appear only in the kinetic part of (7), which results in the linear harmonic equations for $f^{+a}(x, u)$, $\widehat{F}_A^{+r}(x, u)$, $\widehat{g}^{+r}(x, u)$:

$$\begin{aligned} \partial^{++} f^{+a} = 0 &\Rightarrow f^{+a} = u^+ \omega(x), \quad \partial^{++} \widehat{F}_A^{+a} = 0 \Rightarrow \widehat{F}_A^{+a} = \widehat{F}_A^{ri}(x) u_i^+, \\ \partial^{++} \widehat{g}^{+a} = 0 &\Rightarrow \widehat{g}^{+a} = \widehat{g}^{ri}(x) u_i^+, \end{aligned} \quad (15)$$

where we have simultaneously fixed the gauge (9).

Next steps are purely technical and quite similar to those explained in detail in [5] on the examples of the QK extensions of the Taub-NUT and EH metrics. One substitutes the solution (15) back into the action (with the θ and u integrals performed), varies with respect to the rest of non-propagating fields and also substitutes the resulting relations back into the action. At the final stages it proves appropriate to redefine the basic fields once again

$$\widehat{F}_A^{ri} = \frac{1}{\kappa\omega} F_A^{ri}, \quad \widehat{g}^{ri} = \frac{2}{\kappa\omega} g^{ri} \quad (16)$$

and to fully fix the residual gauge freedom of the WZ gauge for the φ transformations (with the singlet gauge parameter $\varphi(x)$), so as to gauge away the singlet part of $g^{ai}(x)$:

$$g^{ai}(x) = g^{(ai)}(x) \quad (17)$$

(the residual $SO(2)$ gauge freedom will be kept for the moment). In particular, in terms of the so defined fields we have the following expressions for the fields ω and $V_m^{(ij)}$ which are obtained by varying the full action (6) with respect to $D(x)$ and $V_m^{(ij)}(x)$:

$$\begin{aligned} \kappa\omega &= \frac{1}{\sqrt{1 - \frac{\lambda}{2}g^2 - 2\lambda F^2}}, \\ V_m^{(ij)} &= -16\lambda^2\omega^2 \left[F_A^{ri} \partial_m F_{rA}^{ij} + \frac{1}{4} g^{a(i} \partial_m g_a^{j)} \right], \end{aligned} \quad (18)$$

where

$$F^2 \equiv F_A^{ri} F_{riA}, \quad g^2 \equiv g^{ai} g_{ai}, \quad \lambda \equiv \frac{\kappa^2}{4}. \quad (19)$$

The final form of the sigma model Lagrangian in terms of the fields $F_A^{ri}(x)$ and $g^{(ik)}(x)$ is as follows (we replaced altogether “ ∂_m ” by “ d ”, thus passing to the distance in the target QK space instead of its x -space pullback)

$$\frac{1}{\mathcal{D}^2} \left\{ \mathcal{D} \left(X + Z + \frac{Y}{4} \right) + \lambda \left(g^2 \cdot \frac{Y}{8} + 2T \right) \right\}. \quad (20)$$

with

$$\begin{aligned}
\mathcal{D} &= 1 - \frac{\lambda}{2} g^2 - 2\lambda F^2, \\
X &= \frac{1}{2} dF_{iaA} dF_{iA}^{ia}, \quad Y = \frac{1}{2} dg_{ij} dg^{ij}, \\
Z &= \frac{1}{4\alpha\beta - \gamma^2} \{ \gamma (J \cdot K) - \alpha (J \cdot J) - \beta (K \cdot K) \}, \\
T &= F_{sB}^i dF_{B}^{js} \left(F_{irA} dF_j^r{}_A + \frac{1}{2} g_{ir} dg^r{}_j \right), \tag{21}
\end{aligned}$$

where

$$J = \frac{1}{2} a^{ab} F_{aA}^i dF_{ibA}, \quad K = -\frac{1}{2} \epsilon_{AB} F^{ia} dF_{iaB} - \frac{\lambda}{2} c_{ij} g^i{}_s dg^{sj}, \tag{22}$$

and

$$\begin{aligned}
\alpha &= \frac{1}{2} \left(\frac{F^2}{4} - \lambda \hat{c}^2 + \frac{\lambda^2}{2} \hat{c}^2 g^2 \right), \\
\beta &= \frac{1}{4} \left(1 + \frac{\hat{a}^2}{4} F^2 - \frac{\lambda}{2} g^2 \right), \\
\gamma &= \frac{1}{4} a^{ab} F_{aA}^i F_{ibB} \epsilon_{AB} - \lambda (c \cdot g). \tag{23}
\end{aligned}$$

where

$$\hat{c}^2 \equiv c^{ik} c_{ik}, \quad \hat{a}^2 = a^{ik} a_{ik}. \tag{24}$$

On top of this, there are two algebraic constraints on the involved fields

$$F_A^{a(i} F_{aB}^{j)} \epsilon_{AB} - \lambda g^{(ij)} g^{(rj)} c_{(tr)} + c^{(ij)} = 0, \tag{25}$$

$$g^{ij} = a^{ab} F_{aB}^i F_{bB}^j, \tag{26}$$

which come out by varying the action with respect to the auxiliary fields $P^{(ik)}(x)$ and $T^{(ik)}(x)$ in the WZ gauge (13). Keeping in mind these 6 constraints and one residual gauge ($SO(2)$) invariance, we are left with just four independent bosonic target coordinates as compared with 11 such coordinates in (20). The problem is now to explicitly solve (25), (26). But before turning to this issue, let us notice that the sought metric includes three parameters. These are the Einstein constant, related to λ , and two breaking parameters : the triplet $c^{(ij)}$, which breaks the $SU(2)_{\text{SUSY}}$ to $U(1)$, and the triplet

$a^{(ab)}$, which breaks the Pauli-Gürsey $SU(2)$ to $U(1)$. The final isometry group is therefore $U(1) \times U(1)$. For convenience we choose the following frame with respect to the broken $SU(2)$ groups

$$c^{12} = ic, \quad c^{11} = c^{22} = 0, \quad a^{12} = ia, \quad a^{11} = a^{22} = 0,$$

with real parameters a and c , and we shift $\lambda \rightarrow \frac{\lambda}{a^2}$. Hereafter we shall use this frame, in which, in particular, the squares (24) become

$$\hat{c}^2 = 2c^2, \quad \hat{a}^2 = 2a^2.$$

4. Solving the constraints We need to find true coordinates to compute the metric. This step is non-trivial, due to the fact that (25) becomes quartic after substitution of (26). Instead of solving this quartic equation, it proves more fruitful to take as independent coordinates just the components of the triplet $g^{(ai)}$

$$g^{12} \equiv iah, \quad \bar{h} = h, \quad g^{11} \equiv g, \quad g^{22} = \bar{g},$$

and one angular variable from F_A^r . Then, relabelling the components of the latter fields as follows

$$\left\{ \begin{array}{l} F^{i=2 \ a=1 \ A=1} = \frac{1}{2}(\mathcal{F} + \mathcal{K}), \quad F^{i=1 \ a=1 \ A=1} = \frac{1}{2}(\mathcal{P} + \mathcal{V}), \\ F^{i=2 \ a=1 \ A=2} = \frac{1}{2i}(\mathcal{F} - \mathcal{K}), \quad F^{i=1 \ a=1 \ A=2} = \frac{1}{2i}(\mathcal{P} - \mathcal{V}), \\ F^{i=1 \ a=2 \ A} = -\overline{F^{i=2 \ a=1 \ A}}, \quad F^{i=2 \ a=2 \ A} = \overline{F^{i=1 \ a=1 \ A}}, \end{array} \right.$$

we substitute this into (25), (26), and find the following general solution (it amounts to solving a quadratic equation and we choose the solution which is regular in the limit $g = \bar{g} = h = 0$)

$$\begin{aligned} \mathcal{P} &= -iM e^{i(\phi+\alpha/\rho_- + \mu\rho_+)}, & \mathcal{F} &= R e^{i(\phi+\mu\rho_-)}, \\ \mathcal{K} &= iS e^{i(\phi-\alpha/\rho_- - \mu\rho_+)}, & \mathcal{V} &= L e^{i(\phi-\mu\rho_-)}, \\ \rho_{\pm} &= 1 \pm 4 \frac{\lambda c}{a^2} \end{aligned} \quad (27)$$

and

$$g = at e^{i(\alpha/\rho_- + 8\lambda c/a^2 \mu)}. \quad (28)$$

The various functions involved are

$$\begin{aligned} L &= \sqrt{\frac{1}{2}(\sqrt{\Delta_-} + B_-)}, & R &= \sqrt{\frac{1}{2}(\sqrt{\Delta_+} + B_+)}, \\ M &= \sqrt{\frac{1}{2}(\sqrt{\Delta_+} - B_+)}, & S &= \sqrt{\frac{1}{2}(\sqrt{\Delta_-} - B_-)}, \end{aligned}$$

with

$$\left\{ \begin{array}{l} A_{\pm} = 1 \pm 2\lambda c h, \quad B_{\pm} = c(1 + \lambda r^2) \pm h A_{\mp}, \\ \Delta_{\pm} = B_{\pm}^2 + t^2 A_{\mp}^2, \quad r^2 = h^2 + t^2, \quad g\bar{g} = a^2 t^2. \end{array} \right.$$

The true coordinates are (ϕ, α, h, t) . An extra angle μ parametrizes the local $SO(2)$ transformations (they as shifts of μ). In view of the gauge invariance of (20), the final

form of the metric should not depend on μ and we can choose the latter at will. For instance, we can change the precise dependence of phases in (27), (28) on the angular coordinates ϕ and α . In what follows we shall stick just to the above parametrization.

5. The resulting metric. To get the full metric is fairly involved and Mathematica was intensively used! The final result is

$$g = \frac{1}{4\mathcal{D}^2} \frac{\mathcal{P}}{\mathcal{A}} \left(d\phi + \frac{\mathcal{Q}}{4\mathcal{P}} d\alpha \right)^2 + \frac{\mathcal{A}}{\mathcal{D}^2} \left(dh^2 + dt^2 + \frac{t^2}{\mathcal{P}} (1 + \lambda r^2)^2 d\alpha^2 \right). \quad (29)$$

It depends on 4 functions

$$\mathcal{D}, \quad \mathcal{A}, \quad \mathcal{P}, \quad \mathcal{Q},$$

given by

$$\begin{aligned} \mathcal{A} &= \frac{a^2}{4} + \frac{1}{8}(1 - 4\lambda c^2)(1 - \lambda r^2) \left(\frac{1}{\sqrt{\Delta_+}} + \frac{1}{\sqrt{\Delta_-}} \right) \\ &\quad - \lambda c h \left(\frac{1}{\sqrt{\Delta_+}} - \frac{1}{\sqrt{\Delta_-}} \right) + \frac{\lambda c^2}{a^2} \frac{4\lambda t^2 - (1 + \lambda r^2)^2}{\sqrt{\Delta_+}\sqrt{\Delta_-}}, \\ \mathcal{P} &= (1 + \lambda r^2)^2 \left(1 - \frac{2\lambda c}{a^2} \left(\frac{h + c(1 - \lambda r^2)}{\sqrt{\Delta_+}} - \frac{h - c(1 - \lambda r^2)}{\sqrt{\Delta_-}} \right) \right)^2 \\ &\quad + \frac{4\lambda^2 c^2 t^2}{a^4} \left(\frac{1 - \lambda r^2 - 4\lambda c h}{\sqrt{\Delta_+}} - \frac{1 - \lambda r^2 + 4\lambda c h}{\sqrt{\Delta_-}} \right)^2, \\ \mathcal{Q} &= -(1 + \lambda r^2)^2 \left(\frac{h + c(1 - \lambda r^2)}{\sqrt{\Delta_+}} + \frac{h - c(1 - \lambda r^2)}{\sqrt{\Delta_-}} \right) \\ &\quad + 4\lambda c t^2 \left(\frac{1 - \lambda r^2 - 4\lambda c h}{\sqrt{\Delta_+}} - \frac{1 - \lambda r^2 + 4\lambda c h}{\sqrt{\Delta_-}} \right). \end{aligned}$$

The overall conformal factor is

$$\mathcal{D} = 1 - \lambda r^2 - 2 \frac{\lambda}{a^2} \left(\sqrt{\Delta_+} + \sqrt{\Delta_-} \right).$$

To simplify matter we first rescale $c \rightarrow c/2$. The relations

$$\Delta_+ = (1 + \lambda c^2) t^2 + (h + c/2(1 - \lambda r^2))^2, \quad \Delta_- = (1 + \lambda c^2) t^2 + (h - c/2(1 - \lambda r^2))^2$$

suggest the following change of coordinates

$$T = \frac{2t}{1 - \lambda r^2}, \quad H = \frac{2h}{1 - \lambda r^2}, \quad \rho = \sqrt{T^2 + H^2}, \quad (30)$$

which has the virtue of reducing the quartic non-linearities according to

$$\Delta_{\pm} = \frac{(1 - \lambda r^2)^2}{4} \delta_{\pm}, \quad \delta_{\pm} = (1 + \lambda c^2) T^2 + (H \pm c)^2.$$

Further, to get rid of the square roots we use spheroidal coordinates (s, x) defined by

$$\sqrt{1 + \lambda c^2} T = \sqrt{(s^2 - c^2)(1 - x^2)}, \quad H = s x, \quad s \geq c, \quad x \in [-1, +1].$$

For convenience reasons we scale the angles ϕ and α according to

$$\frac{\phi}{\sqrt{1 + \lambda c^2}} \Rightarrow \phi, \quad \frac{\alpha}{\sqrt{1 + \lambda c^2}} \Rightarrow \alpha,$$

and to have a smooth limit for $a \rightarrow 0$ we scale

$$\lambda \rightarrow \lambda a^2.$$

Putting these changes together, we get the final form of the metric

$$(4d^2)g = (1 + \lambda a^2 s^2) \frac{P}{A} \left(d\phi + \frac{Q}{4P} d\alpha \right)^2 + \frac{A}{P} (s^2 - c^2)(1 - x^2)(1 + \lambda a^2 c^2 x^2) (d\alpha)^2 + A \left(\frac{ds^2}{(s^2 - c^2)(1 + \lambda a^2 s^2)} + \frac{dx^2}{(1 - x^2)(1 + \lambda a^2 c^2 x^2)} \right), \quad (31)$$

with

$$\begin{cases} d = 1 - 2\lambda s, & Q = -2(1 + \lambda a^2 c^2)(s^2 - c^2)x, \\ 4A = (2 + a^2 s)(s - 2\lambda c^2) - a^2 c^2 d^2 x^2, \\ P = c^2(1 - x^2)(1 + \lambda a^2 c^2 x^2) d^2 + (s^2 - c^2) [1 + \lambda a^2 c^2 x^2 - 4\lambda^2 c^2(1 - x^2)]. \end{cases}$$

The isometry group $U(1) \times U(1)$ acts as translations of ϕ and α .

6. Geometric structure of the metric.

The form (31) of the metric is useful to explicitly check (upon use of Mathematica) that it is Einstein with self-dual Weyl tensor (though this automatically follows from the fact that the metric is QK by construction). Let us define the vierbein

$$e_0 = a(s, x) \left(d\phi + \frac{Q}{4P} d\alpha \right), \quad e_3 = b(s, x) d\alpha, \quad e_1 = \lambda ds, \quad e_2 = \mu dx,$$

with

$$\begin{cases} a(s, x) = \frac{1}{2d} \sqrt{1 + \lambda s^2} \sqrt{\frac{P}{A}}, & \lambda = \frac{1}{2d} \sqrt{\frac{A}{\mathcal{A}}}, & \mathcal{A} = (s^2 - c^2)(1 + \lambda s^2), \\ b(s, x) = \frac{1}{2d} \sqrt{(s^2 - c^2)\mathcal{B}} \sqrt{\frac{A}{P}}, & \mu = \frac{1}{2d} \sqrt{\frac{A}{\mathcal{B}}}, & \mathcal{B} = (1 - x^2)(1 + \lambda c^2 x^2). \end{cases}$$

The spin connection being defined as usual by

$$de_a + \omega_{ab} \wedge e_b = 0, \quad a, b = 0, 1, 2, 3,$$

one gets for the anti-self-dual spin connection

$$\omega_i^- = \omega_i - \frac{1}{2} \epsilon_{ijk} \omega_j \wedge \omega_k, \quad i, j, k = 1, 2, 3,$$

the following structure

$$\begin{cases} \omega_1^- = A_1 \left(d\phi + \frac{Q}{4P} d\alpha \right) + B_2 d\alpha, \\ \omega_2^- = A_2 \left(d\phi + \frac{Q}{4P} d\alpha \right) - B_1 d\alpha, \\ \omega_3^- = C_1 dx - C_2 ds. \end{cases}$$

Explicitly one has

$$A_1 = 4\lambda \frac{1}{d} \sqrt{\frac{s^2 - c^2}{P}} F, \quad A_2 = 4\lambda c^2 x \sqrt{\frac{(1 + \lambda a^2 s^2) \mathcal{B}}{P}}, \quad \text{!!check!!}$$

with

$$F = s(1 + \lambda a^2 c^2 x^2) - 2\lambda c^2(1 - x^2).$$

Defining

$$z = \frac{\mathcal{A}\mathcal{B}}{d^2}, \quad \dot{z} = \frac{dz}{ds}, \quad z' = \frac{dz}{dx},$$

the remaining coefficients are then obtained from

$$\begin{cases} B_1 = \frac{\dot{z}}{\lambda a(s, x)} - \frac{b(s, x)}{a(s, x)} A_1, & B_2 = \frac{z'}{\mu a(s, x)} - \frac{b(s, x)}{a(s, x)} A_2, \\ C_1 = \frac{\dot{\mu}}{\lambda} + \frac{\mu}{\lambda} \frac{\dot{a}(s, x)}{a(s, x)} - \frac{\mu}{a(s, x)} A_1, & C_2 = \frac{\dot{\lambda}}{\mu} + \frac{\lambda}{\mu} \frac{\dot{a}(s, x)}{a(s, x)} - \frac{\lambda}{a(s, x)} A_2. \end{cases}$$

The next step requires the computation of the anti self-dual curvature

$$R_i^- \equiv R_{0i} - \frac{1}{2} \epsilon_{ijk} R_{jk} = d \omega_i^- + \epsilon_{ijk} \omega_j^- \wedge \omega_k^-.$$

One gets the crucial relation

$$R_i^- = -16\lambda (e_0 \wedge e_i - \epsilon_{ijk} e_j \wedge e_k), \quad (32)$$

which shows at the same time that the metric is Einstein, with

$$\text{Ric} = \Lambda g, \quad \frac{\Lambda}{3} = -16\lambda,$$

and that the Weyl tensor is self-dual, i.e. that $W_i^- = 0$.

Let us now consider a few limiting cases.

The quaternionic Taub-NUT limit.

Let us show that in the limit $c \rightarrow 0$ we get the quaternionic Taub-NUT. We first write the metric (29) in the form

$$g(c \rightarrow 0) = \frac{1}{\mathcal{D}^2} \left\{ \frac{(1 + \lambda r^2)^2}{\mathcal{A}_0} \left(d\tau + \frac{h}{r} d\alpha \right)^2 + \mathcal{A}_0 \gamma_0 \right\},$$

with

$$\begin{cases} \psi = -\frac{1}{2} \phi, & \mathcal{A}_0 = a^2 + \frac{1}{r} - \lambda r, & \mathcal{D} = 1 - \lambda r^2 - 4 \frac{\lambda r}{a^2}, \\ \gamma_0 = dh^2 + dt^2 + t^2 d\alpha^2, & r^2 = h^2 + t^2. \end{cases}$$

Switching to the spherical coordinates r, θ, ϕ defined by

$$t \sin \alpha = r \sin \theta \cos \phi, \quad t \sin \alpha = r \sin \theta \sin \phi, \quad h = r \cos \theta$$

allows one to get the final form

$$g(c \rightarrow 0) = \frac{(1 + \lambda r^2)^2}{\mathcal{D}^2} \left\{ \frac{1}{\mathcal{A}_0} \sigma_3^2 + \frac{\mathcal{A}_0}{(1 + \lambda r^2)^2} (dr^2 + r^2(\sigma_1^2 + \sigma_2^2)) \right\}, \quad (33)$$

with

$$\sigma_3 = d\psi + \cos \theta d\phi, \quad \sigma_1^2 + \sigma_2^2 = d\theta^2 + \sin^2 \theta d\phi^2.$$

The derivation of the quaternionic Taub-NUT metric from harmonic superspace was given in [12]. It contains 2 parameters $\tilde{\lambda}, R$, and in the limit $R \rightarrow 0$ it reduces to Taub-NUT.

One can see that (33), upon the identifications

$$s = r, \quad a^2 = 4\tilde{\lambda}^2, \quad \lambda = -R\tilde{\lambda}^2,$$

is the quaternionic Taub-NUT metric.

The quaternionic Eguchi-Hanson limit.

This metric was derived using harmonic superspace in [5], and can be written

$$4C^2 g = \frac{(s^2 - c^2)}{sB} \tilde{\sigma}_3^2 + sB \left(\frac{ds^2}{s^2 - c^2} + \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 \right), \quad (34)$$

where

$$sB = s - \kappa^2 c^2, \quad C = 1 - \kappa^2 s, \quad \tilde{\sigma}_3 = d\phi + \cos \theta d\psi, \quad \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 = d\theta^2 + \sin^2 \theta d\psi^2.$$

The writing (34) is adapted to the Killing ∂_ϕ ; if we

switch to the Killing ∂_ψ we can write the metric

$$\frac{D}{4sBC^2}(d\psi + \mathcal{B})^2 + \frac{sB}{4C^2} \left(\frac{ds^2}{s^2 - c^2} + d\theta^2 + \frac{s^2 - c^2}{D} \sin^2 \theta d\phi^2 \right),$$

with

$$D = (s^2 - c^2) \cos^2 \theta + (sB)^2 \sin^2 \theta, \quad \mathcal{B} = \frac{s^2 - c^2}{D} \cos \theta d\phi.$$

If we now take, in the metric (31), the limit $a \rightarrow 0$ it becomes proportional to the metric (34) upon the coordinates identifications

$$\psi = 2\phi, \quad \phi = -\alpha, \quad \kappa^2 = 2\lambda, \quad x = \cos \theta.$$

The hyperkähler limit.

From relation (32) it is clear that in the $\lambda \rightarrow 0$ limit we recover a Riemann self-dual geometry, which is therefore hyperkähler. At the level of the metric, it is most conveniently discussed using the co-ordinates (30). Indeed we get, up to an overall constant factor, the multicentre structure [16],[14],[15]

$$\frac{1}{V} (dt + \mathcal{A})^2 + V \gamma_0,$$

with the flat 3-metric

$$\gamma_0 = dH^2 + dT^2 + T^2 d\alpha^2.$$

The potential V and the connection \mathcal{A} are respectively

$$V = \frac{a^2}{4} + \frac{1}{8\sqrt{\delta_+}} + \frac{1}{8\sqrt{\delta_-}}, \quad \mathcal{A} = -\frac{1}{4} \left\{ \frac{H-c}{\sqrt{\delta_+}} + \frac{H+c}{\sqrt{\delta_-}} \right\} d\alpha, \quad \delta_\pm = T^2 + (H \pm c)^2.$$

The potential shows two centres and $V(\infty) = a^2/4$. An easy computation shows that

$$dV = - \star_{\gamma_0} d\mathcal{A},$$

which is the fundamental relation of the multicentre metrics. Indeed for $a \neq 0$ we have the so-called double Taub-NUT metric, while for $a = 0$ we are back to the metric of Eguchi-Hanson.

References

- [1] A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, *Ann. Phys. (N.Y.)* **185** (1988) 22.

- [2] A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, *Class. Quantum Grav.* **4** (1987) 1255.
- [3] J.A. Bagger, A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky, *Nucl. Phys. B* **303** (1988) 522.
- [4] A. Galperin, E. Ivanov, O. Ogievetsky, *Ann. Phys. (N.Y.)* **230** (1994) 201.
- [5] E. Ivanov and G. Valent, *Nucl. Phys.*, **B 576** (2000) 543.
- [6] J. Bagger, E. Witten, *Nucl. Phys. B* **222** (1983) 1.
- [7] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky, E. Sokatchev, *Class. Quantum Grav.* **1** (1984) 469.
- [8] A. Galperin, E. Ivanov, V. Ogievetsky, P.K. Townsend, *Class. Quantum Grav.*, **7** (1986) 625.
- [9] K. Galicki, *Nucl. Phys. B* **271** (1986) 402; *ibid B 289* (1987) 573; *Commun. Math. Phys.* **108** (1987) 117; *Class. Quantum Grav.* **8** (1991) 1529.
- [10] T. Eguchi, P.B. Gilkey, A.J. Hanson, *Phys. Reports* **66** (1980) 214.
- [11] K.P. Tod, in *Proceedings*, pp. 307-312.
- [12] E. Ivanov and G. Valent, *Phys. Lett. B* **445** (1998) 60; *Class. Quantum Grav.*, **16** (1999) 1039; *ibid*, 2143E.
- [13] G. Gibbons, D. Olivier, P. Ruback and G. Valent, *Nucl. Phys.*, **B 296** (1988) 679.
- [14] G. Gibbons and S. Hawking, *Phys. Lett.*, **B 78** (1978) 430.
- [15] N. Hitchin, *Math. Proc. Camb. Phil. Soc.*, **85** (1979) 465.
- [16] S. Kloster, M. Som and A. Das, *J. Math. Phys.*, **15** (1974) 1096.