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EXAMPLES OF SYSTEMS OF FUNCTIONS BEING RIESZ BASES IN $L_2(0,1)$. APPLICATION TO A NONLINEAR BOUNDARY VALUE PROBLEM

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1 Introduction. Definitions. Results

In a number of recent author's papers, basis properties in standard spaces of functions for eigenfunctions of nonlinear ordinary differential operators are considered. Earlier, some interesting results in this direction were also established in monograph [1]. Since the question is studied very little and because of the difficulty of the problem, only the simplest ODEs were considered. For example, in [2-4], the following problem is analyzed:

$$-u'' + f(u^2)u = \lambda u, \quad u = u(x), \ x \in (0,1),$$

$$u(0) = u(1) = 0, \quad \int_0^1 u^2(x)dx = 1$$

where all quantities are real, $\lambda \in \mathbb{R}$ is a spectral parameter, and it is assumed that $f(s)$ is a smooth nondecreasing function of $s \geq 0$. In these publications, it is proved that for any integer $n \geq 0$ the problem has an eigenfunction u_n possessing precisely n zeros in $(0,1)$ and that such an eigenfunction is unique up to the coefficient ± 1. The main result states that the sequence of eigenfunctions $\{u_n\}_{n=0,1,2,\ldots}$ is a Bary basis in $L_2 = L_2(0,1)$, i. e., it is a basis and there exists an orthonormal basis $\{e_n\}_{n=0,1,2,\ldots}$ in L_2 for which $\sum\limits_{n=0}^{\infty} ||u_n - e_n||_{L_2}^2 < \infty$. We also note that the author's first paper [2] contains certain errors; corrections to this article were published later in [5]. In [6,7], some modifications of the above nonlinear eigenvalue problem are studied and similar statements on basis properties of their eigenfunctions are obtained. In [8], an analog of the Fourier transform associated with an eigenvalue problem for a nonlinear ordinary differential operator on a half-line is considered.

In the present publication, we aim to present an improvement of our result from [9] where the following nonlinear problem without a spectral parameter is considered:

$$u'' = f(u^2)u, \quad u = u(x), \; x \in (0,1), \tag{1}$$

$$u(0) = u(1) = 0; \tag{2}$$

here all variables are real-valued. Everywhere we assume the following:

(f) *Let $f(u^2)u$ be a continuously differentiable function of $u \in \mathbb{R}$ and let $f(0) \geq 0$ and $f(+\infty) = -\infty$.*

It is very well known now (and partially proved in [9]) that under assumption (f) for each integer $n \geq 0$ problem (1)-(2) has a solution u_n which possesses precisely n zeros in $(0,1)$ and that generally speaking this solution with precisely n zeros is not unique. We call a sequence of solutions $\{u_n\}_{n=0,1,2,...}$ of problem (1)-(2) standard if any nth solution u_n has precisely n zeros in $(0,1)$. The main result in [9] states the existence of a $s_0 < 0$ such that for any $s < s_0$ an arbitrary standard sequence of solutions $\{u_n\}_{n=0,1,2,...}$ is a basis in $H^s(0,1)$; in addition, the sequence $\{u_n/||u_n||_{H^s(0,1)}\}_{n=0,1,2,...}$ is a Riesz basis in $H^s(0,1)$ (here $H^s(0,1)$ are well known Sobolev spaces of negative indices). In the present paper, we in particular improve this result showing the above properties of a standard system $\{u_n\}_{n=0,1,2,...}$ in L_2 (see Theorem 3 below). Partially for this aim, we first obtain a result of general type on bases in L_2 (see Theorem 1 below); we believe that this result is of a separate interest. Also, a direct corollary of Theorem 1 is Theorem 2 below; we establish it for completeness of our presentation.

Now we introduce some notation. First, everywhere by $c, C, C_1, C_2, C', C'', ...$ we denote positive constants. By $L_2(a,b)$ we denote the standard Lebesgue space consisting of functions square integrable between a and b, in which we take the standard inner product $(g,h)_{L_2(a,b)} = = \int_a^b g(x)h(x)dx$ and the corresponding norm $||g||_{L_2(a,b)} = (g,g)^{1/2}_{L_2(a,b)}$. We also denote by $(\cdot,\cdot)$ and $||\cdot||$ the quantities $(\cdot,\cdot)_{L_2(0,1)}$ and $||\cdot||_{L_2(0,1)}$, respectively, and set $L_2 = L_2(0,1)$. Let l_2 be the standard space of square summable sequences $\overline{a} = (a_0, a_1, a_2, ...)$, $\overline{b} = (b_0, b_1, b_2, ...)$ of real numbers $a_n, b_n, ...$ with the scalar product $(\overline{a}, \overline{b})_{l_2} = \sum_{n=0}^{\infty} a_n b_n$ and the corresponding norm $||\overline{a}||_{l_2} = (\overline{a}, \overline{a})^{1/2}_{l_2}$. For a Banach space X with a norm $||\cdot||_X$, let $\mathcal{L}(X;X)$ be the linear space of linear bounded operators acting from X into X, equipped with the norm $||A||_{\mathcal{L}(X;X)} = \sup_{x\in X:\ ||x||_X=1} ||Ax||_X$. We also set $||\cdot||_{\mathcal{L}(L_2;L_2)} = ||\cdot||$.

Now we recall, for convenience of readers, some definitions partially known.

Definition 1 *A system $\{e_n\}_{n=0,1,2,...} \subset L_2(a,b)$ is called a basis in $L_2(a,b)$ if for any $g \in L_2(a,b)$ there exists a unique sequence $\{a_n\}_{n=0,1,2,...}$ of real numbers a_n such that $g = \sum_{n=0}^{\infty} a_n e_n$ in $L_2(a,b)$.*

There are different definitions of Riesz bases. In accordance with the classical paper by N.K. Bary [10], where this concept was introduced for the first time, we accept the following definition.

Definition 2 *Let* $\{e_n\}_{n=0,1,2,...}$ *be a basis in* $L_2(a,b)$. *Then, it is called a Riesz basis in this space if the series* $\sum_{n=0}^{\infty} a_n e_n$ *with real coefficients* a_n *converges in* $L_2(a,b)$ *when and only when* $\sum_{n=0}^{\infty} a_n^2 < \infty$.

Remark 1 It is proved in the above-mentioned paper [10] (see also [4]) that if $\{e_n\}_{n=0,1,2,...}$ is a Riesz basis in $L_2(a,b)$ in the sense of this definition, then there exist constants $0 < c < C$ such that

$$c\sum_{n=0}^{\infty} a_n^2 \leq \left|\left|\sum_{n=0}^{\infty} a_n e_n\right|\right|^2_{L_2(a,b)} \leq C\sum_{n=0}^{\infty} a_n^2$$

for all $\overline{a} = (a_0, a_1, a_2, ...) \in l_2$. These estimates have been often used to define Riesz bases.

We also accept the following two definitions.

Definition 3 *A system* $\{g_n\}_{n=0,1,2,...} \subset L_2(a,b)$ *is called* ω*-linearly independent in* $L_2(a,b)$ *if the equality* $\sum_{n=0}^{\infty} a_n g_n = 0$, *where* a_n *are real numbers, holds in* $L_2(a,b)$ *when and only when* $0 = a_0 = a_1 = a_2 = ...$

Definition 4 *Two systems* $\{h_n\}_{n=0,1,2,...}$ *and* $\{e_n\}_{n=0,1,2,...}$ *belonging to* $L_2(a,b)$ *are called quadratically close to each other in* $L_2(a,b)$ *if* $\sum_{n=0}^{\infty} ||h_n - e_n||^2_{L_2(a,b)} < \infty$.

Our first result is the following.

Theorem 1 *Let* $\{h_n\}_{n=0,1,2,\ldots}$ *be a sequence of real-valued, three times continuously differentiable functions* h_n *of the argument* $x \in \mathbb{R}$ *and let for each integer* $n \geq 0$ *the following take place:*
(a) $h_n\left(x + \frac{1}{n+1}\right) = -h_n(x)$ *and* $h_n\left(\frac{1}{2(n+1)} + x\right) = h_n\left(\frac{1}{2(n+1)} - x\right)$ *for all* $x \in \mathbb{R}$*;*
(b) $h_n'(x) > 0$, $h_n''(x) \leq 0$ *and* $h_n'''(x) \leq 0$ *for all* $x \in \left(0, \frac{1}{2(n+1)}\right)$*;*
(c) there exist $0 < c < C$ *such that* $c < h_n\left(\frac{1}{2(n+1)}\right) < C$ *for all* n*.*
Then, the system $\{h_n\}_{n=0,1,2,\ldots}$ *is a Riesz basis in* L_2*.*

Remark 2 Clearly, it follows from Theorem 1 that if a system of functions $\{h_n\}_{n=0,1,2,\ldots}$ satisfies all conditions of this theorem, maybe except (c), then it is a basis in L_2.

The result below is a direct corollary of Theorem 1.

Theorem 2 *Let* $h(x)$ *be a real-valued three times continuously differentiable function of* $x \in \mathbb{R}$ *satisfying the following:*
(a) $h(1 + x) = -h(x)$ *and* $h(1/2 + x) = h(1/2 - x)$ *for all* $x \in \mathbb{R}$*;*
(b) $h'(x) > 0$, $h''(x) \leq 0$ *and* $h'''(x) \leq 0$ *for all* $x \in (0, 1/2)$*;*
Then, the sequence of functions $h_n(x) = h((n+1)x)$*, where* $n = 0, 1, 2, \ldots$*, is a Riesz basis in* L_2*.*

We apply Theorem 1 to prove the following statement related to problem (1)-(2).

Theorem 3 *Let assumption (f) be valid and let in addition* $f(u^2)+$ $+2u^2 f'(u^2) \leq 0$ *for all sufficiently large* $u > 0$. *Let* $\{u_n\}_{n=0,1,2,\ldots}$ *be an arbitrary standard sequence of solutions of problem (1)-(2). Then, the sequence* $\{||u_n||^{-1}u_n\}_{n=0,1,2,\ldots}$ *is a Riesz basis in* L_2.

When we prove this result, we exploit the following theorem of N.K. Bary.

Bary Theorem *Let* $\{e_n\}_{n=0,1,2,\ldots}$ *be a Riesz basis in* $L_2(a,b)$ *and let a system* $\{h_n\}_{n=0,1,2,\ldots} \subset L_2(a,b)$ *be* ω*-linearly independent and quadratically close to* $\{e_n\}_{n=0,1,2,\ldots}$ *in* $L_2(a,b)$. *Then, the system* $\{h_n\}_{n=0,1,2,\ldots}$ *is a Riesz basis in* $L_2(a,b)$.

This result, in a somewhat weaker form, is proved by N.K. Bary in [10] and, just in the present form, it is proved in [4].

In the next Section 2, we prove Theorem 1, and in Section 3 Theorem 3.

2 Proof of Theorem 1

Let $e_n(x) = \sqrt{2}\sin \pi(n+1)x$, $n = 0,1,2,\ldots$, so that $\{e_n\}_{n=0,1,2,\ldots}$ is an orthonormal basis in L_2.

<u>Lemma 1</u> *Let a function g satisfy condition (a) of Theorem 1 with an integer $n \geq 0$ and let it be positive in $\left(0, \frac{1}{n+1}\right)$. Then,*

$$g(\cdot) = \sum_{m=0}^{\infty} c_m e_m(\cdot) \ \text{ in } \ L_2$$

for some real coefficients c_m where $c_0 = ... = c_{n-1} = 0$ and $c_n > 0$.

<u>Proof</u> repeats the proof of a similar statement from [9]. We have the above expansion in $L_2\left(0, \frac{1}{n+1}\right)$ with $c_m = 0$ if $m \neq (n+1)(l+1)-1$ for all integer $l \geq 0$ (this occurs because the functions $\{e_{(n+1)(m+1)-1}\}_{m=0,1,2,...}$ obviously form an orthogonal basis in $L_2\left(0, \frac{1}{n+1}\right)$). Therefore, in particular $c_0 = ... = c_{n-1} = 0$. We observe that each function $e_{(n+1)(m+1)-1}$ becomes zero at the points $\frac{1}{n+1}, \frac{2}{n+1}, ..., 1$. Further, since due to condition (a) of Theorem 1 the function g is obviously odd with respect to these points and each function $e_{(n+1)(m+1)-1}(x)$ is odd, too, this expansion also holds in each space $L_2\left(\frac{1}{n+1}, \frac{2}{n+1}\right), L_2\left(\frac{2}{n+1}, \frac{3}{n+1}\right), ..., L_2\left(\frac{n}{n+1}, 1\right)$. Finally, $c_n > 0$ because $e_n(x)$ and $g(x)$ are of the same sign everywhere.□

Due to Lemma 1, we have the following sequence of expansions:

$$h_n(\cdot) = \sum_{m=0}^{\infty} a_m^n e_m(\cdot) \text{ in } L_2, \ a_0^n = ... = a_{n-1}^n = 0 \text{ and } a_n^n > 0, \ n = 0, 1, 2, ... \tag{3}$$

<u>Lemma 2</u> *In (3), $(a_n^n)^{-1} |a_{(n+1)(m+1)-1}^n| \leq (m+1)^{-2}$ for all n and m. In addition, $a_{(n+1)(m+1)-1}^n = 0$ if $m = 2l+1$ for some $l = 0, 1, 2, ...$*

<u>Proof.</u> The second claim of this lemma is obvious because $e_{(n+1)(2l+2)-1}(x)$ is odd with respect to the middles of the intervals $\left(0, \frac{1}{n+1}\right)$,

$\left(\frac{1}{n+1}, \frac{2}{n+1}\right), \dots, \left(\frac{n}{n+1}, 1\right)$ and the function $h_n(x)$ is even so that $a^n_{(n+1)(2l+2)-1} = (e_{(n+1)(2l+2)-1}, h_n) = 0$. Let us prove the first one. Due to the properties of the functions h_n and $e_{(n+1)(m+1)-1}$, where $m = 2l$, we have the following sequence of equalities:

$$(a^n_n)^{-1}|a^n_{(n+1)(m+1)-1}| = \frac{\left|\int\limits_0^1 h_n(x)\sin\pi(n+1)(m+1)x dx\right|}{\int\limits_0^1 h_n(x)\sin\pi(n+1)x dx} =$$

$$= \frac{\left|\int\limits_0^{1/2(n+1)} h_n(x)\sin\pi(n+1)(m+1)x dx\right|}{\int\limits_0^{1/2(n+1)} h_n(x)\sin\pi(n+1)x dx} =$$

$$= (m+1)^{-1}\frac{\left|\int\limits_0^{1/2(n+1)} h'_n(x)\cos\pi(n+1)(m+1)x dx\right|}{\int\limits_0^{1/2(n+1)} h'_n(x)\cos\pi(n+1)x dx} =$$

$$= (m+1)^{-1}\frac{\left|\int\limits_0^1 h'_n\left(\frac{s}{2(n+1)}\right)\cos\frac{\pi(m+1)s}{2}ds\right|}{\int\limits_0^1 h'_n\left(\frac{s}{2(n+1)}\right)\cos\frac{\pi s}{2}ds}.$$

It is in fact proved in [8], Lemma 4, that under the conditions $h'_n\left(\frac{s}{2(n+1)}\right) > 0$, $h''_n\left(\frac{s}{2(n+1)}\right) \le 0$ and $h'''_n\left(\frac{s}{2(n+1)}\right) \le 0$ for $s \in (0,1)$ immediately following from assumption (b) of Theorem 1, the expression in the right-hand side becomes maximal when $h'_n(\cdot)$ is a positive constant. Hence, we easily get

$$(a^n_n)^{-1}|a^n_{(n+1)(m+1)-1}| \le (m+1)^{-2}.\square$$

Since due to the conditions of Theorem 1 we have $0 < c \leq |a_n^n| \leq$ $\leq C$, clearly, to prove this theorem, it suffices to prove that the system $\{\overline{h}_n\}_{n=0,1,2,...}$ with $\overline{h}_n = (a_n^n)^{-1} h_n$ is a Riesz basis in L_2.

Lemma 3 *Let $\{g_n\}_{n=0,1,2,...}$ be a sequence of functions such that each nth function g_n satisfies condition (a) of Theorem 1 and is positive in $(0, \frac{1}{n+1})$. Then, the system $\{g_n\}_{n=0,1,2,...}$ is ω-linearly independent in L_2.*

Proof. Suppose the contrary and let $\sum_{n=0}^{\infty} d_n g_n = 0$ in L_2 where real coefficients d_n are not all equal to zero. Let integer $l \geq 0$ be such that $d_0 = ... = d_{l-1} = 0$ and $d_l \neq 0$. Then, we multiply this equality by e_l in L_2 and, due to Lemma 1, we obtain $c_l d_l = 0$ where $c_l \neq 0$ is the lth coefficient in the expansion of g_l over the basis $\{e_n\}_{n=0,1,2,...}$. This contradiction proves Lemma 3.□

Let $b_m^n = (a_n^n)^{-1} a_m^n$, Id be the unit operator in L_2, for each integer $m > 0$ $\quad B_m \in \mathcal{L}(L_2; L_2)$ be the operator mapping any e_n into $b_{(n+1)(m+1)-1}^n e_{(n+1)(m+1)-1}$, and let $B = \sum_{m=1}^{\infty} B_m$. We obviously have for each m:

$$||B_m|| \leq \sup_n |b_{(n+1)(m+1)-1}^n| = b_m.$$

Further, by Lemma 2,

$$\sum_{m=1}^{\infty} b_m \leq \sum_{l=1}^{\infty} (2l+1)^{-2} \leq \int_{1/2}^{\infty} (2x+1)^{-2} dx = 1/4,$$

hence, $B \in \mathcal{L}(L_2; L_2)$ and $||B|| \leq 1/4$; therefore, the operator $A = Id + B$ has a bounded inverse $A^{-1} = Id + \sum_{n=1}^{\infty} (-1)^n B^n$. Note also that $Ae_n = \overline{h}_n$.

Take now an arbitrary $v \in L_2$ and let $u = A^{-1}v = \sum_{n=0}^{\infty} c_n e_n \in L_2$ where c_n are real coefficients. Then, $\sum_{n=0}^{\infty} c_n^2 < \infty$ because $\{e_n\}_{n=0,1,2,...}$ is an orthonormal basis in L_2. Since the series $\sum_{n=0}^{\infty} c_n e_n$ converges in L_2, we have $v = Au = \sum_{n=0}^{\infty} c_n A e_n = \sum_{n=0}^{\infty} c_n \overline{h}_n$ where all infinite sums converge in L_2, too. Therefore, in view of Lemma 3, the system $\{\overline{h}_n\}_{n=0,1,2,...}$ is a basis in L_2 and, if $\sum_{n=0}^{\infty} c_n^2 < \infty$, then the series $\sum_{n=0}^{\infty} c_n h_n$ converges in L_2. Conversely, let a series $u = \sum_{n=0}^{\infty} c_n \overline{h}_n$ converge in L_2. Then, we have: $A^{-1}u = \sum_{n=0}^{\infty} c_n e_n$ in L_2; hence $\sum_{n=0}^{\infty} c_n^2 < \infty$. Thus, $\{\overline{h}_n\}_{n=0,1,2,...}$ is a Riesz basis in L_2, and Theorem 1 is proved.□

3 Proof of Theorem 3

As it is proved in [9] (see Proposition in [9]), any nth solution u_n of problem (1)-(2), that possesses precisely n zeros in $(0,1)$, satisfies condition (a) of Theorem 1 and, in addition, it is strictly monotone $(u_n'(x) \neq 0)$ in the interval $\left(0, \frac{1}{2(n+1)}\right)$. Let $\overline{u} > 0$ be an arbitrary number such that $f(u^2) < 0$ and $f(u^2) + 2u^2 f'(u^2) \leq 0$ for all $u \geq \overline{u}$. Let $\{u_n\}_{n=0,1,2,...}$ be an arbitrary standard system of solutions of problem (1)-(2). We accept that $u_n(x) > 0$ for $x \in (0, \frac{1}{n+1})$ for each n which is possible without the loss of generality due to the invariance of equation (1) with respect to the change $u(x) \rightarrow -u(x)$. Due to the standard comparison theorem $\max_{u \in [0, u_n(1/2(n+1))]} |f(u^2)| \rightarrow +\infty$ as $n \rightarrow \infty$; hence $u_n\left(\frac{1}{2(n+1)}\right) \rightarrow +\infty$ as $n \rightarrow \infty$. For each sufficiently large n denote by $x_n \in \left(0, \frac{1}{2(n+1)}\right)$ the point for which $u_n(x_n) = \overline{u}$.

We have

$$u_n\left(\frac{1}{2(n+1)}\right) - \overline{u} = \int\limits_{x_n}^{1/2(n+1)} u_n'(x)dx = u_n'(\tilde{x}_n)\left(\frac{1}{2(n+1)} - x_n\right)$$

for some $\tilde{x}_n \in \left(x_n, \frac{1}{2(n+1)}\right)$; hence, since $u_n'(x_n) \geq u_n'(\tilde{x}_n)$ (because $f(u^2) < 0$ for $u > \overline{u}$ and, therefore, $u_n''(x) < 0$ for $x \in \left(x_n, \frac{1}{2(n+1)}\right)$), we derive

$$u_n'(x_n) \geq \frac{3}{2} u_n\left(\frac{1}{2(n+1)}\right)(n+1) \tag{4}$$

for all sufficiently large n. Since in view of equation (1) $\sup\limits_n \max\limits_{x\in[0,x_n]} |u_n''(x)| \leq C'$, we have $\min\limits_{x\in[0,x_n]} |u_n'(x)| \geq u_n\left(\frac{1}{2(n+1)}\right)(n+1)$ for all sufficiently large n; therefore,

$$0 < x_n \leq (n+1)^{-1}\left[u_n\left(\frac{1}{2(n+1)}\right)\right]^{-1} \tag{5}$$

for all sufficiently large n.

Take an arbitrary integer sufficiently large $n > 0$. We now want, using the function u_n, to construct a function h_n that satisfies the conditions of Theorem 1. Introduce the linear function $l_n(x) = \frac{\overline{u}}{x_n}x$ equal to 0 at $x = 0$ and to $\overline{u} = u_n(x_n)$ at $x = x_n$. Multiply equation (1), written for $u = u_n$, by $2u_n'(x)$ and integrate the result from 0 to x. Then, we obtain the identity

$$\{[u_n'(x)]^2 + F(u_n^2(x))\}' = 0, \quad x \in \mathbb{R}, \tag{6}$$

where $F(s) = -\int\limits_0^s f(t)dt$. Since due to condition (f) $F(u^2) \to +\infty$ as $u \to \infty$, without the loss of generality we can accept that $\overline{u} > 0$ is so large that $|\overline{u}f(\overline{u}^2)| > |uf(u^2)|$ and $F(\overline{u}^2) > F(u^2)$ for all $u \in [0, \overline{u})$. Then, it immediately follows from (6) that

$$u'_n(x_n) < u'_n(x), \quad x \in [0, x_n), \tag{7}$$

for all sufficiently large n. By (7), we have

$$\overline{u} = \int_0^{x_n} u'_n(x)dx > x_n u'_n(x_n);$$

therefore,

$$u'_n(x_n) < \frac{\overline{u}}{x_n} = l'_n(x) \tag{8}$$

for all sufficiently large n.

Take a sufficiently small $\Delta \in \left(0, \frac{x_n}{2}\right)$ and a continuous function $\omega_1(x)$ equal to $u'''_n(x)$ for $x \in \left[x_n, \frac{1}{2(n+1)}\right]$ and such that $u'''_n(x) \le \omega_1(x) \le 0$ for $x \in [x_n - \Delta, x_n]$ and $\omega_1(x) = 0$ for $x \in [0, x_n - \Delta)$. For $g_1(x)$ we take the function equal to $u_n(x)$ for $x \in \left[x_n, \frac{1}{2(n+1)}\right]$ and for $x \in [0, x_n)$ defined by the rule

$$g''_1(x) = u''_n(x_n) - \int_x^{x_n} \omega_1(t)dt, \quad g'_1(x) = u'_n(x_n) - \int_x^{x_n} g''_1(t)dt,$$

$$g_1(x) = u_n(x_n) - \int_x^{x_n} g'_1(t)dt. \tag{9}$$

Then, obviously $g_1(x)$ is a function three times continuously differentiable in $\left[0, \frac{1}{2(n+1)}\right]$ and satisfying condition (b) of Theorem 1. It is very easy to see that if $\Delta > 0$ is sufficiently small, then $g_1(x_n - \Delta)$ and $g'_1(x_n - \Delta)$ are arbitrary close respectively to $u_n(x_n)$ and $u'_n(x_n)$, and that $g''_1(x)$ is arbitrary close to $u''_n(x_n)$ for all $x \in [0, x_n - \Delta]$. Now, due to our choice of $\overline{u} > 0$, we get that if $\Delta > 0$ is sufficiently small, then $g_1(0)$ is arbitrary close to

$$u_n(x_n) - x_n u_n'(x_n) + \frac{x_n^2}{2} u_n''(x_n) \tag{10}$$

which is negative because

$$0 = u_n(0) = u_n(x_n) - x_n u_n'(x_n) + \int_0^{x_n} dx \int_x^{x_n} u_n''(t)dt$$

where due to our choice of $\overline{u}$ and equation (1) the last term in the right-hand side of this equality is larger than the last term in (10). We fix a function $g_1(x)$ satisfying $g_1(0) < 0$.

Take now a sufficiently small $\Delta \in \left(0, \frac{x_n}{2}\right)$ and a continuous function $\omega_2(x) \leq 0$ equal to $u_n'''(x)$ for $x \in \left[x_n, \frac{1}{2(n+1)}\right]$ and to 0 for $x \in [0, x_n - \Delta)$ such that

$$\int_{x_n-\Delta}^{x_n} \omega_2(x)dx = u_n''(x_n).$$

Then, defining the function $g_2(x)$ just as $g_1(x)$ in (9) with the substitution of ω_2 in place of ω_1 and of g_2 in place of g_1, we get that if $\Delta > 0$ is sufficiently small, then $g_2(x_n - \Delta)$ and $g_2'(x_n - \Delta)$ are arbitrary close, respectively, to $u_n(x_n)$ and $u_n'(x_n)$, and $g_2''(x) = 0$ for $0 \leq x \leq x_n - \Delta$. Therefore, due to (8) $g_2(0) > 0$ if $\Delta > 0$ is sufficiently small, for all sufficiently large n. We fix a function $g_2(x)$ satisfying $g_2(0) > 0$.

Now, consider the family of functions $g_\lambda(x) = \lambda g_1(x) + (1 - \lambda)g_2(x)$ where $\lambda \in [0, 1]$. Clearly, there exists a unique $\lambda_0 \in (0, 1)$ such that $g_{\lambda_0}(0) = 0$; in addition, $g_{\lambda_0}(x)$ satisfies condition (b) of Theorem 1. Continue $g_{\lambda_0}(x)$ onto the entire real line $\mathbb{R}$ by the rules: $g_{\lambda_0}\left(\frac{1}{n+1} + x\right) =$

$= -g_{\lambda_0}(x)$ and $g_{\lambda_0}\left(\frac{1}{2(n+1)}+x\right) = g_{\lambda_0}\left(\frac{1}{2(n+1)}-x\right)$ and take the obtained function for $h_n(x)$. Then, this function h_n satisfies conditions (a) and (b) of Theorem 1.

So, we have constructed, for all sufficiently large n, a sequence of functions h_n. For all other, small values of n for h_n we take arbitrary functions satisfying the conditions of Theorem 1. By construction, the functions $\{h_n\}_{n=0,1,2,\dots}$ satisfy conditions (a) and (b) of Theorem 1.

Let $\alpha_n = \left[h_n\left(\frac{1}{2(n+1)}\right)\right]^{-1}$. Then, by Theorem 1, the system $\{\alpha_n h_n\}_{n=0,1,2,\dots}$ is a Riesz basis in L_2. Further, by Lemma 3, the system $\{\alpha_n u_n\}_{n=0,1,2,\dots}$ is ω-linearly independent in L_2. Also, due to equation (1) and by construction, there exists $C_1 > 0$ such that

$$|u_n''(x)| = \max_{u\in[0,\overline{u}]} |uf(u^2)| \le C_1$$

and

$$\max_{x\in[0,x_n]} |h_n''(x)| = |h_n''(x_n)| = |u_n''(x_n)| = |\overline{u}f(\overline{u}^2)| \le C_1$$

for all sufficiently large n. Hence, $|u_n'(x)-h_n'(x)| \le C_2 x_n$ for all sufficiently large n and for all $x \in [0, x_n]$. Hence, due to (5),

$$||\alpha_n u_n - \alpha_n h_n||^2 \le C_3 x_n^4 \le C_4(n+1)^{-4}$$

for all sufficiently large n and therefore, the systems $\{\alpha_n u_n\}_{n=0,1,2,\dots}$ and $\{\alpha_n h_n\}_{n=0,1,2,\dots}$ are quadratically close in L_2. Thus, in view of the Bary Theorem, Theorem 3 is proved.□

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Жидков П.Е. E5-2001-203

Примеры систем функций, являющихся базисами Рисса в $L_2(0,1)$. Приложение к нелинейной граничной задаче

Представлены примеры систем функций, являющихся базисами Рисса в $L_2(0,1)$. Затем этот результат применен для усиления представленного в работе [9] утверждения: показано, что произвольная «стандартная» система решений некоторой нелинейной граничной задачи, нормированных на единицу в том же пространстве, есть базис Рисса в этом пространстве. Приведенные в настоящей работе доказательства весьма элементарны.

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Zhidkov P.E. E5-2001-203

Examples of Systems of Functions Being Riesz Bases in $L_2(0,1)$. Application to a Nonlinear Boundary Value Problem

We establish examples of systems of functions being Riesz bases in $L_2(0,1)$. We then apply this result to improve a theorem presented in [9] showing that an arbitrary «standard» system of solutions of a nonlinear boundary value problem, normalized to 1 in the same space, is a Riesz basis in this space. The proofs in this work are quite elementary.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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