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P. E. Zhidkov*

ON INVARIANT MEASURES FOR THE VLASOV EQUATION WITH A REGULAR POTENTIAL

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*E-mail: zhidkov@thsun1.jinr.ru

1 Introduction. Notation. Main result

Consider the Vlasov equation

$$\frac{\partial f}{\partial t}+v\cdot\nabla_x f+\nabla_v f\cdot\rho(x,t)=0,\quad (x,v)\in\mathbb{R}^3\times\mathbb{R}^3,\ t\in\mathbb{R},\ f=f(t,x,v), \tag{1}$$

$$\rho(x,t)=-\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\nabla U(x-y)f(t,y,v)dy\,dv, \tag{2}$$

$$f(0,x,v)=f_0(x,v). \tag{3}$$

Hereafter all quantities are real, f is an unknown distribution function of particles, $U(\cdot)$ is a given potential of interactions between particles and $a\cdot b$ means the scalar product of vectors $a,b\in\mathbb{R}^3$ in $\mathbb{R}^3$. In view of the physical sense of f it should be required

$$f\geq 0\quad\text{and}\quad\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}f(t,x,v)dx\,dv=1. \tag{4}$$

Associate also the following system of ODEs with (1)-(4):

$$\dot{x}(t)=v(t), \tag{5}$$

$$\dot{v}(t)=\rho(x(t),t), \tag{6}$$

$$(x(0),v(0))=(x_0,v_0)\in\mathbb{R}^3\times\mathbb{R}^3. \tag{7}$$

Now, we introduce some notation. Let M be the space of nonnegative Borel measures μ in $\mathbb{R}^3\times\mathbb{R}^3$ satisfying $\mu(\mathbb{R}^3\times\mathbb{R}^3)=1$ (we call a measure a Borel measure if it is defined on the Borel sigma-algebra). The space M is equipped with the topology of the weak convergence of measures: a sequence $\{\mu_n\}_{n=1,2,3,\ldots}\subset M$ is called weakly converging to a $\mu\in M$ (we write $\mu_n\to\mu$ as $n\to\infty$) if and only if for any real-valued continuous bounded function φ in $\mathbb{R}^3\times\mathbb{R}^3$ one has:

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\varphi(z)\mu_n(dz)=\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\varphi(z)\mu(dz).$$

Let for a real-valued function φ in $\mathbb{R}^3\times\mathbb{R}^3$

$$||\varphi||_{Lip}=\sup_{z\in\mathbb{R}^3\times\mathbb{R}^3}|\varphi(z)|+\sup_{\substack{z_1,z_2\in\mathbb{R}^3\times\mathbb{R}^3\\ z_1\neq z_2}}\frac{|\varphi(z_1)-\varphi(z_2)|}{|z_1-z_2|}$$

and let L denote the space of all functions φ satisfying $||\varphi||_{Lip} < \infty$. Then, $(L, ||\cdot||_{Lip})$ is a Banach space. For $\mu_1, \mu_2 \in M$, denote

$$d(\mu_1, \mu_2) = \sup_{||\varphi||_{Lip}\leq 1} \int_{\mathbb{R}^3\times\mathbb{R}^3} \varphi(z)[\mu_1(dz) - \mu_2(dz)].$$

In what follows, we apply the following result.

Proposition. *Let $\{\mu_n\} \in M$ and $\mu \in M$. Then, $\mu_n \to \mu$ as $n \to \infty$ if and only if $d(\mu_n, \mu) \to 0$ as $n \to \infty$. In fact, d is a distance in M and $(M; d)$ is a complete metric space.*

In fact, this result is well-known and it is exploited in [1-4] for an analysis of Vlasov equations. Readers may consult with these papers and the references in them for its proof.

For an interval $I \subset \mathbb{R}$, $C(I; M)$ denotes the space of continuous functions from I into $(M; d)$; the space $C(I; M)$ is equipped with the metric $\nu(\mu_1(\cdot), \mu_2(\cdot)) = \sup_{t\in I} d(\mu_1(t), \mu_2(t))$. For a given measure-valued function $\mu(t) \in C(I; M)$, by S_t we denote the operator mapping any $(x_0, v_0) \in \mathbb{R}^3 \times \mathbb{R}^3$ into the point $(x(t), v(t))$ where $(x(\cdot), v(\cdot))$ is the corresponding solution of (5)-(7). As it is shown below (see Lemma in the next section), under the assumptions we make S_t is a diffeomorphism of $\mathbb{R}^3 \times \mathbb{R}^3$ onto itself for any fixed t. By δ we denote the standard Dirac delta-function, interpreting it as a measure.

The standard result on the well-posedness of (1)-(4) well-known now is the following (for its proof, see, for example, [1-4]).

Theorem 1. *Let $U(\cdot)$ be an even two times continuously differentiable function in $\mathbb{R}^3$ all whose first and second partial derivatives are bounded and let $\mu_0 \in M$. Then, problem (1)-(4) has a unique weak solution $\mu(t) \in C(\mathbb{R}; M)$ where $\mu(0) = \mu_0$.*

Remark 1. Here, we do not specify in what sense our weak solution of (1)-(4) is understood because this is not used in what follows. On such a definition, see, for example, [1]. For the indicated solutions relations

(2) are replaced by the following:

$$\rho(x,t) = - \int_{\mathbb{R}^3\times\mathbb{R}^3} \nabla U(x-y)\mu(t)(dy\, dv),$$

and conditions (4) by the nonnegativity of $\mu(t)$ and the equality $\mu(t)(\mathbb{R}^3\times\mathbb{R}^3) \equiv 1$.

Remark 2. It is known (see, for example, [5]) that under the assumptions of Theorem 1 problem (1)-(4) possesses continuously differentiable solutions, too, when f_0 is smooth. The connection of these solutions with solutions of (5)-(7) consists in the relations $f(t, x(t, x_0, v_0), v(t, x_0, v_0)) = f_0(x_0, v_0)$ taking place for all (x_0, v_0) and t where $(x(t, x_0, v_0), v(t, x_0, v_0))$ are the solutions of (5)-(7).

Here, we want to prove the following.

Theorem 2. *Under the assumptions of Theorem 1 one has* $\mu(t)(S_t\Omega) = \mu_0(\Omega)$ *for any Borel set* $\Omega \subset \mathbb{R}^3 \times \mathbb{R}^3$.

Remark 3. Since S_t is a diffeomorphism as a map from $\mathbb{R}^3 \times \mathbb{R}^3$ into itself, it transforms Borel subsets of this space into Borel ones.

Remark 4. In [5], invariant measures for the Vlasov equation analogous to those presented with Theorem 2 are constructed in the case when a solution of this equation is smooth. In the indicated paper, these measures are used to prove the existence of a weak solution for the Vlasov equation with the singular potential of interaction between particles $U(x) = |x|^{-2}$.

2 Proof of Theorem 2 on invariant measures

Consider the following measures: $\mu_0^N = N^{-1} \sum_{n=1}^{N} \delta(x - x_n^0) \times \delta(v - v_n^0)$, where $x_n^0 \in \mathbb{R}^3$ and $v_n^0 \in \mathbb{R}^3$ are arbitrary constants. It is known that for any $\mu_0 \in M$ there exists a sequence of measures of the above kind weakly converging to μ_0. So, we accept that μ_0 and the above sequence are already chosen. The result we apply in the present article states that, denoting by $(x^N(t), v^N(t)) = (x_n^N(t), v_n^N(t))_{n=1,2,\dots,N}$ the sequence of solutions of the following system:

$$\dot{x}_n^N(t) = v_n^N(t),$$

$$\dot{v}_n^N(t) = \rho^N(x^N(t), t) := -N^{-1} \sum_{k=1}^{N} \nabla U(x_n^N(t) - x_k^N(t)),$$

$$x_n^N(0) = x_{n,0}^N \in \mathbb{R}^3, \quad v_n^N(0) = v_{n,0}^N \in \mathbb{R}^3$$

where $\mu_0^N = N^{-1} \sum\limits_{n=1}^{N} \delta(x - x_{n,0}^N) \times \delta(v - v_{n,0}^N)$, one has that for any $T > 0$ the sequence of corresponding measures

$$\mu^N(t) = N^{-1} \sum_{n=1}^{N} \delta(x - x_n^N(t)) \times \delta(v - v_n^N(t))$$

converges to the corresponding solution $\mu(t)$ of (1)-(4) in $C([-T,T]; M)$ (for the proof, see, for example, [1-4]).

Consider now the system

$$\dot{x}_N(t) = v_N(t), \tag{8}$$

$$\dot{v}_N(t) = \rho_N(x_N(t), t) := - \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla U(x_N(t) - y) \mu^N(t)(dy\, dv), \tag{9}$$

$$x_N(0) = x_0 \in \mathbb{R}^3, \quad v_N(0) = v_0 \in \mathbb{R}^3, \tag{10}$$

where now (x_0, v_0) runs over the whole $\mathbb{R}^3 \times \mathbb{R}^3$. Clearly, $(x_N(t), v_N(t)) \equiv$ $\equiv (x_n^N(t), v_n^N(t))$ if $(x_n^N(0), v_n^N(0)) = (x_N(0), v_N(0))$.

Denote by $S_t^N(x_0, v_0) = (x_N(t, x_0, v_0), v_N(t, x_0, v_0))$ the evolution operator and solutions of system (8)-(10) (one can easily verify that the standard local existence and uniqueness theorems hold for this system). Since in view of conditions of Theorem 2 the function $\rho_N(x,t)$ is bounded for each N uniformly in x and t, it easily follows that solutions of (8)-(10) are global, that is, they can be continued onto the entire real line $t \in \mathbb{R}$.

Lemma. *Under the assumptions of Theorem 2 for any fixed t S_t^N and S_t are diffeomorphisms (i. e. they are one-to-one mappings continuously* *differentiable with their inverse) of* $\mathbb{R}^3 \times \mathbb{R}^3$ *onto itself. For any* $T > 0$ *one has* $S_t^N(x_0, v_0) \to S_t(x_0, v_0)$ *as* $N \to \infty$ *uniformly in* $(x_0, v_0, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times [-T, T]$.

Proof is very simple. The fact that S_t^N and S_t for any fixed t are diffeomorphisms follows by the standard existence, uniqueness and continuous differentiability theorems for a system of ODEs. As for the second claim of Lemma, we have by Proposition from Section 1:

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla U(x-y)[\mu(t)(dy\, dv) - \mu^N(t)(dy\, dv)] \to 0$$

as $N \to \infty$ uniformly in $(x, t) \in \mathbb{R}^3 \times [-T, T]$ because $\{\nabla U(x-y)\}_{x \in \mathbb{R}^3}$ is a family of uniformly globally Lipschitz continuous functions. So, $S_t^N(x_0, v_0) \to S_t(x_0, v_0)$ as $N \to \infty$ uniformly in $(x_0, v_0, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times$ $\times [-T, T]$ and Lemma is proved.□

Fix an arbitrary $t \in \mathbb{R}$ and consider a ball $B_r(a) = \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : \ |a - (x, v)| < r\}$ where $a \in \mathbb{R}^3 \times \mathbb{R}^3$ and $r > 0$. It is easy to see that, for a fixed a, there exists at most a countable set of values $r > 0$ for each of which $\mu_0(\partial B_r(a)) > 0$ where ∂A denotes the boundary of a set A (because otherwise we would have $\mu_0(B_r(a)) = +\infty$ for all sufficiently large $r > 0$). By analogy, for a fixed a, there exists at most a countable set of values $r > 0$ for each of which $\mu(t)(\partial(S_t B_r(a))) = \mu(t)(S_t(\partial B_r(a))) >$ > 0. So, for a fixed a, the set of values $r > 0$ such that $\mu_0(\partial(B_r(a))) =$ $= \mu(t)(\partial(S_t B_r(a))) = 0$ is dense in $(0, \infty)$.

Let now $A \subset \mathbb{R}^3 \times \mathbb{R}^3$ be a union of a finite number of open balls such that $\mu_0(\partial A) = \mu(t)(\partial(S_t A)) = 0$. Let us prove that

$$\mu_0(A) = \mu(t)(S_t A). \tag{11}$$

Let $\{\mu_0^N\}_{N=1,2,3,\ldots}$ be the above sequence of measures weakly converging to μ_0. Take an arbitrary $\epsilon > 0$. Then, since $\mu(t)(\partial(S_t A)) = 0$, there exists a neighborhood A_ϵ of $\partial(S_t A)$ of the kind

$$A_\epsilon = \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : \ \text{dist}\ ((x, v); \partial(S_t A)) < c\},$$

where $c > 0$ and $\text{dist}\ (x; B) = \inf_{y \in B} |x - y|$, such that

$$\mu(t)(\overline{A_\epsilon}) < \epsilon/2 \tag{12}$$

(here $\overline{A}_\epsilon$ is the closure of A_ϵ). In addition, as earlier, we can accept that $\mu(t)(\partial A_\epsilon) = 0$ easily varying $c > 0$ in the definition of A_ϵ if necessary. According to the lemma above,

$$(S_t A \setminus A_\epsilon) \subset (S_t^N A) \subset (S_t A \cup A_\epsilon) \tag{13}$$

for all sufficiently large N. In addition, obviously

$$\mu_0^N(A) = \mu^N(t)(S_t^N A). \tag{14}$$

Now, we apply the well-known property of weakly converging sequences of measures according to which

$$\mu^N(s)(B) \to \mu(s)(B) \quad \text{as} \quad N \to \infty,\ s = 0, t, \tag{15}$$

for any Borel set $B \subset \mathbb{R}^3 \times \mathbb{R}^3$ satisfying $\mu(s)(\partial B) = 0,\ s = 0, t$. Due to (12)-(15) we have $|\mu(t)(S_t A) - \mu_0(A)| < \epsilon$ and in view of the arbitrariness of $\epsilon > 0$ (11) follows.

For an arbitrary open set $A \subset \mathbb{R}^3 \times \mathbb{R}^3$ equality (11) can be obtained by approximations of A from inside by the above unions B of finite numbers of balls satisfying $\mu_0(\partial B) = \mu(t)(\partial(S_t B)) = 0$. For an arbitrary Borel set A (11) follows by approximations of A from outside by open sets. So, Theorem 2 is proved.□

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Жидков П. Е. E5-2003-15
Об инвариантных мерах для уравнения Власова
с регулярным потенциалом

Рассматривается уравнение Власова с гладким ограниченным потенциалом взаимодействия между частицами в классе обобщенных решений со значениями в пространстве мер. Для этой задачи построена мера, инвариантная в некотором смысле.

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On Invariant Measures for the Vlasov Equation
with a Regular Potential

We consider a Vlasov equation with a smooth bounded potential of interaction between particles in a class of measure-valued solutions and construct a measure which is invariant for this problem in a sense.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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141980, г. Дубна, Московская обл., ул. Жолио-Кюри, 6.
E-mail: publish@pds.jinr.ru
www.jinr.ru/publish/