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DIVERSITY OF OFF-SHELL TWISTED (4, 4) MULTIPLETS IN $SU(2) \times SU(2)$ HARMONIC SUPERSPACE

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Множество твистованных $(4,4)$ мультиплетов вне массовой оболочки
в $SU(2)\times SU(2)$ гармоническом суперпространстве

Дано детальное описание четырех различных типов твистованных $\mathcal{N}=(4,4)$ супермультиплетов в $SU(2)\times SU(2)$ гармоническом суперпространстве в двух измерениях. В обычном $\mathcal{N}=(4,4)$, 2D суперпространстве эти мультиплеты описываются суперполями $\hat{q}^{\,ia}$, $\hat{q}^{\,\underline{i}a}$, $\hat{q}^{\,i\underline{a}}$, $\hat{q}^{\,\underline{i}\underline{a}}$, которые подчиняются соответствующим дифференциальным связям (дублетные индексы $(i,\underline{i},a,\underline{a})$ относятся к четырем группам $SU(2)$, образующим полную группу $SO(4)_L\times SO(4)_R$ R-симметрии $\mathcal{N}=(4,4)$ суперсимметрии). Для мультиплета каждого типа в аналитическом подпространстве $SU(2)\times SU(2)$ гармонического суперпространства построены сигма-модельные действия вне массовой оболочки, включающие члены кручения, а также соответствующие инвариантные выражения для массовых членов. Общее рассмотрение проиллюстрировано на примере $\mathcal{N}=(4,4)$ суперконформного действия $SU(2)\times U(1)$ ВЗНВ сигма-модели и его массивной деформации для мультиплета $\hat{q}^{\,i\underline{a}}$. Показано, что общее сигма-модельное действие для любой пары различных мультиплетов, инвариантное относительно $\mathcal{N}=(4,4)$ суперсимметрии, распадается в сумму сигма-модельных действий для каждого мультиплета. Аналогичное явление имеет место и при одновременном включении в общее действие большего числа неэквивалентных мультиплетов. Показано, что различные мультиплеты могут взаимодействовать только через смешанные массовые члены, которые возможны лишь для мультиплетов, принадлежащих самодуальным парам $(\hat{q}^{\,ia},\hat{q}^{\,\underline{i}\underline{a}})$ и $(\hat{q}^{\,\underline{i}a},\hat{q}^{\,i\underline{a}})$. Мультиплеты из различных пар не взаимодействуют вовсе. Для двух твистованных мультиплетов, принадлежащих самодуальной паре, приведена наиболее общая форма скалярного потенциала на массовой поверхности.

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Ivanov E. A., Sutulin A. O. E2-2004-28
Diversity of Off-Shell Twisted $(4,4)$ Multiplets
in $SU(2)\times SU(2)$ Harmonic Superspace

We elaborate on four different types of twisted $\mathcal{N}=(4,4)$ supermultiplets in the $SU(2)\times SU(2)$, 2D harmonic superspace. In the conventional $\mathcal{N}=(4,4)$, 2D superspace they are described by the superfields $\hat{q}^{\,ia}$, $\hat{q}^{\,\underline{i}a}$, $\hat{q}^{\,i\underline{a}}$, $\hat{q}^{\,\underline{i}\underline{a}}$, subjected to proper differential constraints, $(i,\underline{i},a,\underline{a})$ being the doublet indices of four groups $SU(2)$ which form the full R-symmetry group $SO(4)_L\times SO(4)_R$ of $\mathcal{N}=(4,4)$ supersymmetry. We construct the torsionful off-shell sigma-model actions for each type of these multiplets, as well as the corresponding invariant mass terms, in an analytic subspace of the $SU(2)\times SU(2)$ harmonic superspace. As an instructive example, $\mathcal{N}=(4,4)$ superconformal extension of the $SU(2)\times U(1)$ WZNW sigma-model action and its massive deformation are presented for the multiplet $\hat{q}^{\,i\underline{a}}$. We prove that $\mathcal{N}=(4,4)$ supersymmetry requires the general sigma-model action of pair of different multiplets to split into a sum of sigma-model actions of each multiplet. This phenomenon also persists if a larger number of non-equivalent multiplets are simultaneously included. We show that different multiplets may interact with each other only through mixed mass terms which can be set up for multiplets belonging to «self-dual» pairs $(\hat{q}^{\,ia},\hat{q}^{\,\underline{i}\underline{a}})$ and $(\hat{q}^{\,\underline{i}a},\hat{q}^{\,i\underline{a}})$. The multiplets from different pairs cannot interact at all. For a «self-dual» pair of the twisted multiplets we give the most general form of the on-shell scalar potential.

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1. INTRODUCTION

The interest in $\mathcal{N} = (4,4)$ supersymmetric two-dimensional sigma models with torsion has a long history and is mainly motivated by the important role the corresponding target spaces play in string theory and AdS/CFT correspondence (see, e. g., [1–5] and refs. therein). The first example of such a model, $\mathcal{N} = (4,4)$ supersymmetric (and superconformal) extension of $SU(2) \times U(1)$ WZNW model, was discovered in Ref. [6] as a special case of $\mathcal{N} = (4,4)$ super-Liouville system (see also [7]). A more general class of these sigma models was presented in [8,9]. In [10], $\mathcal{N} = (4,4)$ superextensions of other group-manifold WZNW models were constructed, and the exhaustive list of group-manifolds for which such extensions exist was given (they are those admitting a quaternionic structure). Superfield formulations of $\mathcal{N} = (4,4)$ models were given in $\mathcal{N} = (2,2)$ superspace [8], in ordinary $\mathcal{N} = (4,4)$ superspace [11, 12], in the projective superspace [13, 14] and in the $\mathcal{N} = (4,4)$, $SU(2) \times SU(2)$ bi-harmonic superspace [15–18]. The $\mathcal{N} = (4,4)$ superfield formulations are most appropriate, as they make manifest and off-shell the full amount of the underlying supersymmetry. As argued in [15], the bi-harmonic formulations are especially advantageous because they manifest not only supersymmtery but also internal *R*-symmetry $SU(2)_L \times SU(4)_R$ of $\mathcal{N} = (4,4)$, 2D Poincaré supersymmetry, and allow one to control how this symmetry is broken in various sigma-model actions.

The bosonic target geometry of general $\mathcal{N} = (4,4)$ supersymmetric sigma models of the considered type was studied, e. g., in [8, 19, 20]. One of its versions is characterized by two sets of mutually commuting covariantly constant quaternionic structures. Any sigma model of this sort can be described off-shell by the «twisted» $\mathcal{N} = (4,4)$, 2D supermultiplets with the off-shell content $(\mathbf{4}, \mathbf{8}, \mathbf{4})$. As for $\mathcal{N} = (4,4)$ sigma models with *non-commuting* left and right complex structures, which involve, e. g., most of the group manifold $\mathcal{N} = (4,4)$ WZNW sigma models (with exception of $SU(2) \times U(1)$ and some other product manifolds including $SU(2)$ and $U(1)$ as the factors [10, 19]), not too much is known about their superfield description. What is certainly known is that it is impossible to formulate them in terms of $\mathcal{N} = (4,4)$ twisted multiplets alone [21]. In terms of some other multiplets, models of this kind were discussed in the framework of $\mathcal{N} = (2,2)$ superspace [13, 21, 22] and in the bi-harmonic $\mathcal{N} = (4,4)$ superspace

[17] (in the latter case, superfields with infinitely many auxiliary fields have to be involved, and the Poisson structures on the target space naturally appear).

Yet in the case of $\mathcal{N}=(4,4)$ sigma models based on twisted multiplets there is a subtlety related to the existence of few types of these multiplets which differ in the transformation properties of their component fields with respect to the full *R*-symmetry group $SO(4)_L \times SO(4)_R$ of $\mathcal{N}=(4,4)$, 2D Poincaré superalgebra. This degeneracy of twisted multiplets was first noticed in $[11,23,24]$ *. It is clearly seen just in the $\mathcal{N}=(4,4)$ superfield language where the various twisted multiplets are represented by the properly constrained superfields [24]

$$\hat{q}^{\,i\,a}\,, \quad \hat{q}^{\,i\,\underline{a}}\,, \quad \hat{q}^{\,\underline{i}\,a}\,, \quad \hat{q}^{\,\underline{i}\,\underline{a}}\,. \tag{1.1}$$

Here, the external doublet indices $i, \underline{i}$ and $a, \underline{a}$ refer to two left and two right $SU(2)$ constituents of the *R*-symmetry groups $SO(4)_L$ and $SO(4)_R$, respectively. While looking at these superfields from the perspective of the diagonal subgroup $SU(2)_{\rm diag}$ in the product $SU(2)_L \times SU(2)_R$, with the $SU(2)$ factors being realized on the indices i and a, these four types of twisted multiplets amount, respectively, to a sum of the $SU(2)_{\rm diag}$ singlet and triplet superfields, two complex doublet superfields and a sum of four $SU(2)_{\rm diag}$ singlet superfields. These sets provide an off-shell extension of what was called scalar multiplets SM-II, SM-III, SM-IV and SM-I in [23,25]. Two of these superfields, $\hat{q}^{\,i\,a}$ and $\hat{q}^{\,\underline{i}\,\underline{a}}$, comprise just what was termed TM-II and TM-I twisted multiplets in [12,23]. In [11,12,23] the superfield kinetic actions were written for such multiplets, as well as the invariant mass (potential) terms, and it was observed (see also [7]) that the mixed mass terms can be composed only of the two multiplets «dual» to each other. In our notation, such «self-dual» pairs are formed by the first and fourth, or second and third superfields from the above set. The natural question is as to what is the most general self-interaction of these four different species of twisted multiplets, both in regard the sigma-model type of it (generalizing free kinetic terms) and superpotential type (generalizing the mass terms). In both $\mathcal{N}=(2,2)$ [8] and $\mathcal{N}=(4,4)$ [15] superspace approaches only the general Lagrangians of *one kind* of twisted multiplet were considered and the appropriate restrictions on the relevant bosonic target metric and torsion were deduced.

One of the purposes of the present paper is to answer the question just mentioned, using the bi-harmonic $SU(2)\times SU(2)$ approach of Refs. [15,16,18]. As a prerequisite, we give how these four different twisted multiplets are described within this setting. Only one of them (just the one comprised by the superfield $\hat{q}^{\,i\,a}$ from the above set) is presented by an analytic bi-harmonic superfield, and it is just the multiplet the general $\mathcal{N}=(4,4)$ actions for which were given

*As observed in [23], even further proliferation of non-equivalent twisted multiplets can be achieved by grading their components in different ways under 2D space-time parity.

in [15]. The remaining three multiplets have a more complicated description. We firstly construct the general invariant superfield sigma-model-type actions for these multiplets, including an example of the superconformal action which is the appropriate $\mathcal{N} = (4, 4)$ superextension of $SU(2) \times U(1)$ WZNW model. Then we study the mixed case when multiplets of different kind could interact with each other. We find that $\mathcal{N} = (4, 4)$ supersymmetry requires the corresponding actions to split into a sum of actions for separate multiplets, and this phenomenon is one of the basic findings of our paper. Another one concerns the structure of admissible superpotential terms. We find that such terms can be constructed for each separate multiplet and/or for a pair of multiplets «dual» to each other. These additional terms are defined in a unique way, and the form of the corresponding component potential is uniquely fixed by the bosonic target metric, like in the cases considered in [11, 15]. Once again, $\mathcal{N} = (4, 4)$ supersymmetry forbids possible superpotential terms composed of the multiplets belonging to different «self-dual» pairs. Thus these pairs cannot «talk» to each other at all.

The paper is organized as follows. We start in Sec. 2 with recollecting the basic facts about the $SU(2) \times SU(2)$ HSS and off-shell description of the twisted analytic $q^{1,1}$ multiplet in its framework (corresponding to the superfield $\hat{q}^{\,i\,a}$ from the set (1.1)). We also recall the realization of $\mathcal{N} = 4$, 2D superconformal groups in the analytic subspace of this HHS. In Sec. 3 we give the description of the remaining three twisted multiplets in $SU(2) \times SU(2)$ HSS and show that in the analytic subspace they are presented by some analytic superfunctions having nontrivial transformation properties under the supersymmetry. Due to the latter circumstance, the supersymmetric actions of these multiplets in the analytic subspace are written through the Lagrangians subjected to some differential constraints required by supersymmetry, and these actions are invariant up to a shift of the Lagrangians by a total derivative. The relevant component actions are shown to be completely specified by the metric on the physical bosons manifold. They reveal the same target geometry as in the $q^{1,1}$ case [15]. In Sec. 4, on the example of the multiplet $\hat{q}^{\,i\,\underline{a}}$, we show that the requirement of invariance under one of the four «small» $SU(2)$ superconformal groups which can be defined in the analytic subspace uniquely specifies the relevant action to be that of $\mathcal{N} = (4, 4)$ extension of the group manifold $SU(2) \times U(1)$ WZNW sigma model. In Sec. 5 we construct massive extensions of the superfield sigma-model actions for two separate multiplets and show that these extra terms are uniquely fixed by supersymmetry, like in the $q^{1,1}$ case [15]. Section 6 is devoted to possible mixed interactions of different multiplets. A careful analysis shows that the general sigma-model actions always split into a sum of actions for separate multiplets, while the crossing-interaction through the mass terms is possible only for twisted multiplets belonging to the same «self-dual» pair. We present the most general form of the component potential term for such a pair arising as a result of elimination of the auxiliary fields in the full action. This potential gets contributions from the three sources: mass terms for each separate multiplet and the mixed mass term.

2. $SU(2)\times SU(2)$ HARMONIC SUPERSPACE

We begin by recalling basics of $\mathcal{N}=(4,4)$, 2D supersymmetry. The standard real $\mathcal{N}=(4,4)$, 2D superspace is parametrized by the following set of the light-cone coordinates

$$\mathbf{R}^{(1,1|4,4)} = (\,Z\,) = (\,z^{++}\,, z^{--}\,, \theta^{+i\underline{k}}\,, \theta^{-a\underline{b}}\,)\,.$$

Here $+,-$ are the light-cone indices and i, $\underline{k}$, a, $\underline{b}$ are the doublet indices of four commuting $SU(2)$ groups which constitute the full automorphism group $SO(4)_L\times SO(4)_R$ of $\mathcal{N}=(4,4)$, 2D Poincaré superalgebra. The corresponding covariant spinor derivatives obey the following algebra

$$\{\,D_{i\underline{k}}\,, D_{j\underline{l}}\,\} = 2i\,\varepsilon_{i\,j}\,\varepsilon_{\underline{k}\,\underline{l}}\,\partial_{++}\,, \quad \{\,D_{a\underline{b}}\,, D_{c\underline{d}}\,\} = 2i\,\varepsilon_{a\,c}\,\varepsilon_{\underline{b}\,\underline{d}}\,\partial_{--} \tag{2.1}$$

where

$$D_{i\underline{k}} = \frac{\partial}{\partial\theta^{i\underline{k}}} + i\theta_{i\underline{k}}\,\partial_{++}\,, \qquad D_{a\underline{b}} = \frac{\partial}{\partial\theta^{a\underline{b}}} + i\theta_{a\underline{b}}\,\partial_{--} \tag{2.2}$$

(hereafter, we omit the light-cone indices of the Grassmann coordinates, keeping in mind the rule that the doublet indices $i,\underline{k}$ refer to the left sector, while $a,\underline{b}$ to the right one). Here we use the quartet notation for spinor derivatives and Grassmann coordinates. Its relation to the complex notation of Ref. [15] is as follows

$$\theta^{i\underline{k}} \equiv (\theta^i, \bar\theta^i)\,, \quad D_{i\underline{k}} \equiv (D_i\,, \bar D_i)\,, \quad \theta^{a\underline{b}} \equiv (\theta^a, \bar\theta^a)\,, \quad D_{a\underline{b}} \equiv (D_a\,, \bar D_a)\,. \tag{2.3}$$

The complex conjugation rules are

$$(\theta^{i\underline{k}})^\dagger = \varepsilon_{i\,l}\,\varepsilon_{\underline{k}\,\underline{n}}\,\theta^{l\underline{n}}\,, \quad (D_{i\underline{k}})^\dagger = -\varepsilon^{i\,l}\,\varepsilon^{\underline{k}\,\underline{n}}\,D_{l\underline{n}} \tag{2.4}$$

(and the same for the objects from the right sector).

The $\mathcal{N}=(4,4)$ $SU(2)\times SU(2)$ harmonic superspace (HSS) introduced in [15] is an extension of the real 2D superspace defined above by two independent sets of harmonic variables $u^{\pm1}_i$ and $v^{\pm1}_a$ associated with one of the $SU(2)$ factors of the $SO(4)_L$ and $SO(4)_R$ automorphism groups of the left and right sectors of $\mathcal{N}=(4,4)$ supersymmetry, respectively (we denote them by $SU(2)_L$ and $SU(2)_R$, this choice of $SU(2)$ subgroups is optional). The $SU(2)\times SU(2)$ HSS formalism enables one to keep both these $SU(2)$ symmetries manifest at each step and to control their breakdown. We define the central basis of this HSS as

$$\mathbf{HR}^{(1+2,1+2|4,4)} = (Z,u,v) = \mathbf{R}^{(1,1|4,4)} \otimes (u^{\pm 1}_i, v^{\pm 1}_a),\ u^{1i}u^{-1}_i = 1,\ v^{1a}v^{-1}_a = 1. \tag{2.5}$$

The analytic basis in the same $\mathcal{N}=(4,4)$ $SU(2)\times SU(2)$ HSS amounts to the following choice of coordinates

$$\mathbf{HR}^{(1+2,1+2|4,4)} = (X,u,v) = (\,x^{++}\,, x^{--}\,, \theta^{\pm 1,0\,\underline{i}}\,, \theta^{0,\pm 1\,\underline{a}}\,, u^{\pm 1}_i\,, v^{\pm 1}_a\,), \tag{2.6}$$

where

$$\theta^{\pm 1,0\,\underline{i}} = \theta^{k\underline{i}}\,u^{\pm 1}_k\,, \quad \theta^{0,\pm 1\,\underline{a}} = \theta^{b\underline{a}}\,v^{\pm 1}_b\,.$$

The precise relation between $x^{\pm\pm}$ and $z^{\pm\pm}$ can be found in [15]. The main feature of the analytic basis is that it visualizes the existence of the *analytic subspace* in the $SU(2)\times SU(2)$ HSS:

$$\mathbf{AR}^{(1+2,1+2|2,2)} = (\zeta,u,v) = (\,x^{++}\,, x^{--}\,, \theta^{1,0\,\underline{i}}\,, \theta^{0,1\,\underline{a}}\,, u^{\pm 1}_i\,, v^{\pm 1}_a\,)\,, \tag{2.7}$$

which is closed under the $\mathcal{N}=(4,4)$ supersymmetry transformations. The existence of the analytic subspace matches with the form of covariant spinor derivatives in the analytic basis

$$D^{1,0\,\underline{i}} = -\frac{\partial}{\partial\theta^{-1,0}_{\underline{i}}}\,, \quad D^{0,1\,\underline{a}} = -\frac{\partial}{\partial\theta^{0,-1}_{\underline{a}}}\,, \tag{2.8}$$

where

$$D^{\pm 1,0\,\underline{i}} \equiv D^{k\underline{i}}\,u^{\pm 1}_k\,, \quad D^{0,\pm 1\,\underline{a}} \equiv D^{b\underline{a}}\,v^{\pm 1}_b\,. \tag{2.9}$$

The «shortness» of $D^{1,0\,\underline{i}}\,, D^{0,1\,\underline{a}}$ means that the Grassmann-analytic bi-harmonic superfields $\Phi^{\,q,\,p}$,

$$D^{1,0\,\underline{i}}\,\Phi^{\,q,\,p} = D^{0,1\,\underline{a}}\,\Phi^{\,q,\,p} = 0\,, \tag{2.10}$$

do not depend on $\theta^{-1,0\,\underline{i}}\,, \theta^{0,-1\,\underline{a}}$ in the analytic basis, i.e. are defined on the analytic superspace (2.7)

$$\Phi^{\,q,\,p} = \Phi^{\,q,\,p}(\zeta,u,v)\,. \tag{2.11}$$

The pair of superscripts 'q,p' on $\Phi^{\,q,\,p}$ in (2.10), as well as analogous superscripts on other quantities, stands for the values of two independent harmonic $U(1)$ charges which, as in the case of $SU(2)$ HSS [26, 27], are assumed to be strictly preserved. As a consequence of this requirement, all superfields (or superfunctions), defined on (2.7), i.e. the $SU(2)\times SU(2)$ *analytic* $\mathcal{N}=(4,4)$ superfields (or superfunctions), are assumed to admit expansions in the double harmonic series on the product of two 2-spheres $SU(2)_L/U(1)_L \times SU(2)_R/U(1)_R\,$.

The extra doublet indices $\underline{i}$, $\underline{a}$ of Grassmann coordinates in (2.7) refer to two additional $SU(2)$ automorphism groups of $\mathcal{N}=(4,4)$, 2D Poincaré supersymmetry which, together with $SU(2)_L$ and $SU(2)_R$, constitute the full automorphism group $SO(4)_L \times SO(4)_R$ of the latter. We prefer not to «harmonize» these additional $SU(2)$ groups in order to avoid unwanted complications in the notation.*

In the bi-harmonic superspace one can define two sets of mutually commuting harmonic derivatives, the left and right ones, each forming an $SU(2)$ algebra [15]. Here we will need to know the explicit expressions only for the derivatives with positive $U(1)$ charges which commute with $D^{1,0\,\underline{i}}, D^{0,1\,\underline{a}}$ and so preserve the harmonic analyticity. In the analytic basis, these derivatives read

$$\bigtriangledown^{2,0} = D^{2,0} + \theta^{1,0\,\underline{i}}\frac{\partial}{\partial\theta^{-1,0\,\underline{i}}}\,, \quad \bigtriangledown^{0,2} = D^{0,2} + \theta^{0,1\,\underline{a}}\frac{\partial}{\partial\theta^{0,-1\,\underline{a}}}, \tag{2.12}$$

where

$$D^{2,0} = \partial^{2,0} + i\theta^{1,0\,\underline{i}}\theta^{1,0}_{\underline{i}}\partial_{++}\,, \quad D^{0,2} = \partial^{0,2} + i\theta^{0,1\,\underline{a}}\theta^{0,1}_{\underline{a}}\partial_{--} \tag{2.13}$$

and

$$\partial^{2,0} = u^{1i}\frac{\partial}{\partial u^{-1i}}\,, \quad \partial^{0,2} = v^{1a}\frac{\partial}{\partial v^{-1a}}\,.$$

When acting on the analytic superfields, $\bigtriangledown^{2,0}$ and $\bigtriangledown^{0,2}$ are reduced to $D^{2,0}$ and $D^{0,2}$.

The main advantage of using the $SU(2)\times SU(2)$ HSS (as compared, e. g., with the standard $SU(2)$, 2D HSS obtained as a dimensional reduction of $\mathcal{N}=2$, 4D HSS [26,27]) consists in the fact that it provides a natural superfield description for the important class of $\mathcal{N}=(4,4)$ supersymmetric sigma models, those with torsion on the bosonic manifold, such that the whole amount of the underlying $\mathcal{N}=(4,4)$ supersymmetry is manifest and off-shell.

Here we recall the HSS off-shell formulation of the first type of $\mathcal{N}=(4,4)$ twisted multiplets from the set (1.1), postponing the HSS treatment of the remaining three types to the next Section. This multiplet was used in [15] to construct the $\mathcal{N}=(4,4)$ superspace version of the general $\mathcal{N}=(2,2)$ superspace action of one sort of $\mathcal{N}=(4,4)$ twisted multiplet [8]. In HSS this multiplet is described by a real analytic $\mathcal{N}=(4,4)$ superfield $q^{1,1}(\zeta,u,v)$ subjected to the harmonic constraints

$$D^{2,0}q^{1,1} = 0\,, \quad D^{0,2}q^{1,1} = 0\,. \tag{2.14}$$

These constraints leave $(8+8)$ independent components in $q^{1,1}$ [15], which is just the irreducible *off-shell* component content of $\mathcal{N}=(4,4)$ twisted multiplet. In the central basis the constraints (2.14) and the analyticity conditions imply

$$q^{1,1} = \hat{q}^{\,i\,a}u^1_i v^1_a\,, \quad D^{(k\underline{k}}\hat{q}^{\,i)\,a} = D^{(b\underline{b}}\hat{q}^{\,k\,a)} = 0\,, \tag{2.15}$$

*$\mathcal{N}=(4,4)$, 2D HSS with three sets of $SU(2)$ harmonics was considered in [28].

and we end up with the first type of twisted multiplet from the set (1.1) (the form of constraints as in (2.15) was exhibited for the first time in [6]). The analytic basis solution of the harmonic constraints (2.14) is given by

$$
\begin{aligned}
q^{1,1} &= q^{i\,a}u^1_i v^1_a \\
&+ \theta^{1,0\,\underline{i}}\varphi^a_{\underline{i}} v^1_a + \theta^{0,1\,\underline{a}}\eta^i_{\underline{a}} u^1_i - i(\theta^{1,0})^2\partial_{++}q^{i\,a}u^{-1}_i v^1_a - i(\theta^{0,1})^2\partial_{--}q^{i\,a}u^1_i v^{-1}_a \\
&+ \theta^{1,0\,\underline{i}}\theta^{0,1\,\underline{a}}F_{\underline{i}\,\underline{a}} - i\theta^{1,0\,\underline{i}}(\theta^{0,1})^2\partial_{--}\varphi^a_{\underline{i}} v^{-1}_a - i(\theta^{1,0})^2\theta^{0,1\,\underline{a}}\partial_{++}\eta^i_{\underline{a}} u^{-1}_i \\
&- (\theta^{1,0})^2(\theta^{0,1})^2\partial_{++}\partial_{--}q^{i\,a}u^{-1}_i v^{-1}_a\,,
\end{aligned} \tag{2.16}
$$

where $(\theta^{1,0})^2 = \theta^{1,0\,\underline{k}}\,\theta^{1,0}_{\underline{k}}$, $(\theta^{0,1})^2 = \theta^{0,1\,\underline{a}}\,\theta^{0,1}_{\underline{a}}$.

The general *off-shell* action of n such superfields $q^{1,1\,M}$ $(M = 1, 2, ...n)$ can be written as the following integral over the analytic superspace (2.7)

$$
S^{\rm gen} = \int \mu^{-2,-2}\,\mathcal{L}^{2,2}(q^{1,1\,M}, u, v), \tag{2.17}
$$

where

$$
\mu^{-2,-2} = d^2x\,d^2\theta^{1,0}\,d^2\theta^{0,1}\,du\,dv \tag{2.18}
$$

is the analytic superspace integration measure (see Appendix for its precise definition). In general, the dimensionless analytic superfield Lagrangian $\mathcal{L}^{2,2}$ bears an arbitrary dependence on its arguments, the only restriction being a compatibility with its external $U(1)$ charges $(2, 2)$. The free action is given by

$$
S^{\rm free} \sim \int \mu^{-2,-2}\,q^{1,1\,M}q^{1,1\,M}\,, \tag{2.19}
$$

so for consistency we are led to assume

$$
\det\left(\frac{\partial^2\mathcal{L}^{2,2}}{\partial q^{1,1\,M}\,\partial q^{1,1\,N}}\right)\Bigg|_{q^{1,1}=0} \neq 0\,.
$$

The passing to the component form of the action (2.17) is straightforward [15]. The relevant bosonic sigma-model action consists of two parts related to each other by $\mathcal{N} = (4, 4)$ supersymmetry: the metric part and the part which includes the torsion potential. These terms obey the same constraints as in the $\mathcal{N} = (2, 2)$ superspace description of general torsionful sigma models associated with twisted $\mathcal{N} = (4, 4)$ multiplets of one type [8]. So (2.17) provides a manifestly $\mathcal{N} = (4, 4)$ supersymmetric form of the general action of such multiplets.

As the last topic of this Section we remind some details of the $\mathcal{N} = 4$, 2D superconformal groups. As discussed in [15], in the $SU(2) \times SU(2)$ analytic HSS one can realize two different infinite-parameter «small» $\mathcal{N} = 4$, $SU(2)$ superconformal groups (in each light-cone sector), having as their closure the «large» $\mathcal{N}=4$, $SO(4)\times U(1)$ superconformal group. One of these $\mathcal{N}=4$, $SU(2)$ superconformal groups acts on all coordinates of the analytic HSS, including the harmonic coordinates [15, 29]

$$\delta x^{++}=\tilde{\Lambda}_{(I)L},\ \delta u^1_i=\Lambda^{2,0}u^{-1}_i,\ \delta u^{-1}_i=0,\ \delta\theta^{1,0\,\underline{i}}=\Lambda^{1,0\,\underline{i}}_{(I)},\ \delta D^{2,0}=-\Lambda^{2,0}D^{0,0}_u, \tag{2.20}$$

$$\delta x^{--}=\tilde{\Lambda}_{(I)R},\ \delta v^1_a=\Lambda^{0,2}v^{-1}_a,\ \delta v^{-1}_a=0,\ \delta\theta^{0,1\,\underline{a}}=\Lambda^{0,1\,\underline{a}}_{(I)},\ \delta D^{0,2}=-\Lambda^{0,2}D^{0,0}_v. \tag{2.21}$$

Here $D^{0,0}_u$ and $D^{0,0}_v$ are the left and right $U(1)$ charge-counting operators and

$$\tilde{\Lambda}_{(I)L}=a_L-\frac{1}{2}\,\partial^{-2,0}D^{2,0}a_L\,,\ \ \Lambda^{2,0}=D^{2,0}\Lambda_{(I)L}\,,\ \ \Lambda_{(I)L}=-\frac{1}{2}\,\partial_{++}a_L\,,$$
$$\Lambda^{1,0\,\underline{i}}_{(I)}=-\varepsilon^{\underline{i}\,\underline{k}}\,\frac{i}{4}\,\frac{\partial}{\partial\theta^{1,0\,\underline{k}}}\,D^{2,0}a_L\,,\ \ (D^{2,0})^2\,a_L=0\,, \tag{2.22}$$

$$\tilde{\Lambda}_{(I)R}=a_R-\frac{1}{2}\,\partial^{0,-2}D^{0,2}a_R\,,\ \ \Lambda^{0,2}=D^{0,2}\Lambda_{(I)R}\,,\ \ \Lambda_{(I)R}=-\frac{1}{2}\,\partial_{--}a_R\,,$$
$$\Lambda^{0,1\,\underline{a}}_{(I)}=-\varepsilon^{\underline{a}\,\underline{b}}\,\frac{i}{4}\,\frac{\partial}{\partial\theta^{0,1\,\underline{b}}}\,D^{0,2}a_R\,,\ \ (D^{0,2})^2\,a_R=0\,. \tag{2.23}$$

The superparameter functions a_L and a_R depend only on the left and right light-cone coordinates, respectively, i.e., $a_L=a_L(\zeta_L,u)$ and $a_R=a_R(\zeta_R,v)$, where $\zeta_L=(\,x^{++}\,,\theta^{1,0\,\underline{i}}\,)$, $\zeta_R=(\,x^{--}\,,\theta^{0,1\,\underline{a}}\,)$. The explicit form of these functions can be found in [15]. In what follows we shall need the identities

$$\Lambda^{1,0\,\underline{i}}_{(I)}=D^{2,0}\Lambda^{-1,0\,\underline{i}}_{(I)}-\theta^{1,0\,\underline{i}}\Lambda_{(I)L}\,,\quad \Lambda^{0,1\,\underline{a}}_{(I)}=D^{0,2}\Lambda^{0,-1\,\underline{a}}_{(I)}-\theta^{0,1\,\underline{a}}\Lambda_{(I)R}, \tag{2.24}$$

where

$$\Lambda^{-1,0\,\underline{i}}_{(I)}=-\varepsilon^{\underline{i}\,\underline{k}}\,\frac{i}{4}\,\frac{\partial a_L}{\partial\theta^{1,0\,\underline{k}}}\,,\quad \Lambda^{0,-1\,\underline{a}}_{(I)}=-\varepsilon^{\underline{a}\,\underline{b}}\,\frac{i}{4}\,\frac{\partial a_R}{\partial\theta^{0,1\,\underline{b}}}\,. \tag{2.25}$$

They can be proved using (2.22).

Another $\mathcal{N}=4$, $SU(2)$ superconformal group (also consisting of two mutually commuting left and right components) does not affect harmonic variables

$$\delta x^{++}=\Lambda_{(II)L}\,,\quad \delta\theta^{1,0\,\underline{i}}=\Lambda^{1,0\,\underline{i}}_{(II)}\,,\quad \delta u^{\pm1}_i=0\,,$$
$$\delta x^{--}=\Lambda_{(II)R}\,,\quad \delta\theta^{0,1\,\underline{a}}=\Lambda^{0,1\,\underline{a}}_{(II)}\,,\quad \delta v^{\pm1}_a=0, \tag{2.26}$$

and is fully specified by requiring $D^{2,0}$, $D^{0,2}$ to be invariant

$$\delta D^{2,0}=0\,,\qquad \delta D^{0,2}=0\,. \tag{2.27}$$

The latter equations imply

$$D^{2,0}\Lambda^{1,0\,\underline{i}}_{(II)}=0\,,\quad D^{2,0}\Lambda_{(II)L}=2i\,\theta^{1,0}_{\underline{i}}\Lambda^{1,0\,\underline{i}}_{(II)}\,,$$
$$D^{0,2}\Lambda^{0,1\,\underline{a}}_{(II)}=0\,,\quad D^{0,2}\Lambda_{(II)R}=2i\,\theta^{0,1}_{\underline{a}}\Lambda^{0,1\,\underline{a}}_{(II)}\,. \tag{2.28}$$

The general solution to Eqs. (2.28) is provided by

$$
\begin{aligned}
&\Lambda^{1,0\,\underline{i}}_{(II)} = \lambda^{k\,\underline{i}} u^1_k + \theta^{1,0\,\underline{k}}\,(\lambda^{\underline{i})}_{(\underline{k}} - \frac{1}{2}\,\delta^{\underline{i}}_{\underline{k}}\,\partial_{++}\lambda_L) - i\,(\theta^{1,0})^2\partial_{++}\lambda^{k\,\underline{i}}u^{-1}_k\,,\\
&\Lambda_{(II)L} = \lambda_L + 2i\,\theta^{1,0}_{\underline{i}}\lambda^{k\,\underline{i}}u^{-1}_k\,,\\
&\Lambda^{0,1\,\underline{a}}_{(II)} = \lambda^{b\,\underline{a}} v^1_b + \theta^{0,1\,\underline{b}}\,(\lambda^{\underline{a})}_{(\underline{b}} - \frac{1}{2}\,\delta^{\underline{a}}_{\underline{b}}\,\partial_{--}\lambda_R) - i\,(\theta^{0,1})^2\partial_{--}\lambda^{b\,\underline{a}}v^{-1}_b\,,\\
&\Lambda_{(II)R} = \lambda_R + 2i\,\theta^{0,1}_{\underline{a}}\lambda^{b\,\underline{a}}v^{-1}_b\,.
\end{aligned}
\tag{2.29}
$$

We also quote the identities to be used in what follows:

$$
D^{2,0}\Lambda^{-1,0\,\underline{i}}_{(II)} = \Lambda^{1,0\,\underline{i}}_{(II)} - \theta^{1,0\,\underline{k}}\,\frac{\partial\Lambda^{1,0\,\underline{i}}_{(II)}}{\theta^{1,0\,\underline{k}}}\,, \quad D^{0,2}\Lambda^{0,-1\,\underline{a}}_{(II)} = \Lambda^{0,1\,\underline{a}}_{(II)} - \theta^{0,1\,\underline{b}},\frac{\partial\Lambda^{0,1\,\underline{a}}_{(II)}}{\theta^{0,1\,\underline{b}}}, \tag{2.30}
$$

where

$$
\Lambda^{-1,0\,\underline{i}}_{(II)} \equiv \lambda^{k\,\underline{i}}u^{-1}_k\,, \quad \Lambda^{0,-1\,\underline{a}}_{(II)} \equiv \lambda^{b\,\underline{a}}v^{-1}_b\,. \tag{2.31}
$$

They can be proved using the explicit expressions (2.29).

The analytic superfield $q^{1,1}$ defined by Eqs. (2.14) behaves as a scalar of conformal weight zero under the action of the $\mathcal{N}=4$, $SU(2)$ superconformal group II, but possesses nontrivial transformation properties under the $\mathcal{N}=4$, $SU(2)$ superconformal group I [15]:

$$
\delta_{II}\,q^{1,1} = 0\,, \quad \delta_I\,q^{1,1} = \Lambda_{(I)L}\,q^{1,1} \tag{2.32}
$$

(the transformation rules with respect to the right branches of these superconformal groups are the same up to the change $L \to R$).

In the realization on $q^{1,1}$, the basic difference between the superconformal groups I and II manifests itself in the action of their $SU(2)$ subgroups. The left and right $SU(2)$'s belonging to the superconformal group I act on the indices i and a and so possess a nontrivial action on both the physical bosons $q^{i\,a} = \hat{q}^{\,i\,a}|$ and the fermions; at the same time, $SU(2)$ subgroups from the superconformal group II act on the indices $\underline{i}, \underline{a}$ and so affect only fermions and auxiliary fields in $q^{1,1}$. Note that the above pairing of two left and two right $\mathcal{N}=4$, $SU(2)$ superconformal groups into the $\mathcal{N}=(4,4)$ superconformal groups I and II is optional: one could alternatively assemble one of such $\mathcal{N}=(4,4)$ supergroups as a direct product of the left branch of the superconformal group I and the right branch of the superconformal group II, and the second $\mathcal{N}=(4,4)$ superconformal group as the product of the right branch of I and the left branch of II. These different possibilities of composing $\mathcal{N}=(4,4)$ superconformal groups out of the mutually commuting left and right pairs of $\mathcal{N}=4$, $SU(2)$ superconformal groups are directly related to the existence of four different sorts of twisted multiplets, as given in (1.1) and explained in more details in next Sections. For every of these multiplets one can single out the proper $\mathcal{N}=(4,4)$ superconformal groups which act on them precisely in the same way as the superconformal groups I and II defined above act on the multiplet $\hat{q}^{\,i\,a} \sim q^{1,1}$.

Finally, we wish to mention that the analytic superspace integration measure (2.18) is invariant with respect to both superconformal groups [15].

3. NEW TYPES OF TWISTED MULTIPLETS IN $SU(2)\times SU(2)$ HSS

3.1. Constraints in the General and Analytic Superspaces. As mentioned in Introduction, in the standard real $\mathcal{N}=(4,4)$, 2D superspace we can define four types of twisted multiplets in accord with the four possibilities to pair the doublet indices of various $SU(2)$ factors of the left and right *R*-symmetry groups $SO(4)_L$ and $SO(4)_R$. The first type is $\hat{q}^{\,i\,a}$ studied in [15]. Its HSS description was reminded in the previous Section. The remaining three ones are

$$\text{(a)}\;\; \hat{q}^{\,i\,\underline{a}}\,, \quad \text{(b)}\;\; \hat{q}^{\,\underline{i}\,a}\,, \quad \text{(c)}\;\; \hat{q}^{\,\underline{i}\,\underline{a}}\,, \tag{3.1}$$

$$(\hat{q}^{\,i\,\underline{a}})^\dagger = \varepsilon_{i\,k}\,\varepsilon_{\underline{a}\,\underline{b}}\,\hat{q}^{\,k\,\underline{b}}\,, \quad (\hat{q}^{\,\underline{i}\,a})^\dagger = \varepsilon_{\underline{i}\,\underline{k}}\,\varepsilon_{a\,b}\,\hat{q}^{\,\underline{k}\,b}\,, \quad (\hat{q}^{\,\underline{i}\,\underline{a}})^\dagger = \varepsilon_{\underline{i}\,\underline{k}}\,\varepsilon_{\underline{a}\,\underline{b}}\,\hat{q}^{\,\underline{k}\,\underline{b}}\,.$$

The real quartet superfields in (3.1) are subjected to the irreducibility conditions which are quite similar to those defining $\hat{q}^{\,i\,a}$ in the central basis (see Eq. (2.15))

$$\begin{aligned} &\text{(a)}\;\; D^{(k\underline{k}}\hat{q}^{\,i)\,\underline{a}} = D^{b(\underline{b}}\hat{q}^{\,i\,\underline{a})} = 0\,, \quad \text{(b)}\;\; D^{k(\underline{k}}\hat{q}^{\,\underline{i})\,a} = D^{(b\underline{b}}\hat{q}^{\,\underline{i}\,a)} = 0\,, \\ &\text{(c)}\;\; D^{k(\underline{k}}\hat{q}^{\,\underline{i})\,\underline{a}} = D^{b(\underline{b}}\hat{q}^{\,\underline{i}\,\underline{a})} = 0\,. \end{aligned} \tag{3.2}$$

Clearly, these constraints like (2.15) imply that all superfields (3.1) carry out the same off-shell content $(8+8)$, though with a different assignment of the component fields with respect to four $SU(2)$ groups which form $SO(4)_L \times SO(4)_R$. Converting the $SU(2)$ indices of the superfields $\hat{q}^{\,i\,\underline{a}}$, $\hat{q}^{\,\underline{i}\,a}$, $\hat{q}^{\,\underline{i}\,\underline{a}}$ and spinor derivatives in (3.2) with the harmonics u^1_i, v^1_a, we can rewrite these constriants in the analytic basis (2.6) of the $SU(2)\times SU(2)$ HSS as

$$\begin{aligned} &\text{(a)}\;\; D^{1,0\,\underline{k}}\,\hat{q}^{\,1,0\,\underline{a}} = D^{0,1\,(\underline{b}}\,\hat{q}^{\,1,0\,\underline{a})} = 0\,, \quad \text{(b)}\; D^{1,0\,(\underline{k}}\,\hat{q}^{\,0,1\,\underline{i})} = D^{0,1\,\underline{b}}\,\hat{q}^{\,0,1\,\underline{i}} = 0\,, \\ &\text{(c)}\;\; D^{1,0\,(\underline{k}}\,\hat{q}^{\,\underline{i})\,\underline{a}} = D^{0,1\,(\underline{b}}\,\hat{q}^{\,\underline{i}\,\underline{a})} = 0, \end{aligned} \tag{3.3}$$

where, in the central basis,

$$\hat{q}^{\,1,0\,\underline{a}} = \hat{q}^{\,i\,\underline{a}} u^1_i\,, \quad \hat{q}^{\,0,1\,\underline{i}} = \hat{q}^{\,\underline{i}\,a} v^1_a\,. \tag{3.4}$$

Using the analytic basis form of $D^{1,0\,\underline{i}}$, $D^{0,1\,\underline{a}}$, Eq. (2.8), and expanding the superfields in the non-analytic odd coordinates $\theta^{-1,0\,\underline{i}}$, $\theta^{0,-1\,\underline{a}}$, one can solve

(3.3) in the analytic basis as

$$(a)\ \hat{q}^{\,1,0\,\underline{a}}(X,u,v) = q^{1,0\,\underline{a}} + \theta^{0,-1\,\underline{a}}\, g^{1,1},$$
$$(b)\ \hat{q}^{\,0,1\,\underline{i}}(X,u,v) = q^{0,1\,\underline{i}} + \theta^{-1,0\,\underline{i}}\, f^{1,1}, \tag{3.5}$$
$$(c)\ \hat{q}^{\,\underline{i}\,\underline{a}}(X,u,v) = \tilde{q}^{\,\underline{i}\,\underline{a}} + \theta^{-1,0\,\underline{i}}\, f^{1,0\,\underline{a}} + \theta^{0,-1\,\underline{a}}\, h^{0,1\,\underline{i}} + \theta^{-1,0\,\underline{i}}\,\theta^{0,-1\,\underline{a}}\, t^{1,1}, \tag{3.6}$$

where all the coefficients depend only on the analytic coordinates (ζ, u, v).

Thus the harmonic superfields $\hat{q}^{\,1,0\,\underline{a}}$, $\hat{q}^{\,0,1\,\underline{i}}$, $\hat{q}^{\,\underline{i}\,\underline{a}}$ bear the explicit dependence on the non-analytic Grassmann coordinates with the negative $U(1)$ charge and so are not harmonic-analytic, as opposed to the superfield $q^{1,1}$. On the other hand, all the components in their expansions over the non-analytic coordinates are defined on the analytic subspace of the $SU(2)\times SU(2)$ HSS. The basic difference between them and the analytic superfield $q^{1,1}$ consists in their supersymmetry transformation properties. Keeping in mind that $\hat{q}^{\,1,0\,\underline{a}}$, $\hat{q}^{\,0,1\,\underline{i}}$, $\hat{q}^{\,\underline{i}\,\underline{a}}$ are the general harmonic superfields (here δ is the variation under supersymmetry)

$$\delta\hat{q}^{\,1,0\,\underline{a}} = \delta\hat{q}^{\,0,1\,\underline{i}} = \delta\hat{q}^{\,\underline{i}\,\underline{a}} = 0\,, \tag{3.7}$$

and the supertranslations of θ's are $\delta\theta^{-1,0\,\underline{i}} = \varepsilon^{-1,0\,\underline{i}}$, $\delta\theta^{0,-1\,\underline{a}} = \varepsilon^{0,-1\,\underline{a}}$, it is easy to find how these analytic components are transformed

$$(a)\ \delta q^{1,0\,\underline{a}} = -\varepsilon^{0,-1\,\underline{a}}\, g^{1,1}, \delta g^{1,1} = 0,\ (b)\ \delta q^{0,1\,\underline{i}} = -\varepsilon^{-1,0\,\underline{i}}\, f^{1,1},\ \delta f^{1,1} = 0, \tag{3.8}$$

$$(c)\ \delta\tilde{q}^{\,\underline{i}\,\underline{a}} = -\varepsilon^{-1,0\,\underline{i}}\, f^{1,0\,\underline{a}} - \varepsilon^{0,-1\,\underline{a}}\, h^{0,1\,\underline{i}},\ \delta f^{1,0\,\underline{a}} = -\varepsilon^{0,-1\,\underline{a}}\, t^{1,1},$$
$$\delta h^{0,1\,\underline{i}} = -\varepsilon^{-1,0\,\underline{i}}\, t^{1,1},\ \delta t^{1,1} = 0. \tag{3.9}$$

Looking at (3.8), (3.9), one observes that the highest components of $\hat{q}^{\,1,0\,\underline{a}}$, $\hat{q}^{\,0,1\,\underline{i}}$, $\hat{q}^{\,\underline{i}\,\underline{a}}$ are the genuine analytic superfields, while $q^{1,0\,\underline{a}}$, $q^{0,1\,\underline{i}}$ and $\tilde{q}^{\,\underline{i}\,\underline{a}}$, despite being functions of (ζ, u, v), are not analytic superfields in the rigorous sense since they possess non-standard transformation properties under supersymmetry. Note that $q^{1,0\,\underline{a}}$ and $q^{0,1\,\underline{i}}$ are still superfields of the left and right light-cone Poincaré supersymmteries, respectively, while $\tilde{q}^{\,\underline{i}\,\underline{a}}$ possesses non-standard transformation properties under both supersymmetries.

Let us now re-express the central-basis property that the superfields $\hat{q}^{\,1,0\,\underline{a}}$, $\hat{q}^{\,0,1\,\underline{i}}$ and $\hat{q}^{\,\underline{i}\,\underline{a}}$ have the constrained dependence on the harmonics ($\hat{q}^{\,1,0\,\underline{a}}$ and $\hat{q}^{\,0,1\,\underline{i}}$ are linear in harmonics, while $\hat{q}^{\,\underline{i}\,\underline{a}}$ does not depend on them at all) as the following equivalent harmonic constraints in the analytic basis

$$\bigtriangledown^{2,0}\hat{q}^{\,1,0\,\underline{a}} = \bigtriangledown^{0,2}\hat{q}^{\,1,0\,\underline{a}} = \bigtriangledown^{2,0}\hat{q}^{\,0,1\,\underline{i}} = \bigtriangledown^{0,2}\hat{q}^{\,0,1\,\underline{i}} = \bigtriangledown^{2,0}\hat{q}^{\,\underline{i}\,\underline{a}} = \bigtriangledown^{0,2}\hat{q}^{\,\underline{i}\,\underline{a}} = 0\,. \tag{3.10}$$

Here, $\bigtriangledown^{2,0}$ and $\bigtriangledown^{0,2}$ are the full analytic-basis harmonic derivatives defined in Eqs. (2.12), (2.13). Substituting the expansions (3.5), (3.6) for $\hat{q}^{\,1,0\,\underline{a}}$, $\hat{q}^{\,0,1\,\underline{i}}$, $\hat{q}^{\,\underline{i}\,\underline{a}}$ into (3.10), we can rewrite the latter in a more detailed form as

$$D^{2,0}q^{1,0\,\underline{a}}=0,\; D^{0,2}q^{1,0\,\underline{a}}+\theta^{0,1\,\underline{a}}g^{1,1}=0,\; D^{2,0}g^{1,1}=D^{0,2}g^{1,1}=0, \quad (3.11)$$

$$D^{2,0}q^{0,1\,\underline{i}}+\theta^{1,0\,\underline{i}}f^{1,1}=0,\; D^{0,2}q^{0,1\,\underline{i}}=0,\; D^{2,0}f^{1,1}=D^{0,2}f^{1,1}=0, \quad (3.12)$$

$$\begin{aligned}
&D^{2,0}\tilde{q}^{\,\underline{i}\,\underline{a}}+\theta^{1,0\,\underline{i}}f^{1,0\,\underline{a}}=0\,, \quad D^{0,2}\tilde{q}^{\,\underline{i}\,\underline{a}}+\theta^{0,1\,\underline{a}}h^{0,1\,\underline{i}}=0\,,\\
&D^{2,0}f^{1,0\,\underline{a}}=0\,, \quad D^{0,2}f^{1,0\,\underline{a}}+\theta^{0,1\,\underline{a}}t^{1,1}=0,\\
&D^{2,0}h^{0,1\,\underline{i}}-\theta^{1,0\,\underline{i}}t^{1,1}=0, \quad D^{0,2}h^{0,1\,\underline{i}}=0,\\
&D^{2,0}t^{1,1}=D^{0,2}t^{1,1}=0\,. \qquad (3.13)
\end{aligned}$$

These constraints are solved by

$$\begin{aligned}
q^{1,0\,\underline{a}} &= q^{i\,\underline{a}}u_i^1+\theta^{1,0\,\underline{i}}\alpha_{\underline{i}}^{\underline{a}}+\theta^{0,1\,\underline{a}}\beta^{i\,a}u_i^1v_a^{-1}+\theta^{1,0\,\underline{i}}\theta^{0,1\,\underline{a}}F_{\underline{i}}^{a}v_a^{-1}\\
&- i(\theta^{1,0})^2\partial_{++}q^{i\,\underline{a}}u_i^{-1}-i(\theta^{1,0})^2\theta^{0,1\,\underline{a}}\partial_{++}\beta^{i\,a}u_i^{-1}v_a^{-1}\,, \qquad (3.14)
\end{aligned}$$

$$\begin{aligned}
q^{0,1\,\underline{i}} &= q^{\underline{i}\,a}v_a^1+\theta^{1,0\,\underline{i}}\rho^{i\,a}u_i^{-1}v_a^1+\theta^{0,1\,\underline{a}}\gamma_{\underline{a}}^{\underline{i}}+\theta^{1,0\,\underline{i}}\theta^{0,1\,\underline{a}}F_{\underline{a}}^{i}u_i^{-1}\\
&- i(\theta^{0,1})^2\partial_{--}q^{\underline{i}\,a}v_a^{-1}-i\theta^{1,0\,\underline{i}}(\theta^{0,1})^2\partial_{--}\rho^{i\,a}u_i^{-1}v_a^{-1}\,, \qquad (3.15)
\end{aligned}$$

$$\tilde{q}^{\,\underline{i}\,\underline{a}}=q^{\underline{i}\,\underline{a}}-\theta^{1,0\,\underline{i}}\psi^{i\,\underline{a}}u_i^{-1}-\theta^{0,1\,\underline{a}}\xi^{\underline{i}\,a}v_a^{-1}+\theta^{1,0\,\underline{i}}\theta^{0,1\,\underline{a}}F^{i\,a}u_i^{-1}v_a^{-1}, \quad (3.16)$$

where all the coefficients are 2D fields, $q^{i\,\underline{a}}=q^{i\,\underline{a}}(x)$, etc. The expressions for the remaining analytic components are given in Appendix. It is important to realize that all off-shell fields are collected already in $q^{1,0\,\underline{a}}$, $q^{0,1\,\underline{i}}$ and $\tilde{q}^{\,\underline{i}\,\underline{a}}$, while the remaining analytic superfunctions contain no new fields. The component expansions (3.14) – (3.16) are to be compared with that of $q^{1,1}$ obtained by solving the harmonic constraints (2.14) in the analytic basis (Eq. (2.16)).

We observe a sort of duality inside the pairs $(q^{1,1}\,,\ \hat{q}^{\,\underline{i}\,\underline{a}})$ and $(\hat{q}^{\,1,0\,\underline{a}}\,,\ \hat{q}^{\,0,1\,\underline{i}})$: the $SU(2)$ assignments of the physical and auxiliary bosonic fields in the first and second superfields within each pair are reversed with respect to each other, while the assignments of fermions are the same. As we shall see later, only these mutually «dual» twisted multiplets can interact (through the proper superpotential terms).

3.2. General Actions of the Superfields $\hat{q}^{\,1,0\,\underline{a}}$ and $\hat{q}^{\,\underline{i}\,\underline{a}}$. As given in Sec. 2, the most general action (2.17) of n twisted multiplets carried out by the analytic superfields $q^{1,1\,M}(M=1,\ldots n)$ is written as an analytic superspace integral of the Lagrangian which is a generic charge $(2,2)$ function of $q^{1,1\,M}$ and harmonic variables $u_i^{\pm1}, v_a^{\pm1}$. Being an analytic superfield, such a Lagrangian is manifestly invariant under the supersymmetry transformations. On the other hand, the analytic superfunctions representing other three twisted multiplets are not the standard superfields, therefore their functionals are not superfields as well. As a result, constructing the general supersymmetric actions of these multiplets in the analytic superspace is not so straightforward as in the case of $q^{1,1}$.

The only primary principles (besides reality) of such a construction are: i) The preservation of two harmonic $U(1)$ charges whence it follows that the relevant Lagrangian density should have the $U(1)$ charges $(2,2)$ for the action to be chargeless; ii) 2D Lorentz covariance which implies the Lagrangian density $\mathcal{L}^{2,2}$ to be Lorentz singlet; iii) Dimensionality reasoning which imply $\mathcal{L}^{2,2}$ to have the «engineering dimension» zero (i. e., the same as the superfields $\hat{q}$ in (3.1) have).

After constructing a general $\mathcal{L}^{2,2}$ obeying these criteria, one should examine which additional constraints are to be imposed on it for the action to be invariant under the transformations (3.8) and (3.9) (possibly, up to a shift of the Lagrangian by a total derivative).

Before turning to the general case of the actions which simultaneously contain a few different twisted $\mathcal{N}=(4,4)$ multiplets, we consider the actions with only one type of the non-standard twisted multiplets (3.1) as proper analogs of the $q^{1,1}$ action (2.17). Without loss of generality, it suffices to examine such actions only for $\hat{q}^{\,i\,\underline{a}}$ and $\hat{q}^{\underline{i}\,\underline{a}}$ since the action for $\hat{q}^{\underline{i}\,a}$ can be recovered from that for $\hat{q}^{\,i\,\underline{a}}$ via simple substitutions including the replacement $v_a^{\pm1} \leftrightarrow u_i^{\pm1}$.

We firstly consider the actions of single multiplets.

In accordance with the primary principles above, the most general candidate actions of the superfields $\hat{q}^{\,1,0\,\underline{a}}$, $\hat{q}^{\underline{i}\,\underline{a}}$ can be chosen as the following integrals over the analytic superspace

$$S^{\rm gen}_{(a)} = \int \mu^{-2,-2}\,\mathcal{L}^{2,2}_{(a)}\,(q^{1,0\,\underline{a}}, g^{1,1}, \theta^{0,1\,\underline{a}}, u, v)\,, \tag{3.17}$$

$$S^{\rm gen}_{(c)} = \int \mu^{-2,-2}\,\mathcal{L}^{2,2}_{(c)}\,(\tilde{q}^{\underline{i}\,\underline{a}}, f^{1,0\,\underline{a}}, h^{0,1\,\underline{i}}, t^{1,1}, \theta^{1,0\,\underline{i}}, \theta^{0,1\,\underline{a}}, u, v), \tag{3.18}$$

where $\mu^{-2,-2}$ is the analytic superspace integration measure defined in (2.18). Note that the left Grassmann coordinates $\theta^{1,0\,\underline{i}}$ cannot explicitly appear in $\mathcal{L}^{2,2}_{(a)}$ since $q^{1,0\,\underline{a}}$ and $g^{1,1}$ are superfields with respect to the left $\mathcal{N}=4$ supersymmetry. On the other hand, under the right supersymmtery $q^{1,0\,\underline{a}}$ has non-standard transformation properties (see (3.8a)), and this is the reason why $\theta^{0,1\,\underline{a}}$ is included as a possible explicit argument in $\mathcal{L}^{2,2}_{(a)}$. Both types of Grassmann coordinates are admissible as explicit arguments in $\mathcal{L}^{2,2}_{(c)}$ since $\tilde{q}^{\underline{i}\,\underline{a}}$, $f^{1,0\,\underline{a}}$ and $h^{0,1\,\underline{i}}$ possess non-standard transformation properties with respect to both left and right supersymmetries.

In order to further specify the Lagrangians in (3.17), (3.18), we can resort to the following reasoning. First, we must require that all possible terms in them are Lorentz invariant. Second, we can rule out the dependence on all involved superfields and superfunctions in (3.17), except for $q^{1,0\,\underline{a}}$ and harmonics, and in (3.18), except for $\tilde{q}^{\underline{i}\,\underline{a}}$ and harmonics. Indeed, Lorentz invariance requires that, e. g., $g^{1,1}$ could enter (3.17) only as $\theta^{0,1\,\underline{a}}g^{1,1}$ (also taking into account that the fermionic superfield $g^{1,1}$ is nilpotent and, hence, its any degree is vanishing). Then the harmonic constraints (3.11) imply that such a term can be written as $D^{0,2}q^{1,0\,\underline{a}}$.

Taking into account this reasoning (and a similar one for the $\hat{q}^{\underline{i}\,\underline{a}}$ multiplet), we can cast the Lagrangians in (3.17), (3.18) in the following more detailed form

$$
\begin{aligned}
S^{\mathrm{gen}}_{(a)} \;&=\; \int \mu^{-2,-2}\,\Big\{\mathcal{L}^{2,2}_{0}(q^{1,0\,\underline{a}},u,v) + \mathcal{L}^{1,0}_{\underline{a}}\,(q^{1,0\,\underline{a}},u,v)\,D^{0,2}q^{1,0\,\underline{a}} \\
&+\; (\theta^{0,1})^2\hat{\mathcal{L}}^{1,0}_{\underline{a}}\,(q^{1,0\,\underline{a}},u,v)\,\partial_{--}q^{1,0\,\underline{a}}\Big\}\,, \qquad (3.19)
\end{aligned}
$$

$$
\begin{aligned}
S^{\mathrm{gen}}_{(c)} = -\int \mu^{-2,-2}\,\Big\{&\mathcal{L}^{2,2}_{0}(\tilde{q}^{\underline{i}\,\underline{a}},u,v) + \mathcal{L}^{2,0}_{\underline{i}\,\underline{a}}\,(\tilde{q}^{\underline{i}\,\underline{a}},u,v)\,D^{0,2}\tilde{q}^{\underline{i}\,\underline{a}} \\
&+ \mathcal{L}^{0,2}_{\underline{i}\,\underline{a}}\,(\tilde{q}^{\underline{i}\,\underline{a}},u,v)\,D^{2,0}\tilde{q}^{\underline{i}\,\underline{a}} + \mathcal{L}_{\underline{i}\,\underline{a}\,\underline{k}\,\underline{b}}\,(\tilde{q}^{\underline{i}\,\underline{a}},u,v)\,D^{2,0}\tilde{q}^{\underline{i}\,\underline{a}}\,D^{0,2}\tilde{q}^{\underline{k}\,\underline{b}}\Big\}\,. \qquad (3.20)
\end{aligned}
$$

The reason why possible terms with x-derivatives are not included in (3.20) will become clear soon.

Now we turn to discussing the properties of the Lagrangians $\mathcal{L}^{2,2}_{(a)}$ and $\mathcal{L}^{2,2}_{(c)}$ under supersymmetry.

We start with (3.19). As we know, $q^{1,0\,\underline{a}}$ is not a superfield. It has a nontrivial transformation law (3.8) under the supersymmetry. So we need to find which constraints should be imposed on the functions in the r. h. s. of (3.19) for ensuring the latter to be invariant. The requirement of invariance amounts to the condition that the variation of (3.19) under the supersymmetry transformations is a sum of total derivatives of arbitrary functions,

$$
\delta\,\mathcal{L}^{2,2}_{(a)} = D^{0,2}(\varepsilon^{0,-1\,\underline{a}}\,F^{2,1}_{\underline{a}}) + \partial_{--}(\varepsilon^{0,-1\,\underline{a}}\,G^{2,3}_{\underline{a}} + \varepsilon^{0,1\,\underline{a}}\,H^{2,1}_{\underline{a}})\,, \qquad (3.21)
$$

which depend on the same arguments as the Lagrangian in (3.19). A possible extra term which could be added to the r. h. s. of (3.21),

$$
D^{0,2}(\varepsilon^{0,1\,\underline{a}}\,A^{2,-1}_{\underline{a}})\,, \qquad (3.22)
$$

is reduced to the one already included, after representing $\varepsilon^{0,1\,\underline{a}} = D^{0,2}\varepsilon^{0,-1\,\underline{a}}$ and integrating by parts.

Using the transformation rules (3.8) of $q^{1,0\,\underline{a}}$ and those of θ's, it is easy to compute the explicit form of the supersymmetry variation in the l. h. s. of

(3.21) and to find that only one constraint is actually required for the action to be invariant:

$$\frac{\partial \mathcal{L}_0^{2,2}}{\partial q^{1,0\,\underline{a}}} = \partial^{0,2} \mathcal{L}_{\underline{a}}^{1,0}, \tag{3.23}$$

where the partial harmonic derivative acts only on the explicit harmonics v in $\mathcal{L}_{\underline{a}}^{1,0}$. A corollary of this constraint is the following condition on $\mathcal{L}^{1,0\,\underline{a}}$:

$$\partial^{0,2} \left(\frac{\partial \mathcal{L}^{1,0\,\underline{a}}}{\partial q^{1,0\,\underline{a}}} \right) = 0\,. \tag{3.24}$$

Another consequence of the invariance condition (3.21) is that the last term in (3.19) is a total x-derivative and so makes no contribution. Indeed, the most general form of the Grassmann functions $G_{\underline{a}}^{2,3}$ and $H_{\underline{a}}^{2,1}$ in (3.21), compatible with the Lorentz covariance and the fact that these functions have the dimension $-1/2$, is

$$G_{\underline{a}}^{2,3} = (\theta^{0,1})^2\, g^{1,1}\, G_{\underline{a}}^{1,0}\,(q^{1,0\,\underline{a}}, u, v)\,, H_{\underline{a}}^{2,1} = \theta^{0,1\,\underline{b}}\, H_{\underline{a}\,\underline{b}}^{2,0}\,(q^{1,0\,\underline{a}}, u, v)\,. \tag{3.25}$$

Substituting (3.25) into the r. h. s. of (3.21) and computing the explicit form of $\delta\,\mathcal{L}_{(a)}^{2,2}$, we find the relations

$$G_{\underline{a}}^{1,0} = -\hat{\mathcal{L}}_{\underline{a}}^{1,0}, \qquad \hat{\mathcal{L}}_{\underline{a}}^{1,0} = \varepsilon^{\underline{c}\,\underline{d}}\, \frac{\partial H_{\underline{d}\,\underline{c}}^{2,0}}{\partial q^{1,0\,\underline{a}}}\,. \tag{3.26}$$

The second relation implies that the last term in (3.19) can be expressed as x^{--}-derivative of the function $H_{\underline{a}\,\underline{b}}^{2,0}$. The same phenomenon occurs for the Lagrangian (3.20), and it was the reason why we have omitted possible terms of this kind in (3.20) from the very beginning.

Keeping in mind this remark, let us deduce the analogous constraints on $\mathcal{L}_{(c)}^{2,2}$. For the action (3.18) (or (3.20)) to be invariant, the variation of this Lagrangian should be as follows

$$\delta\,\mathcal{L}_{(c)}^{2,2} = D^{2,0}(\varepsilon^{-1,0\,\underline{i}}\, A_{\underline{i}}^{1,2} + \varepsilon^{0,-1\,\underline{a}}\, B_{\underline{a}}^{0,3}) + D^{0,2}(\varepsilon^{-1,0\,\underline{i}}\, C_{\underline{i}}^{3,0} + \varepsilon^{0,-1\,\underline{a}}\, D_{\underline{a}}^{2,1})\,. \tag{3.27}$$

Writing the variation in the l. h. s. of (3.27) in the explicit form, it is straightforward to find that the component functions in (3.20) should obey the following system of constraints:

$$\frac{\partial\mathcal{L}^{2,2}_0}{\partial\tilde{q}^{\,i\,\underline{a}}} = \partial^{2,0}\mathcal{L}^{0,2}_{\underline{i}\;\underline{a}} + \partial^{0,2}\mathcal{L}^{2,0}_{\underline{i}\;\underline{a}}\,, \quad \mathcal{L}_{(\underline{i}\;\underline{l})(\underline{a}\;\underline{c})} = 0\,, \tag{3.28}$$

$$\frac{\partial\mathcal{L}^{2,0}_{\underline{i}\;\underline{a}}}{\partial\tilde{q}^{\,\underline{l}\,\underline{c}}} - \frac{\partial\mathcal{L}^{2,0}_{\underline{l}\;\underline{c}}}{\partial\tilde{q}^{\,\underline{i}\,\underline{a}}} = \partial^{2,0}\mathcal{L}_{\underline{l}\;\underline{c}\;\underline{i}\;\underline{a}}\,, \quad \frac{\partial\mathcal{L}^{0,2}_{\underline{i}\;\underline{a}}}{\partial\tilde{q}^{\,\underline{l}\,\underline{c}}} - \frac{\partial\mathcal{L}^{0,2}_{\underline{l}\;\underline{c}}}{\partial\tilde{q}^{\,\underline{i}\,\underline{a}}} = \partial^{0,2}\mathcal{L}_{\underline{i}\;\underline{a}\;\underline{l}\;\underline{c}}\,, \tag{3.29}$$

$$\frac{\partial\mathcal{L}_{\underline{i}\;\underline{a}\;\underline{k}\;\underline{b}}}{\partial\tilde{q}^{\,\underline{l}\,\underline{c}}} - \frac{\partial\mathcal{L}_{\underline{l}\;\underline{c}\;\underline{k}\;\underline{b}}}{\partial\tilde{q}^{\,\underline{i}\,\underline{a}}} = \frac{\partial\mathcal{L}_{(\underline{i}\;\underline{l})[\underline{a}\;\underline{c}]}}{\partial\tilde{q}^{\,\underline{k}\,\underline{b}}}\,, \quad \frac{\partial\mathcal{L}_{\underline{i}\;\underline{a}\;\underline{k}\;\underline{b}}}{\partial\tilde{q}^{\,\underline{l}\,\underline{c}}} - \frac{\partial\mathcal{L}_{\underline{i}\;\underline{a}\;\underline{l}\;\underline{c}}}{\partial\tilde{q}^{\,\underline{k}\,\underline{b}}} = \frac{\partial\mathcal{L}_{[\underline{l}\;\underline{k}](\underline{c}\;\underline{b})}}{\partial\tilde{q}^{\,\underline{i}\,\underline{a}}}, \tag{3.30}$$

where we introduced the notation

$$\mathcal{L}_{\underline{i}\;\underline{a}\;\underline{l}\;\underline{c}} = \mathcal{L}_{(\underline{i}\;\underline{l})\;(\underline{a}\;\underline{c})} + \mathcal{L}_{(\underline{i}\;\underline{l})\;[\underline{a}\;\underline{c}]} + \mathcal{L}_{[\underline{i}\;\underline{l}]\;(\underline{a}\;\underline{c})} + \mathcal{L}_{[\underline{i}\;\underline{l}]\;[\underline{a}\;\underline{c}]} \tag{3.31}$$

(the symbols ( ) and [ ] mean symmetrization and antisymmetrization with the factor $1/2$).

Below we shall identify the θ-independent piece of the last term in (3.31) with the bosonic target metric of sigma model, while the antisymmetric part of $\mathcal{L}_{\underline{i}\;\underline{a}\;\underline{l}\;\underline{c}}|_{\theta=0}$ will be identified with the torsion potential.

It is straightforward to substitute the component expansion of $q^{1,0\,\underline{a}}$, Eq. (3.14), into (3.19) and that of $\tilde{q}^{\,\underline{i}\,\underline{a}}$, Eq. (3.16), into (3.20), to integrate over θ's and harmonics with the help of the constraints (3.23) and (3.28), (3.29), and to eventually obtai the component form of the actions in x-space. We give here only those parts which involve the physical bosonic and auxiliary fields.

For the action (3.19) these pieces are as follows

$$S^{\rm phb}_{(a)} = \frac{1}{2}\int d^2x\,\Big\{ G_{i\,\underline{a}\,j\,\underline{b}}(q)\,\partial_{++}q^{i\,\underline{a}}\,\partial_{--}q^{j\,\underline{b}} + 2\,B_{i\,\underline{a}\,j\,\underline{b}}(q)\,\partial_{++}q^{i\,\underline{a}}\,\partial_{--}q^{j\,\underline{b}}\Big\}, \tag{3.32}$$

$$S^{\rm auxb}_{(a)} = \frac{1}{8}\int d^2x\; G(q)\; F^a_{\underline{i}}\,F^{\underline{i}}_a, \tag{3.33}$$

where

$$G_{i\,\underline{a}\,j\,\underline{b}}(q) = G(q)\;\varepsilon_{i\,j}\,\varepsilon_{\underline{a}\,\underline{b}}\,, \quad G(q) = \int du\; g(q^{1,0},u)\,, \tag{3.34}$$

$$B_{i\,\underline{a}\,j\,\underline{b}}(q) = \int du\; g(q^{1,0},u)\; u^1_{(i}u^{-1}_{j)}\;\varepsilon_{\underline{b}\,\underline{a}}\,, \tag{3.35}$$

$$g(q^{1,0},u) = \left.\frac{\partial\mathcal{L}^{1,0\,\underline{a}}}{\partial q^{1,0\,\underline{a}}}\right|_{\theta=0}, \quad q^{1,0\,\underline{a}}|_{\theta=0} = q^{i\,\underline{a}}(x)\,u^1_i\,. \tag{3.36}$$

The $v^{\pm1}_a$-independence of the function $g(q^{1,0},u)$ in (3.36) and, hence, of the torsion potential, follows from the constraint (3.24).

Analogous terms for the action (3.20) read

$$S^{\rm phb}_{(c)} = \int d^2x \, \Big\{ G_{\underline{i}\,\underline{a}\,\underline{j}\,\underline{b}} + B_{\underline{i}\,\underline{a}\,\underline{j}\,\underline{b}} \Big\}\, \partial_{++} q^{\underline{i}\,\underline{a}}\, \partial_{--} q^{\underline{j}\,\underline{b}}, \tag{3.37}$$

$$S^{\rm auxb}_{(c)} = \frac{1}{4}\int d^2x \, \hat{G}\, F^a_i\, F^i_a, \tag{3.38}$$

where the involved objects are the appropriate $\theta = 0$ projections

$$G_{\underline{i}\,\underline{a}\,\underline{j}\,\underline{b}} = \mathcal{L}_{[\,\underline{i}\,\underline{j}\,]\,[\,\underline{a}\,\underline{b}\,]}|_{\theta=0} \equiv \varepsilon_{\underline{i}\,\underline{j}}\, \varepsilon_{\underline{a}\,\underline{b}}\, \hat{G}\,, \quad \hat{G} = \frac{1}{4}\, \varepsilon^{\underline{i}\,\underline{j}}\, \varepsilon^{\underline{a}\,\underline{b}}\, G_{\underline{i}\,\underline{a}\,\underline{j}\,\underline{b}}\,, \tag{3.39}$$

$$B_{\underline{i}\,\underline{a}\,\underline{j}\,\underline{b}} = \Big(\mathcal{L}_{[\,\underline{i}\,\underline{j}\,]\,(\,\underline{a}\,\underline{b}\,)} + \mathcal{L}_{(\,\underline{i}\,\underline{j}\,)\,[\,\underline{a}\,\underline{b}\,]} \Big)\,|_{\theta=0}\,. \tag{3.40}$$

With the help of constraints (3.28), (3.29) one can show that the scalar metric and torsion potential in this case are independent of both sets of harmonic variables modulo a gauge transformation of $B_{\underline{i}\,\underline{a}\,\underline{j}\,\underline{b}}$. Indeed, the second constraint in (3.28) and constraints (3.29) together imply

$$\partial^{2,0} \mathcal{L}_{[\,\underline{i}\,\underline{j}\,]\,[\,\underline{a}\,\underline{b}\,]}|_{\theta=0} = \partial^{0,2} \mathcal{L}_{[\,\underline{i}\,\underline{j}\,]\,[\,\underline{a}\,\underline{b}\,]}|_{\theta=0} = 0\,,$$

whence $\partial^{2,0}\hat{G} = \partial^{0,2}\hat{G} = 0$. Further, we can rewrite the $\theta = 0$ projection of (3.29) as

$$\frac{\partial \mathcal{L}^{2,0}_{\underline{i}\,\underline{a}}}{\partial q^{\underline{l}\,\underline{c}}} - \frac{\partial \mathcal{L}^{2,0}_{\underline{l}\,\underline{c}}}{\partial q^{\underline{i}\,\underline{a}}} = \partial^{2,0}\, B_{\underline{l}\,\underline{c}\,\underline{i}\,\underline{a}}\,, \quad \frac{\partial \mathcal{L}^{0,2}_{\underline{i}\,\underline{a}}}{\partial q^{\underline{l}\,\underline{c}}} - \frac{\partial \mathcal{L}^{0,2}_{\underline{l}\,\underline{c}}}{\partial q^{\underline{i}\,\underline{a}}} = \partial^{0,2}\, B_{\underline{i}\,\underline{a}\,\underline{l}\,\underline{c}}\,. \tag{3.41}$$

Also, one should take into account that $B_{\underline{i}\,\underline{a}\,\underline{l}\,\underline{c}}$ is defined up to the gauge transformation

$$\delta B_{\underline{i}\,\underline{a}\,\underline{l}\,\underline{c}} = \frac{\partial X_{\underline{i}\,\underline{a}}}{\partial q^{\underline{l}\,\underline{c}}} - \frac{\partial X_{\underline{l}\,\underline{c}}}{\partial q^{\underline{i}\,\underline{a}}}, \tag{3.42}$$

where $X_{\underline{i}\,\underline{a}}$ is an arbitrary function of $q^{\underline{i}\,\underline{a}}(x)$ and harmonics (such addition to B in (3.37) produces a total x-derivative). Exploiting this gauge freedom together with the constraints (3.41) and the $\theta = 0$ projection of the first constraint in (3.28), one can show that it is possible to choose a gauge in which $B_{\underline{i}\,\underline{a}\,\underline{l}\,\underline{c}}$ satisfies the homogeneous harmonic constraints

$$\partial^{2,0} \tilde{B}_{\underline{i}\,\underline{a}\,\underline{l}\,\underline{c}} = 0\,, \quad \partial^{0,2} \tilde{B}_{\underline{i}\,\underline{a}\,\underline{l}\,\underline{c}} = 0\,, \tag{3.43}$$

and so indeed does not depend on harmonics.

The objects $G_{i\,\underline{a}\,j\,\underline{b}}$, $B_{i\,\underline{a}\,j\,\underline{b}}$ ($G_{\underline{i}\,\underline{a}\,\underline{j}\,\underline{b}}, B_{\underline{i}\,\underline{a}\,\underline{j}\,\underline{b}}$) are, respectively, symmetric and antisymmetric under the simultaneous permutation of the indices $i \leftrightarrow j$, $\underline{a} \leftrightarrow \underline{b}$ ($\underline{i} \leftrightarrow \underline{j}$, $\underline{a} \leftrightarrow \underline{b}$), and so they can be identified with the metric and torsion potential on the target space. Sometimes it is advantageous to express the second terms in (3.32) and (3.37) through the torsion field strengths which are defined by

$$H_{i\,\underline{a}\,j\,\underline{b}\,k\,\underline{c}} = \frac{\partial B_{i\,\underline{a}\,j\,\underline{b}}}{\partial q^{k\,\underline{c}}} + \frac{\partial B_{k\,\underline{c}\,i\,\underline{a}}}{\partial q^{j\,\underline{b}}} + \frac{\partial B_{j\,\underline{b}\,k\,\underline{c}}}{\partial q^{i\,\underline{a}}} \tag{3.44}$$

and

$$H_{\underline{i}\,\underline{a}\,\underline{j}\,\underline{b}\,\underline{k}\,\underline{c}} = \frac{\partial B_{\underline{i}\,\underline{a}\,\underline{j}\,\underline{b}}}{\partial q^{\underline{k}\,\underline{c}}} + \frac{\partial B_{\underline{k}\,\underline{c}\,\underline{i}\,\underline{a}}}{\partial q^{\underline{j}\,\underline{b}}} + \frac{\partial B_{\underline{j}\,\underline{b}\,\underline{k}\,\underline{c}}}{\partial q^{\underline{i}\,\underline{a}}}\,. \tag{3.45}$$

They are totally antisymmetric with respect to permutations of the quartet pairs $i\;\underline{a}\,,\;\; j\;\underline{b}\,,\;\; k\;\underline{c}$ in (3.44) and $\underline{i}\;\underline{a},\;\; \underline{j}\;\underline{b},\;\; \underline{k}\;\underline{c}$ in (3.45). For $B_{i\,\underline{a}\,j\,\underline{b}}$ given by Eq. (3.35) and $B_{\underline{i}\,\underline{a}\,\underline{j}\,\underline{b}}$ given by Eq. (3.40), the corresponding torsion field strengths $H_{i\,\underline{a}\,j\,\underline{b}\,k\,\underline{c}}$ and $H_{\underline{i}\,\underline{a}\,\underline{j}\,\underline{b}\,\underline{k}\,\underline{c}}$ are reduced to

$$H_{i\,\underline{a}\,j\,\underline{b}\,k\,\underline{c}} = \varepsilon_{\underline{b}\,\underline{c}}\,\varepsilon_{i\,(j}\,\frac{\partial G}{\partial q^{k)\,\underline{a}}} + \varepsilon_{\underline{a}\,\underline{b}}\,\varepsilon_{k\,(i}\,\frac{\partial G}{\partial q^{j)\,\underline{c}}} + \varepsilon_{\underline{c}\,\underline{a}}\,\varepsilon_{j\,(k}\frac{\partial G}{\partial q^{i)\,\underline{b}}} \equiv 3\tilde{H}_{i\,\underline{a}\,j\,\underline{b}\,k\,\underline{c}}, \tag{3.46}$$

where

$$\tilde{H}_{i\,\underline{a}\,j\,\underline{b}\,k\,\underline{c}} = \varepsilon_{i\,j}\,\varepsilon_{\underline{a}\,\underline{c}}\,\frac{\partial G}{\partial q^{k\,\underline{b}}} - \varepsilon_{i\,k}\,\varepsilon_{\underline{a}\,\underline{b}}\,\frac{\partial G}{\partial q^{j\,\underline{c}}} \tag{3.47}$$

and

$$H_{\underline{i}\,\underline{a}\,\underline{j}\,\underline{b}\,\underline{k}\,\underline{c}} = \varepsilon_{\underline{b}\,\underline{a}}\,\varepsilon_{\underline{k}\,(\underline{i}}\,\frac{\partial \hat{G}}{\partial q^{\underline{j})\,\underline{c}}} + \varepsilon_{\underline{a}\,\underline{c}}\,\varepsilon_{\underline{j}\,(\underline{k}}\,\frac{\partial \hat{G}}{\partial q^{\underline{i})\,\underline{b}}} + \varepsilon_{\underline{c}\,\underline{b}}\,\varepsilon_{\underline{i}\,(\underline{j}}\,\frac{\partial \hat{G}}{\partial q^{\underline{k})\,\underline{a}}}\,. \tag{3.48}$$

When deducing (3.48), we essentially used the constraints (3.30).

We would like to point out that in both considered cases the geometric target space objects (metric and torsion) are expressed through single scalar functions $G(q^{i\,\underline{a}})$ or $\hat{G}(q^{\underline{i}\,\underline{a}})$ defined, respectively, by Eqs. (3.34) and (3.39). The only constraint they satisfy is the four-dimensional Laplace equation

$$\text{(a)}\;\; \frac{\partial^2 G}{\partial q^{i\,\underline{a}}\,\partial q_{i\,\underline{a}}} = 0\,, \quad \text{(b)}\;\; \frac{\partial^2 \hat{G}}{\partial q^{\underline{i}\,\underline{a}}\,\partial q_{\underline{i}\,\underline{a}}} = 0\,. \tag{3.49}$$

Eq. (3.49a) follows from the definition of G in (3.34) and the property

$$\frac{\partial^2}{\partial q^{i\,\underline{a}}\,\partial q_{i\,\underline{a}}} \sim \frac{\partial^2}{\partial q^{1,0\,\underline{a}}\,\partial q_{\underline{a}}{}^{-1,0}} - \frac{\partial^2}{\partial q^{-1,0\,\underline{a}}\,\partial q_{\underline{a}}{}^{1,0}}\,,$$

which is a consequence of the completeness relation $u^1_i u^{-1}_k - u^{-1}_i u^1_k = \varepsilon_{i\,k}\,$, whereas Eq. (3.49b) is implied by the constraints (3.30) and the second constraint in (3.28). The same bosonic target geometry was found in the case of the analytic twisted multiplet $q^{1,1}$ in [15] and, in the $\mathcal{N} = (2,2)$ superspace formulation, in [8]. Thus we conclude that the most general *off-shell* $\mathcal{N} = (4,4)$ sigma models associated with each twisted multiplet from the four-entry set $\hat q^{\,i\,a}$, $\hat q^{\,i\,\underline{a}}$, $\hat q^{\,\underline{i}\,a}$, $\hat q^{\,\underline{i}\,\underline{a}}$ defined in Eq. (1.1) show up equivalent target geometries. In the next Subsection, on the example of the multiplet $\hat q^{\,i\,\underline{a}}$, we shall see that the same is true for the cases when a few multiplets of the same sort are present.

Finally, as a particular case of the above general actions, we quote the supersymmetric free actions of $\hat q^{\,1,0\,\underline{a}}$ and $\hat q^{\,\underline{i}\,\underline{a}}$

$$S^{\rm free}_{(a)} \sim \int \mu^{-2,-2}\, q^{1,0}_{\underline{a}} D^{0,2} q^{1,0\,\underline{a}}\,, \tag{3.50}$$

$$S^{\rm free}_{(c)} \sim \int \mu^{-2,-2}\, D^{2,0}\tilde q_{\underline{i}\,\underline{a}}\, D^{0,2}\tilde q^{\,\underline{i}\,\underline{a}}\,. \tag{3.51}$$

3.3. Generalization to the Case of Several $\hat q^{\,1,0\,\underline{a}}$. Generalizing the action of a single $q^{1,1}$ superfield to the case of n self-interacting superfields $q^{1,1\,M}$ $(M = 1,\ldots,n)$ is straightforward, see Eq. (2.17). Now we are going to generalize the supersymmetric action of single $\hat q^{\,1,0\,\underline{a}}$ to the general case of several self-interacting $\hat q^{\,1,0\,\underline{a}\,M}$. The supersymmeric transformation properties of $q^{1,0\,\underline{a}\,M}$ are

$$\delta\, q^{1,0\,\underline{a}\,M} = -\varepsilon^{0,-1\,\underline{a}}\, g^{1,1\,M}\,, \quad \delta\, g^{1,1\,M} = 0\,. \tag{3.52}$$

The defining harmonic constraints for each value of the index M have the form (3.11). Solving them, we find the bosonic component content of $q^{1,0\,\underline{a}\,M}$ as

$$q^{1,0\,\underline{a}\,M} = q^{i\,\underline{a}\,M} u^1_i + \theta^{1,0\,\underline{i}}\theta^{0,1\,\underline{a}} F^{a\,M}_{\underline{i}} v^{-1}_a - i(\theta^{1,0})^2\partial_{++} q^{i\,\underline{a}\,M} u^{-1}_i\,. \tag{3.53}$$

Following the same line of arguments as in the construction of (3.17), (3.19), we can again take the candidate general action as an analytic superspace integral of some function $\mathcal{L}^{2,2}_{(a)}$:

$$S^{\rm Gen}_{(a)} = \int \mu^{-2,-2}\, \mathcal{L}^{2,2}_{(a)}\,(q^{1,0\,\underline{a}\,M}, g^{1,1M}, \theta^{0,1\,\underline{a}}, u, v), \tag{3.54}$$

and then specify it according to the harmonic constraints as

$$\begin{aligned} \mathcal{L}^{2,2}_{(a)} \;&=\; \mathcal{L}^{2,2}_{0}(q^{1,0\,\underline{a}\,M}, u, v) + \mathcal{L}^{1,0\,M}_{\underline{a}}(q^{1,0\,\underline{a}\,M}, u, v)\, D^{0,2} q^{1,0\,\underline{a}\,M} \\ &+\; \mathcal{L}^{0,-2\,MN}_{\underline{a}\,\underline{b}}\,(q^{1,0\,\underline{a}\,M}, u, v) D^{0,2} q^{1,0\,\underline{a}\,M} D^{0,2} q^{1,0\,\underline{b}\,N}\,. \end{aligned} \tag{3.55}$$

Like in the case of single $\hat q^{\,1,0\,\underline{a}}$, we omit a possible term with explicit x-derivative (and explicit θ's) in (3.55) because it can be shown to be a total x-derivative as a consequence of requiring (3.54) to be supersymmetric. Demanding that the variation of the Lagrangian (3.55) under the supersymmetry transformations is a total harmonic derivative of an arbitrary function depending on the same arguments as the Lagrangian itself,

$$\delta\, \mathcal{L}^{2,2}_{(a)} = D^{0,2}(\varepsilon^{0,-1\,\underline{a}}\, F^{2,1}_{\underline{a}})\,, \tag{3.56}$$

one finds the set of constraints for the considered case:

$$\frac{\partial \mathcal{L}_0^{2,2}}{\partial q^{1,0\,\underline{a}\,M}} = \partial^{0,2}\mathcal{L}_{\underline{a}}^{1,0\,M}\,, \tag{3.57}$$

$$\frac{\partial \mathcal{L}_{\underline{b}}^{1,0\,N]}}{\partial q^{1,0\,\underline{a}\,[M}} - \frac{\partial \mathcal{L}_{\underline{a}}^{1,0\,[M}}{\partial q^{1,0\,\underline{b}\,N]}} = \partial^{0,2}\mathcal{L}_{\underline{a}\,\underline{b}}^{0,-2\,[MN]} \tag{3.58}$$

(one automatically gains antisymmetrization in indices M,N in (3.58) since the latter actually emerges multiplied by $g^{1,1\,M}g^{1,1\,N}$). The constraint (3.57) has the same form as in the case of single $\hat{q}^{\,1,0\,\underline{a}}$. The constraint (3.58) is new. Let us discuss what it means. First, from the structure of the last term in (3.55) one can derive that the function $\mathcal{L}_{\underline{a}\,\underline{b}}^{0,-2MN}$ is antisymmetric with respect to the permutation of each pair of its indices

$$\mathcal{L}_{\underline{a}\,\underline{b}}^{0,-2\,MN} = -\mathcal{L}_{\underline{a}\,\underline{b}}^{0,-2\,NM} = -\mathcal{L}_{\underline{b}\,\underline{a}}^{0,-2\,MN} = \mathcal{L}_{\underline{b}\,\underline{a}}^{0,-2\,NM}\,. \tag{3.59}$$

Second, one can rewrite the l. h. s. of (3.58) as

$$\frac{\partial \mathcal{L}_{\underline{b}}^{1,0\,N]}}{\partial q^{1,0\,\underline{a}\,[M}} - \frac{\partial \mathcal{L}_{\underline{a}}^{1,0\,[M}}{\partial q^{1,0\,\underline{b}\,N]}} = \frac{1}{2}\,\frac{\partial \mathcal{L}_{(\underline{b}}^{1,0\,N]}}{\partial q^{1,0\,\underline{a})\,[M}}\,. \tag{3.60}$$

Thus we see that (3.58) actually amounts to the two independent constraints

$$\frac{\partial \mathcal{L}_{(\underline{a}}^{1,0\,[M}}{\partial q^{1,0\,\underline{b})\,N]}} = 0\,, \quad \partial^{0,2}\mathcal{L}_{\underline{a}\,\underline{b}}^{0,-2\,[MN]} = 0 \;\;\Rightarrow\;\; \mathcal{L}_{\underline{a}\,\underline{b}}^{0,-2\,[MN]} = 0\,. \tag{3.61}$$

Now we are prepared to write the component form of the bosonic sector of the general action (3.55). After integrating over θ's with the help of (3.57), (3.61), one finds

$$S_{(a)}^{\rm phb} = \frac{1}{2}\int d^2x\,\Big\{ G_{i\,\underline{a}\,j\,\underline{b}}^{MN}(q)\,\partial_{++}q^{j\,\underline{b}\,N}\,\partial_{--}q^{i\,\underline{a}\,M} + 2B_{i\,\underline{a}\,j\,\underline{b}}^{MN}(q)\,\partial_{++}q^{j\,\underline{b}\,N}\,\partial_{--}q^{i\,\underline{a}\,M}\Big\}\,, \tag{3.62}$$

$$S_{(a)}^{\rm auxb} = \frac{1}{8}\int d^2x\,G^{MN}(q)\,F_{\underline{i}}^{\,a\,M}\,F_{a}^{\,\underline{i}\,N}, \tag{3.63}$$

where

$$G_{i\,\underline{a}\,j\,\underline{b}}^{MN}(q) = \int du\;g^{MN}(q,u)\,\varepsilon_{i\,j}\,\varepsilon_{\underline{a}\,\underline{b}} = \varepsilon_{i\,j}\,\varepsilon_{\underline{a}\,\underline{b}}\,G^{MN}(q)\,,$$

$$B_{i\,\underline{a}\,j\,\underline{b}}^{MN}(q) = \int du\;g^{MN}(q,u)\,u^{1}_{(\,i}u^{-1}_{j\,)}\,\varepsilon_{\underline{a}\,\underline{b}}\,,$$

$$G^{MN}(q) = \int du\;g^{MN}(q,u)\,, \quad g^{MN}(q,u) = \left.\frac{\partial \mathcal{L}^{1,0\,\underline{a}\,(M}}{\partial q^{1,0\,\underline{a}\,N)}}\right|_{\theta=0}. \tag{3.64}$$

The objects $G^{MN}_{i\,\underline{a}\,j\,\underline{b}}(q)$, $B^{MN}_{i\,\underline{a}\,j\,\underline{b}}(q)$ are, respectively, symmetric and antisymmetric under the simultaneous permutation of the indices $i \leftrightarrow j,\ \underline{a} \leftrightarrow \underline{b},\ M \leftrightarrow N$, and so they can be identified with the target space metric and torsion potential. The torsion field strength is given by

$$H^{MNT}_{i\,\underline{a}\,j\,\underline{b}\,k\,\underline{c}} = \varepsilon_{\underline{b}\,\underline{c}}\,\varepsilon_{i\,(j}\,\frac{\partial G^{NT}}{\partial q^{k)\,\underline{a}\,M}} + \varepsilon_{\underline{a}\,\underline{b}}\,\varepsilon_{k\,(i}\,\frac{\partial G^{MN}}{\partial q^{j)\,\underline{c}\,T}} + \varepsilon_{\underline{c}\,\underline{a}}\,\varepsilon_{j\,(k}\,\frac{\partial G^{TM}}{\partial q^{i)\,\underline{b}\,N}}\,. \quad (3.65)$$

An analog of the basic functions $G(q)$, $\hat{G}(q)$ of the one-multiplet case is the symmetric $n \times n$ matrix function $G^{MN}(q)$ through which both the metric and torsion are expressed. From its definition (3.64) and the first constraint in (3.61) it is easy to find analogs of the constraint (3.49) for the considered case

$$\text{(a)}\ \ \frac{\partial^2 G^{MN}}{\partial q^{i\,\underline{a}\,T}\,\partial q_{i\,\underline{a}}{}^{F}} = 0\,, \quad \text{(b)}\ \ \frac{\partial G^{MN}}{\partial q^{i\,\underline{a}\,T}} - \frac{\partial G^{TN}}{\partial q^{i\,\underline{a}\,M}} = 0\,. \quad (3.66)$$

This is the manifestly $SU(2) \times SU(2)$ covariant form of the similar constraints obtained in [8] for the case of a few twisted multiplets in the $\mathcal{N} = (2,2)$ superfield approach. They also coincide with the constraints for a similar $n\times n$ metric for the case of n superfields $q^{1,1\,M}$ [15]. This metric is defined as the $\theta = 0$ projection of

$$\tilde{G}^{MN}(q) = \int du\,dv\,\frac{\partial^2 \mathcal{L}^{2,2}(q^{1,1\,T}, u, v)}{\partial q^{1,1\,M}\,\partial q^{1,1\,N}}\,.$$

It is straightforward to check that it satisfies the same constraints (3.66), up to the replacement $q^{i\,\underline{a}\,T} \to q^{i\,a\,T}$. Thus in both cases we are facing the same sort of the bosonic target HKT geometry.

One can construct an analogous off-shell superfield action also for several twisted multiplet $\hat{q}^{\,\underline{i}\,\underline{a}}$, though such a construction is somewhat more involved. The corresponding component action and bosonic target geometry are the same as in the case of $q^{1,1\,M}$ or $\hat{q}^{1,0\,\underline{a}\,M}$ (up to the proper rearrangement of $SU(2)$ indices of the component fields).

4. $SU(2) \times U(1)$ WZNW SIGMA MODEL OF $\hat{q}^{\,1,0\,\underline{a}}$ MULTIPLET

In this Section we present one more explicit example of off-shell action for $\hat{q}^{\,1,0\,\underline{a}}$ (besides the free action (3.50)). We shall show that the requirement of invariance under one of two $\mathcal{N} = (4,4)$ $SU(2)$ superconformal groups defined in Sec. 2 uniquely fixes the $\hat{q}^{\,1,0\,\underline{a}}$ sigma-model action to be that of $\mathcal{N} = (4,4)$ $SU(2) \times U(1)$ WZNW sigma model.

As a first step, we must find the transformation properties of $\hat{q}^{\,1,0\,\underline{a}}$ under both $SU(2)$ superconformal groups defined in Sec. 2. These transformation laws are uniquely fixed by the requirement of preserving the harmonic constraints (3.11). Since these constraints do not respect a symmetry under the permutation of the left and right light-cone sectors (as opposed to the $q^{1,1}$ defining constraints (2.14)), the left and right components of superconformal groups have a different action on the set $(q^{1,0\,\underline{a}},\, g^{1,1})$.

The left light-cone branches of two $\mathcal{N}=(4,4)$ $SU(2)$ superconformal groups act on $q^{1,0\,\underline{a}}$ and $g^{1,1}$ in the very simple manner

$$\delta_{(I)L}\, q^{1,0\,\underline{a}} = \Lambda_{(I)L}\, q^{1,0\,\underline{a}}\,, \quad \delta_{(I)L}\, g^{1,1} = \Lambda_{(I)L}\, g^{1,1}\,,$$
$$\delta_{(II)L}\, q^{1,0\,\underline{a}} = 0\,, \quad \delta_{(II)L}\, g^{1,1} = 0, \tag{4.1}$$

where the parameter $\Lambda_{(I)L}$ was defined in (2.22). The requirement of preserving the harmonic constraints (3.11) under the action of right light-cone branches of these superconformal groups results in the following transformation laws

$$\delta_{(I)R}\, q^{1,0\,\underline{a}} = -\Lambda^{0,-1\,\underline{a}}_{(I)}\, g^{1,1}\,, \quad \delta_{(I)R}\, g^{1,1} = \Lambda_{(I)R}\, g^{1,1}\,, \tag{4.2}$$

$$\delta_{(II)R}\, q^{1,0\,\underline{a}} = -\Lambda^{0,-1\,\underline{a}}_{(II)}\, g^{1,1} + q^{1,0\,\underline{b}}\, \frac{\partial \Lambda^{0,1\,\underline{a}}_{(II)}}{\partial \theta^{0,1\,\underline{b}}} - q^{1,0\,\underline{a}}\, \frac{\partial \Lambda^{0,1\,\underline{b}}_{(II)}}{\partial \theta^{0,1\,\underline{b}}}\,,$$
$$\delta_{(II)R}\, g^{1,1} = g^{1,1}\, \partial_{--}\Lambda_{(II)R} + 2i q^{1,0\,\underline{a}}\, \partial_{--}\Lambda^{0,1}_{(II)\,\underline{a}}\,. \tag{4.3}$$

All the involved parameters Λ were defined in Sec. 2.

Let us now specialize to a single $\hat{q}^{\,1,0\,\underline{a}}$ and construct for it an action invariant under the superconformal groups defined by Eqs. (2.20) – (2.30), (4.1) – (4.3). The free action (3.50) does not respect the full superconformal invariance, it is invariant only under the $\mathcal{N}=4$ superconformal group II of the left sector and the $\mathcal{N}=4$ superconformal group I of the right sector (the corresponding *R*-symmetry $SU(2)$ groups act only on fermions and auxiliary fields in $\hat{q}^{\,1,0\,\underline{a}}$). So, if we wish to have invariance also under those $\mathcal{N}=4$ superconformal groups whose $SU(2)$ subgroups affect physical bosonic fields, the corresponding invariant action should necessarily include self-interaction. To find its precise form, we apply the procedure which has been employed earlier in [30] for constructing an action of improved $\mathcal{N}=2$, 4D tensor multiplet in the harmonic analytic $\mathcal{N}=2$, 4D superspace and in [15] for constructing a superconformal action of $q^{1,1}$ in the $SU(2)\times SU(2)$ HSS. Let us introduce

$$\tilde{q}^{\,1,0\,\underline{a}} = q^{1,0\,\underline{a}} - c^{1,0\,\underline{a}},\; c^{\pm1,0\,\underline{a}} = c^{i\,\underline{a}} u^{\pm1}_i,\; X = \tilde{q}^{\,1,0\,\underline{a}} c^{-1,0}_{\underline{a}},\; c^2 \equiv \frac{1}{2}\, c^{i\,\underline{a}} c_{i\,\underline{a}}.$$

These newly defined quantities have the following inhomogeneous transformation law under the action of the first superconformal group

$$\delta_{(I)L}\, c^{1,0\,\underline{a}} = \Lambda^{2,0} c^{-1,0\,\underline{a}}\,, \quad \delta_{(I)L}\, \tilde{q}^{\,1,0\,\underline{a}} = \Lambda_{(I)L}\, (\tilde{q}^{\,1,0\,\underline{a}} + c^{1,0\,\underline{a}}) - \Lambda^{2,0} c^{-1,0\,\underline{a}}\,. \tag{4.4}$$

Now we represent the sought superconformal action as a series in X

$$S^{\rm sc} = \frac{1}{2\kappa^2}\int \mu^{-2,-2}\,\tilde{q}^{\,1,0\,\underline{a}}\,D^{0,2}\,\tilde{q}^{\,1,0}_{\underline{a}}\sum_{n=0}^{\infty} a_n X^n\,. \tag{4.5}$$

Using the relation

$$c^{1,0}_{\underline{a}}c^{-1,0}_{\underline{b}} - c^{1,0}_{\underline{b}}c^{-1,0}_{\underline{a}} = \varepsilon_{\underline{a}\,\underline{b}}\,c^2\,, \tag{4.6}$$

one can rewrite the prefactor in (4.5) also in terms of X :

$$\tilde{q}^{\,1,0\,\underline{a}}\,D^{0,2}\,\tilde{q}^{\,1,0}_{\underline{a}} = \frac{1}{c^2}\left(D^{2,0}XD^{0,2}X - XD^{2,0}D^{0,2}X\right). \tag{4.7}$$

Now, keeping in mind that the newly introduced analytic superfunction X transforms inhomogeneously under the superconformal transformation, one concludes that there is a possibility to achieve the invariance of (4.5) by requiring that the variations of the terms of different order in X cancel each other up to total harmonic derivatives. Namely, we take into account the invariance of the integration measure and then demand the homogeneous part of the variation of the second-order term to be cancelled by the inhomogeneous part of the variation of the third-order term, etc. Proceeding in this way, one finally proves that the action (4.5) is invariant provided the following recurrence relations between the coefficients a_n hold

$$a_{n+1} = -\frac{1}{c^2}\frac{(n+2)^2}{(n+1)(n+3)}a_n\,, \tag{4.8}$$

whence one finds

$$a_n = (-1)^n\left(\frac{1}{c^2}\right)^n\frac{2(n+1)}{(n+2)}a_0\,. \tag{4.9}$$

Introducing new variable t ,

$$t = \frac{X}{c^2}\,, \tag{4.10}$$

it is straightforward to show that the series in (4.5) is summed up into the expression

$$S^{\rm sc} = \frac{1}{\kappa^2}\int \mu^{-2,-2}\,\tilde{q}^{\,1,0\,\underline{a}}\,D^{0,2}\,\tilde{q}^{\,1,0}_{\underline{a}}\left\{\frac{\ln(1+t)}{t^2} - \frac{1}{t(1+t)}\right\}. \tag{4.11}$$

The Lagrangian in (4.11) is the sought superconformally invariant Lagrangian of the $\hat{q}^{\,1,0\,\underline{a}}$ multiplet. Integrating by parts with the help of formulas (4.6), (4.7), one can rewrite the action (4.11) in the more concise equivalent form

$$S^{\rm sc} = \frac{1}{\kappa^2}\int \mu^{-2,-2}\,\frac{1}{(1+t)^2}\,D^{2,0}t\,D^{0,2}t\,. \tag{4.12}$$

Using the transformation law (4.2) of $q^{1,0\,\underline{a}}$, it is easy to check the invariance of the action (4.11) also under the right light-cone branch of the considered superconformal group. This action is also invariant with respect to the second of two $\mathcal{N} = (4,4)$ supeconformal groups defined in Sec. 2. To demonstrate this, one should take the action in the form (4.12) and use the identity (4.6), the constraints (3.11) combined with the constraints on $\Lambda^{0,1\,\underline{a}}_{(II)}$, Eqs. (2.29), and the following commutator

$$[D^{0,2}, \frac{\partial}{\theta^{0,1\,\underline{a}}}] = -2i\,\theta^{0,1}_{\underline{a}}\partial_{--}\,. \tag{4.13}$$

In order to be convinced that the action (4.11) indeed describes $\mathcal{N} = (4,4)$ superextension of $SU(2) \times U(1)$ WZNW model, we give here its component form.

Let us begin with the bosonic part of the action. It is given by a sum of the physical and auxiliary bosonic field terms which, after integrating over Grassmann variables, take the form

$$S_{\rm phb} = \frac{1}{2\kappa^2}\int d^2x \left\{ G_{i\,\underline{a}\,j\,\underline{b}}(q)\;\partial_{++}q^{i\,\underline{a}}\,\partial_{--}q^{j\,\underline{b}} + 2\,B_{i\,\underline{a}\,j\,\underline{b}}(q)\;\partial_{++}q^{i\,\underline{a}}\,\partial_{--}q^{j\,\underline{b}} \right\},$$

$$S_{\rm auxb} = \frac{1}{8\kappa^2}\int d^2x\; G(q)\; F^{a}_{\underline{i}}\, F^{\underline{i}}_{a}, \tag{4.14}$$

where

$$G_{i\,\underline{a}\,j\,\underline{b}}(q) = \int du\,dv\; g(t)\;\varepsilon_{i\,j}\,\varepsilon_{\underline{a}\,\underline{b}} \tag{4.15}$$

$$B_{i\,\underline{a}\,j\,\underline{b}}(q) = \int du\,dv\; g(t)\; u^{1}_{(i}u^{-1}_{j)}\;\varepsilon_{\underline{a}\,\underline{b}} \tag{4.16}$$

$$G(q) = \int du\,dv\; g(t)\,, \quad g(t) = \left.\frac{1}{(1+t)^2}\right|_{\theta=0}. \tag{4.17}$$

It turns out that all the target geometry quantities present in the Lagrangian (including its fermionic part) are eventually expressed through the single object $G(q)$. So, in order to find an explicit formula for the metric $G(q)$ we need to calculate the harmonic integral in (4.17). Following Ref. [15] and choosing $c^2 = 1$, one can fix the freedom with respect to two independent rigid $SU(2)$ groups realized on the indices i and $\underline{a}$, as well as with respect to two rigid $SU(2)$ groups from the left and right branches of the superconformal group *I* (recall the transformation rule (4.4)), in such a way that

$$c^{i\,\underline{a}} = \varepsilon^{i\,\underline{a}}\,, \quad q^{i\,\underline{a}} = \varepsilon^{i\,\underline{a}}\,\rho(x)\,, \quad \rho^2 = \frac{1}{2}\,(q^{i\,\underline{a}}\,q_{i\,\underline{a}})\,. \tag{4.18}$$

In this frame, using (4.6), we find that

$$t = (q^{1,0\,\underline{a}} - c^{1,0\,\underline{a}})\, c^{-1,0}_{\underline{a}} = (\rho - 1)\,. \tag{4.19}$$

Then, the calculation of the harmonic integral yields the simple expression for the metric $G(q)$ as

$$G(q) = \int du\, dv\; g(t) = \rho^{-2}\,. \tag{4.20}$$

Parameterizing the 4×4 matrix of physical bosons as

$$q^{i\,\underline{a}}(x) = e^{u(x)}\, \tilde{q}^{\,i\,\underline{a}}(x), \tag{4.21}$$

where $\tilde{q}^{\,i\,\underline{a}}(x)$ is an unitary $SU(2)$ matrix,

$$\tilde{q}^{\,i\,\underline{a}}\, \tilde{q}^{\,k}_{\underline{a}} = \varepsilon^{k\,i}\,, \quad \tilde{q}^{\,i\,\underline{a}}\, \tilde{q}^{\,\underline{b}}_{i} = \varepsilon^{\underline{b}\,\underline{a}}\,, \tag{4.22}$$

we find that

$$G(q) = e^{-2u(x)}\,. \tag{4.23}$$

So, the metric term in (4.14) is reduced to a sum of the free Lagrangian of the field $u(x)$ and the Lagrangian of the $SU(2)$ principal chiral field sigma model

$$G(q)\, \partial_{++} q^{i\,\underline{a}}\, \partial_{--} q_{i\,\underline{a}} = 2\, \partial_{++} u\, \partial_{--} u + \partial_{++} \tilde{q}^{\,i\,\underline{a}}\, \partial_{--} \tilde{q}_{\,i\,\underline{a}}\,. \tag{4.24}$$

In the present case, the totally antisymmetric (with respect to the permutations of pairs of the indices $i\,\underline{a}\,, j\,\underline{b}\,, ...$) torsion field strength $H_{i\,\underline{a}\,j\,\underline{b}\,k\,\underline{c}}$ defined by the general formula (3.46) is given by the simple expression

$$H_{i\,\underline{a}\,j\,\underline{b}\,k\,\underline{c}} = \varepsilon_{i\,j}\, \varepsilon_{\underline{a}\,\underline{c}}\, \frac{\partial G}{\partial q^{k\,\underline{b}}} - \varepsilon_{i\,k}\, \varepsilon_{\underline{a}\,\underline{b}}\, \frac{\partial G}{\partial q^{j\,\underline{c}}}\,, \tag{4.25}$$

which, taking into account (4.20), is reduced to

$$H_{i\,\underline{a}\,j\,\underline{b}\,k\,\underline{c}} = \rho^{-4} (\varepsilon_{i\,k}\, \varepsilon_{\underline{a}\,\underline{b}}\, q_{j\,\underline{c}} - \varepsilon_{i\,j}\, \varepsilon_{\underline{a}\,\underline{c}}\, q_{k\,\underline{b}})\,. \tag{4.26}$$

After substituting this expression into the torsion term

$$B_{i\,\underline{a}\,j\,\underline{b}}\, \partial_{++} q^{i\,\underline{a}}\, \partial_{--} q^{j\,\underline{b}} = \int_0^1 dt\; H_{i\,\underline{a}\,j\,\underline{b}\,k\,\underline{c}}\, \partial_t\, q^{i\,\underline{a}}\, \partial_{++} q^{j\,\underline{b}}\, \partial_{--} q^{k\,\underline{c}} \tag{4.27}$$

and passing to the parametrization (4.21), the r. h. s. of (4.27) takes the form

$$\int_0^1 dt\; \partial_t\, \tilde{q}_{\,i\,\underline{a}}\, \tilde{q}_{\,j\,\underline{b}}\, (\, \partial_{++} \tilde{q}^{\,j\,\underline{a}}\, \partial_{--} \tilde{q}^{\,i\,\underline{b}} - \partial_{++} \tilde{q}^{\,i\,\underline{b}}\, \partial_{--} \tilde{q}^{\,j\,\underline{a}}\,)\,, \tag{4.28}$$

which is the standard $SU(2)$ WZNW term.

Summing up the above contributions, one may write the final expression for the bosonic part of the action (4.12) as

$$
\begin{aligned}
S^{\rm sc}_{\rm bos} &= \frac{1}{\kappa^2}\int d^2x \,\Big\{ (\,\partial_{++}u\,\partial_{--}u + \frac{1}{2}\,\partial_{++}\tilde q^{\,i\,\underline{a}}\,\partial_{--}\tilde q_{\,i\,\underline{a}}\,) \\
&+ \int_0^1 dt\,\partial_t\,\tilde q_{\,i\,\underline{a}}\,\tilde q_{\,j\,\underline{b}}\,(\,\partial_{++}\tilde q^{\,j\,\underline{a}}\,\partial_{--}\tilde q^{\,i\,\underline{b}} - \partial_{++}\tilde q^{\,i\,\underline{b}}\,\partial_{--}\tilde q^{\,j\,\underline{a}}\,) \\
&+ \frac{1}{8}\,e^{-2u}\,F^{a}_{\underline{k}}\,F^{\underline{k}}_{a}\Big\}\,. \qquad (4.29)
\end{aligned}
$$

Let us now turn to the fermionic sector. The fermionic part of the component action consists of three pieces

$$
S^{\rm sc}_{\rm ferm} = S_{\rm 4f} + S_{\rm auxf} + S_{\rm kinf} \qquad (4.30)
$$

which correspond, respectively, to the term quartic in fermionic fields, a term involving auxiliary fields and the kinetic term of fermions. These are as follows

$$
S_{\rm 4f} = \frac{1}{16\kappa^2}\int d^2x\,\frac{\partial^2 G}{\partial q^{i\,\underline{a}}\,\partial q^{k\,\underline{b}}}\,\beta^i_a\,\beta^{k\,a}\,\alpha^{\underline{a}}_{\underline{n}}\,\alpha^{\underline{n}\,\underline{b}}\,, \qquad (4.31)
$$

$$
S_{\rm auxf} = \frac{1}{4\kappa^2}\int d^2x\,\frac{\partial G}{\partial q^{i\,\underline{a}}}\,F^{\underline{n}\,a}\,\beta^i_a\,\alpha^{\underline{a}}_{\underline{n}}\,, \qquad (4.32)
$$

$$
\begin{aligned}
S_{\rm kinf} = \frac{1}{4i\kappa^2}\int d^2x\,\Big\{ & G\,(\alpha^{\underline{a}}_{\underline{n}}\,\partial_{--}\alpha^{\underline{n}}_{\underline{a}} + \beta^i_a\,\partial_{++}\beta^a_i) \\
& - \frac{\partial G}{\partial q^{i\,\underline{a}}}\,(\alpha^{\underline{a}}_{\underline{n}}\,\alpha^{\underline{n}}_{\underline{b}}\,\partial_{--}q^{i\,\underline{b}} + \beta^i_a\,\beta^a_k\,\partial_{++}q^{k\,\underline{a}})\Big\}\,. \qquad (4.33)
\end{aligned}
$$

Using the explicit expressions (4.18), (4.20), (4.23) for G, one observes:

i) After the field redefinition

$$
\tilde F^{\underline{n}\,a} = F^{\underline{n}\,a} + e^{-u}\,\tilde q^{\,k\,\underline{b}}\,\beta^a_k\,\alpha^{\underline{n}}_{\underline{b}} \qquad (4.34)
$$

the sum of $S_{\rm 4f}$ and $S_{\rm auxf}$ is entirely cancelled by the contribution coming from $S^{\rm sc}_{\rm bos}$. Thus the *off-shell* action does not contain 4-fermionic term which is present in the generic action. The full auxiliary fields part of the action takes the simple form

$$
S^{\rm sc}_{\rm aux} = \frac{1}{8\kappa^2}\int d^2x\,e^{-2u}\,\tilde F^{a}_{\underline{k}}\,\tilde F^{\underline{k}}_{a}\,. \qquad (4.35)
$$

ii) Being written through redefined fermionic fields

$$\alpha^{i\,\underline{n}} = e^{-u}\,\tilde{q}^{\,i\,\underline{a}}\,\alpha^{\underline{n}}_{\underline{a}}\,, \quad \beta^{a\underline{b}} = e^{-u}\,\tilde{q}^{\,i\,\underline{b}}\,\beta^{a}_{i}\,, \tag{4.36}$$

$S_{\rm kinf}$ is reduced to a sum of the free fermionic terms

$$S_{\rm kinf} = \frac{i}{4\kappa^2}\int d^2x\,(\alpha^{i\,\underline{n}}\,\partial_{--}\alpha_{i\,\underline{n}} + \beta^{a\,\underline{b}}\,\partial_{++}\beta_{a\,\underline{b}})\,. \tag{4.37}$$

Up to a redefinition of $SU(2)$ indices, the full action coincides with the component action of $\mathcal{N}=(4,4)$ WZNW model based on the multiplet $q^{1,1}$ [15].

5. POTENTIAL TERMS OF $\hat{q}^{\,1,0\,\underline{a}}$ AND $\hat{q}^{\,\underline{i}\,\underline{a}}$

Usually, the potential (or mass) terms and, in particular, mass term in $\mathcal{N}=2$, $4D$ sigma-models actions are generated as a result of including central charges into the algebra of supersymmetry [31,32]. It was observed for the first time in [6] that in the $\mathcal{N}=(4,4)$ sigma models there exists a possibility to construct such terms without changing the supersymmetry algebra. The explicitly elaborated example is the $\mathcal{N}=4$ $SU(2)$ WZNW – Liouville system of Refs. [6, 7, 11] which is a superconformally invariant deformation of $\mathcal{N}=(4,4)$ supersymmetric $SU(2)\times U(1)$ WZNW model. As shown from [15], in the $SU(2)\times SU(2)$ HSS the off-shell mass terms of $q^{1,1}$ multiplet are defined in a unique way and result, at the component level, in the scalar potential fully specified by the bosonic target metric. Here we construct similar terms for $\hat{q}^{\,1,0\,\underline{a}}$ and $\hat{q}^{\,\underline{i}\,\underline{a}}$ multiplets and, as an example, present a massive deformation of the superconformal action (4.11). Mass terms which involve twisted multiplets of different types will be discussed in Sec. 6.

Keeping in mind that the superfunction $q^{1,0\,\underline{a}}$ and the integration measure $\mu^{-2,-2}$ are dimensionless, the only way to construct a mass term is to allow explicit θ's in the action. The simplest term of this kind for the multiplet $\hat{q}^{\,1,0\,\underline{a}}$ reads

$$S^{m}_{(a)} = m\int \mu^{-2,-2}\,\theta^{1,0\,\underline{i}}\,\theta^{0,1\,\underline{a}}\,C^{0,1}_{\underline{i}\,\underline{a}\,\underline{b}}(u,v)\,q^{1,0\,\underline{b}}, \tag{5.1}$$

where the harmonic dependence of $C^{0,1}_{\underline{i}\,\underline{a}\,\underline{b}}(u,v)$ is unspecified for the moment. It is easy to show that the requirement of invariance of this term under the supersymmetry transformation constrains $C^{0,1}_{\underline{i}\,\underline{a}\,\underline{b}}$ in the following way

$$C^{0,1}_{\underline{i}\,\underline{a}\,\underline{b}} = \varepsilon_{\underline{a}\,\underline{b}}\,C^{0,1}_{\underline{i}}\,, \quad D^{0,2}C^{0,1}_{\underline{i}} = 0\,, \quad D^{2,0}C^{0,1}_{\underline{i}} = 0\,. \tag{5.2}$$

To this end, one represents the supertranslations of $\theta^{0,1\,\underline{a}}$ and $\theta^{1,0\,\underline{i}}$ as

$$\delta\theta^{0,1\,\underline{a}} = \varepsilon^{b\,\underline{a}}\,v^{1}_{b} = D^{0,2}\varepsilon^{b\,\underline{a}}\,v^{-1}_{b}\,, \quad \delta\theta^{1,0\,\underline{i}} = \varepsilon^{k\,\underline{i}}\,u^{1}_{k} = D^{2,0}\varepsilon^{k\,\underline{i}}\,u^{-1}_{k}\,, \tag{5.3}$$

integrates by parts with respect to $D^{0,2}$, $D^{2,0}$ and uses the harmonic constraint (3.11). The general solution of (5.2) is

$$C^{0,1}_{\underline{k}} = C^{a}_{\underline{k}}\, v^1_a, \tag{5.4}$$

where $C^a_{\underline{k}}$ are some constants.

Let us examine how adding of (5.1) to the generic action (3.19) affects the component structure of the latter. After integrating over Grassmann and harmonic variables, one finds the *off-shell* component action

$$S^m_{(a)} = -\frac{m}{4}\int d^2x\, C^{\underline{k}\,a}\, F_{\underline{k}\,a}\,. \tag{5.5}$$

Then, after eliminating the auxiliary fields in the sum $S_q + S^m_{(a)}$, the physical component action acquires new terms which are expressed through the scalar metric $G(q)$ defined in Eq. (3.34). We present here only the potential term of $q^{1,0\,\underline{a}}$

$$S^{\rm pot}_{(a)} = \frac{m^2}{8}\int d^2x\, G^{-1}(q)\, C^{\underline{k}\,a}\, C_{\underline{k}\,a}\,. \tag{5.6}$$

It is accompanied by proper Yukawa-type couplings with fermionic fields. Note that (5.6) yields a nontrivial scalar potential for bosonic fields (including a possible mass term) only for non-constant function $G(q)$; so no mass terms can be generated in this way starting from the free kinetic action of single twisted multiplet. Yet, such terms can be generated in the system of two twisted multiplets of different types, see Sec. 6.

We wish to point out that the off-shell term (5.1), being simplest, is at the same time unique. Allowing any higher powers of $q^{1,0\,\underline{a}}$, and/or of analytic Grassmann coordinates, would require that harmonic functions with negative $U(1)$ charges must be included, and the harmonic differential constraints imposed on these functions by supersymmetry can be shown to make them vanish.

The mass term for $\hat{q}^{\,\underline{i}\,\underline{a}}$ multiplet can be written in the following form

$$S^m_{(c)} = m\int \mu^{-2,-2}\,\theta^{1,0\,\underline{i}}\,\theta^{0,1\,\underline{a}}\,C^{1,1}_{\underline{i}\,\underline{a}\,\underline{k}\,\underline{b}}(u,v)\,\tilde{q}^{\underline{k}\,\underline{b}}\,, \tag{5.7}$$

with C being a set of harmonic-dependent constants which are arbitrary for the moment. As in the case of $\hat{q}^{\,1,0\,\underline{a}}$, it is easy to show that the supersymmetry condition

$$\delta\, S^m_{(c)} = 0 \tag{5.8}$$

is satisfied provided the harmonic functions C have the structure

$$C^{1,1}_{\underline{i}\,\underline{a}\,\underline{k}\,\underline{b}} = C^{\,1,1}\varepsilon_{\underline{i}\,\underline{k}}\,\varepsilon_{\underline{a}\,\underline{b}} \tag{5.9}$$

and obey the harmonic constraints

$$D^{2,0}C^{\,1,1} = 0\,, \quad D^{0,2}C^{\,1,1} = 0\,, \tag{5.10}$$

which have the simple unique solution $C^{\,1,1} = C^{i\,a}\,u^1_i\,v^1_a$. After integrating over Grassmann and harmonic variables the corresponding *off-shell* component action reads

$$S^m_{(c)} = -\frac{m}{4}\int d^2x\; C^{k\,a}\,F_{k\,a}\,. \tag{5.11}$$

The potential term of $\hat q^{\,\underline{i}\,\underline{a}}$, which arises in the sum of $S_q + S^m_{(c)}$ after eliminating the auxiliary fields, is expressed through the metric $\hat G(q)$ (3.39)

$$S^{\rm pot}_{(c)} = \frac{m^2}{16}\int d^2x\; \hat G^{-1}\,C^{k\,a}\,C_{k\,a}\,. \tag{5.12}$$

We observe that the potential (mass) terms of new types of twisted multiplet have the same form as that given in [15] for the case of the $q^{1,1}$ multiplet (up to the proper replacements of $SU(2)$ indices).

As the last example of this Section, we discuss a massive deformation of the superconformal action (4.12):

$$S^{\rm sc}_m = \frac{1}{\kappa^2}\int \mu^{-2,-2}\left\{\frac{1}{(1+t)^2}\,D^{2,0}t\,D^{0,2}t + 2\,m\,\theta^{1,0\,\underline{i}}\,\theta^{0,1\,\underline{a}}\,C^{\,0,1}_{\underline{i}}\,q^{1,0}_{\underline{a}}\right\}. \tag{5.13}$$

In terms of component fields the mass term in (5.13) is

$$S_m = -\frac{m}{2}\int d^2x\; F^{\underline{i}\,a}\,\varepsilon_{\underline{i}\,a}\,. \tag{5.14}$$

After eliminating auxiliary fields, it gives rise to the following physical component action of deformed $SU(2)\times U(1)$ WZNW sigma model

$$S^{\rm sc}_m = S^{\rm sc}_{\rm bos}(F_{\underline{i}\,a}=0) + S_{\rm kinf} + S_m, \tag{5.15}$$

where $S^{\rm sc}_{\rm bos}$ and $S_{\rm kinf}$ are given by Eqs. (4.35), (4.37) and

$$S_m = \frac{1}{\kappa^2}\int d^2x\;\left\{m^2\,e^{2u} + \frac{m}{2}\,e^{-u}\,\varepsilon^{\underline{k}\,b}\,\tilde q_{i\,\underline{a}}\,\beta^i_b\,\alpha^{\underline{a}}_{\underline{k}}\right\}. \tag{5.16}$$

After rescaling the fields as

$$\alpha\to 2\alpha\,,\quad \beta\to 2\beta\,,\quad F\to 2F\,, \tag{5.17}$$

one finds that the resulting piece in the full action coincides, up to an overall normalization, with the analogous one for the superconformal WZNW model of $q^{1,1}$ [15]. The superconformal properties of the modified action are also the same as in [15], up to the proper reshuffling of superconformal groups in the left and right sectors.

6. THE ACTIONS WITH FEW TYPES OF TWISTED MULTIPLETS

This Section is devoted to the proof that the general sigma-model action of a pair of two different twisted multiplets is split into a sum of two actions, each involving only one multiplet. From this result (and its extension to the case of larger number of multiplets) we conclude that it is impossible to construct a nontrivial supersymmetric sigma-model action which would contain interactions among different twisted multiplets. Nevertheless, it turns out possible to construct mass terms including the pairs of multiplets which are «dual» to each other in the sense defined in Sec. 3. We show that the structure of these terms is uniquely fixed by supersymmetry, as in the case of mass terms for separate multiplets discussed in the previous Section.

6.1. Sigma-Model Actions. To proceed, let us again apply to the description of our four twisted supermultiplets in the original $\mathcal{N}=(4,4)$, 2D superspace. As given in (1.1), each of these multiplets carries two indices of the full $SO(4)_L \times SO(4)_R$ automorphism group of the theory. One of these indices, the left index i or $\underline{i}$, corresponds to one of two $SU(2)$ factors of $SO(4)_L$, while another, right index a or $\underline{a}$, corresponds to one of two $SU(2)$ factors of $SO(4)_R$. Since $SU(2)$ groups in a given light-cone sector are on completely equal footing, and the left and right sectors are related to each other via the reflection $+ \leftrightarrow -$, there exist only *two* non-equivalent options of singling out a pair in the set (1.1). One of these possibilities is to pair multiplets having one $SU(2)$ index in common, e. g. $\hat{q}^{\,i\,a}$ and $\hat{q}^{\,i\,\underline{a}}$, $\hat{q}^{\,i\,a}$ and $\hat{q}^{\,\underline{i}\,a}$, etc. Another possibility is to pair those having no $SU(2)$ indices in common at all. In this case there are only two variants: $\hat{q}^{\,i\,a}$ and $\hat{q}^{\,\underline{i}\,\underline{a}}$, or $\hat{q}^{\,i\,\underline{a}}$ and $\hat{q}^{\,\underline{i}\,a}$.

Keeping in mind these equivalences between various choices, we, without loss of generality, can restrict our attention to considering most general actions of the pairs $(\hat{q}^{\,i\,a}\,,\ \hat{q}^{\,i\,\underline{a}})$ and $(\hat{q}^{\,i\,\underline{a}}\,,\ \hat{q}^{\,\underline{i}\,a})$ as the two essentially different possibilities. In the $SU(2)\times SU(2)$ analytic HSS the most general candidate Lagrangians for these two options are given by

$$(I) \quad \mathcal{L}^{2,2}_{I}\left(q^{1,1}\,,\ q^{1,0\,\underline{a}}\,,\ g^{1,1}\,,\ \theta^{0,1\,\underline{a}}, u^{\pm 1}_{i}, v^{\pm 1}_{b}\right)\,, \tag{6.1}$$

$$(II) \quad \mathcal{L}^{2,2}_{II}\left(q^{0,1\,\underline{i}}\,,\ q^{1,0\,\underline{a}}\,,\ f^{1,1}\,,\ g^{1,1}\,,\ \theta^{1,0\,\underline{i}}, \theta^{0,1\,\underline{a}}, u^{\pm 1}_{i}, v^{\pm 1}_{b}\right)\,. \tag{6.2}$$

This choice of two non-equivalent pairs is optional. We prefer it just because it is technically easier compared to other possible choices.

Before turning to the general case, let us start with instructive simple examples of actions which are bilinear in the twisted multiplet superfields. There are two essentially different types of such actions: the actions containing only one kind of multiplet and those containing two different kinds.

As an example of the first type of actions, we consider the most general quadratic action of the $\hat{q}^{\,1,0\,\underline{a}}$ multiplet. Taking into account the harmonic constraints (3.11) and the freedom of integrating by parts with respect to harmonic derivatives, it can be written in the analytic HSS as

$$S^{\rm quad} = \int \mu^{-2,-2} \left\{ C^{0,2}_{(\underline{a}\,\underline{b})}(v)\, q^{1,0\,\underline{a}} q^{1,0\,\underline{b}} + C^{0,0}_{[\underline{a}\,\underline{b}]}(v)\, q^{1,0\,\underline{b}} D^{0,2} q^{1,0\,\underline{a}} \right\}. \quad (6.3)$$

Requirement of $\mathcal{N}=(4,4)$ supersymmetry amounts to the following conditions on the harmonic-dependent functions $C(v)$

$$C^{0,2}_{(\underline{a}\,\underline{b})} = 0\,, \quad D^{0,2} C^{0,0}_{[\underline{a}\,\underline{b}]} = 0\,, \quad \Rightarrow \quad C^{0,0}_{[\underline{a}\,\underline{b}]} \sim \varepsilon_{\underline{a}\,\underline{b}} \quad (6.4)$$

whence the free action (3.50) is unambiguously recovered. The general actions quadratic in other sorts of multiplets from the set (1.1) are also reduced to the relevant free actions.

As an example of the second possibility, we consider a bilinear action of the pair $(q^{1,1}, q^{1,0\,\underline{a}})$

$$\tilde{S}^{\rm quad} = \int \mu^{-2,-2} \left\{ C^{0,1}_{\underline{a}} q^{1,1} q^{1,0\,\underline{a}} + B^{0,-1}_{\underline{a}} q^{1,1} D^{0,2} q^{1,0\,\underline{a}} \right\}, \quad (6.5)$$

where C and B are arbitrary harmonic constants. Keeping in mind the defining constraint (2.14) for $q^{1,1}$, the second term is reduced to the first one modulo a total harmonic derivative. Then the requirement of $\mathcal{N}=(4,4)$ supersymmetry leads to

$$C^{0,1}_{\underline{a}} = 0\,, \quad (6.6)$$

i. e. the supersymmetry requires (6.5) to vanish.

Inspecting the bilinear actions of other pairs of different multiplets, it is easy to prove that the requirement of invariance under the $\mathcal{N}=(4,4)$ supersymmetry transformations also implies these actions to vanish.

Action for $\hat{q}^{\,i\,a}$ *and* $\hat{q}^{\,i\,\underline{a}}$. We start with examining the first option in (6.1), (6.2). As in the case of one multiplet $\hat{q}^{\,1,0\,\underline{a}}$ (Sec. 3), the use of constraints (3.11) leads us to the following most general form of the action with $\mathcal{L}^{2,2}_I$ given in (6.1)

$$S = \int \mu^{-2,-2} \left\{ \mathcal{L}^{2,2}_0(q^{1,1}, q^{1,0\,\underline{a}}, u, v) + \mathcal{L}^{1,0}_{\underline{a}}(q^{1,1}, q^{1,0\,\underline{a}}, u, v)\, D^{0,2} q^{1,0\,\underline{a}} \right\}. \, (6.7)$$

The superfield $q^{1,1}$ enters the superpotentials in (6.7) as an extra argument, and this dependence cannot affect the process of deriving the constraints imposed on these potentials by supersymmetry since $q^{1,1}$ is a scalar superfield. Thus, the requirement that variation of the action (6.7) under the supersymmetry transformations is proportional to a total harmonic derivative gives rise to a single constraint which coincides with (3.23).

To reveal all consequences of this constraint in the case under consideration, we should plug into the component structure of the action (6.7). We limit our consideration to the bosonic part; the conclusions we shall arrive at are equally valid for the fermionic part.

After integrating over the Grassmann coordinates, we find that the terms with x-derivatives contain both diagonal and non-diagonal pieces

$$\begin{aligned} S^{\rm bos}_{\rm kin} = \int d^2x\, du\, dv\, \Bigg\{ & -\frac{\partial^2 \mathcal{L}^{2,2}_0}{(\partial q^{1,1})^2}\, \partial_{++} q^{i\,a}\, \partial_{--} q^{j\,b}\, u^{-1}_i u^1_j v^1_a v^{-1}_b \\ & + \frac{\partial \mathcal{L}^{1,0}_{\underline{a}}}{\partial q^{1,0\,\underline{b}}}\, \partial_{++} q^{j\,\underline{b}}\, \partial_{--} q^{i\,\underline{a}}\, u^1_i u^{-1}_j \\ - \frac{\partial^2 \mathcal{L}^{2,2}_0}{\partial q^{1,1} \partial q^{1,0\,\underline{a}}}\, \partial_{++} q^{i\,\underline{a}}\, \partial_{--} q^{j\,b}\, u^1_j u^{-1}_i v^{-1}_b & + \frac{\partial \mathcal{L}^{1,0}_{\underline{a}}}{\partial q^{1,1}}\, \partial_{++} q^{j\,b}\, \partial_{--} q^{i\,\underline{a}}\, u^1_i u^{-1}_j v^1_b \\ & - \frac{\partial \mathcal{L}^{2,2}_0}{\partial q^{1,1}}\, \partial_{++}\partial_{--} q^{i\,a}\, u^{-1}_i v^{-1}_a + \mathcal{L}^{1,0}_{\underline{a}}\, \partial_{++}\partial_{--} q^{i\,\underline{a}}\, u^{-1}_i \Bigg\}. \end{aligned} \quad (6.8)$$

Integrating by parts in the last two terms in (6.8) and using the constraint (3.23), one can easily check that the kinetic bosonic part of the action takes a diagonal form, i. e. reduces to a sum of kinetic terms of the physical bosons from the multiplets $q^{1,1}$ and $\hat{q}^{\,1,0\,\underline{a}}$. To obtain the auxiliary field part of the bosonic component action, we integrate over Grassmann coordinates with the help of the constraint (3.23) and find that this part is also reduced to a sum of two diagonal pieces. Each of these pieces coincides with the auxiliary field part of the bosonic component action of the relevant multiplet. Thus, collecting the results of our calculations for the kinetic and the auxiliary pieces of the action, we conclude that the bosonic component action for the pair of multiplets splits into a sum of two parts

$$\begin{aligned} S_1 = \frac{1}{2} \int d^2x\, \Big\{ & G_{i\,a\,j\,b}\, \partial_{++} q^{i\,a}\, \partial_{--} q^{j\,b} \\ & + 2\, B_{i\,a\,j\,b}\, \partial_{++} q^{i\,a}\, \partial_{--} q^{j\,b} + \frac{1}{4}\, G_1\, F^{\underline{k}}_{\underline{b}}\, F^{\underline{b}}_{\underline{k}} \Big\}, \end{aligned} \quad (6.9)$$

$$\begin{aligned} S_2 = \frac{1}{2} \int d^2x\, \Big\{ & G_{i\,\underline{a}\,j\,\underline{b}}\, \partial_{++} q^{i\,\underline{a}}\, \partial_{--} q^{j\,\underline{b}} \\ & + 2\, B_{i\,\underline{a}\,j\,\underline{b}}\, \partial_{++}\, q^{i\,\underline{a}} \partial_{--} q^{j\,\underline{b}} + \frac{1}{4}\, G_2\, F^{\underline{i}}_{a}\, F^{a}_{\underline{i}} \Big\}, \end{aligned} \quad (6.10)$$

where

$$\begin{aligned} G_{i\,a\,j\,b} &= \int du\, dv\, g_1(q,u,v)\, \varepsilon_{i\,j}\, \varepsilon_{a\,b}, \\ B_{i\,a\,j\,b} &= \int du\, dv\, g_1(q,u,v)\, \varepsilon_{b\,a}\, u^1_{(i} u^{-1}_{j)}\,, \end{aligned} \quad (6.11)$$

$$G_{i\,\underline{a}\,j\,\underline{b}} = \int du\, dv\, g_2(q,u,v)\, \varepsilon_{i\,j}\, \varepsilon_{\underline{a}\,\underline{b}}\,,$$
$$B_{i\,\underline{a}\,j\,\underline{b}} = \int du\, dv\, g_2(q,u,v)\, \varepsilon_{\underline{b}\,\underline{a}}\, u^1_{(i}u^{-1}_{j)}\,, \tag{6.12}$$

$$G_1 = \int du\, dv\, g_1\,, \quad g_1(q,u,v) = \left.\frac{\partial^2 \mathcal{L}^{2,2}_0(q^{1,1}, q^{1,0\,\underline{a}}, u, v)}{\partial q^{1,1}\,\partial q^{1,1}}\right|_{\theta=0}\,,$$
$$G_2 = \int du\, dv\, g_2\,, \quad g_2(q,u,v) = \left.\frac{\partial \mathcal{L}^{1,0\,\underline{a}}(q^{1,1}, q^{1,0\,\underline{a}}, u)}{\partial q^{1,0\,\underline{a}}}\right|_{\theta=0}$$

(see (2.16) for the definition of bosonic components of $q^{1,1}$).

Let us make a remark about the structure of the torsion potential. The expression for $B_{i\,a\,j\,b}$ given by Eq. (6.11) differs from the one originally obtained in [15] (we denote it $\tilde{B}_{i\,a\,j\,b}$). On the other hand, the torsion potential is defined up to the gauge transformation

$$B'_{i\,a\,j\,b} = B_{i\,a\,j\,b} + \frac{\partial X_{i\,a}}{\partial q^{j\,b}} - \frac{\partial X_{j\,b}}{\partial q^{i\,a}}\,, \tag{6.13}$$

which leaves invariant the torsion field strength. It is easy to see that the difference between $B_{i\,a\,j\,b}$ defined in (6.11) and $\tilde{B}_{i\,a\,j\,b}$ of Ref. [15] is just the above gauge transformation corresponding to the special choice of gauge parameter

$$X_{i\,a} = \int du\, dv\, \frac{\partial \mathcal{L}^{2,2}}{\partial q^{1,1}}\, u^1_i v^1_a\,. \tag{6.14}$$

Keeping in mind this remark, we conclude that the actions S_1 and S_2 given by Eqs. (6.9) and (6.10) have the same structure as the sigma-model actions of separate multiplets. But, unlike the previous case, the scalar functions G_1 and G_2, as well as the torsion potentials, can, in principle, bear dependence on physical bosonic fields of both multiplets. Nevertheless, it is easy to check that both scalar functions and torsion potentials are independent of $q^{i\,a}$ and $q^{i\,\underline{a}}$, for S_2 and S_1, respectively.

Indeed, using the constraint (3.23) at $\theta = 0$ and integrating by parts in harmonic integral, one finds

$$\begin{aligned}
\frac{\partial G_1}{\partial q^{i\,\underline{a}}} &= \int du\, dv\, \frac{\partial^3 \mathcal{L}^{2,2}_0}{(\partial q^{1,1})^2\, q^{1,0\,\underline{a}}}\, u^1_i = \int du\, dv\, \partial^{0,2}\, \frac{\partial^2 \mathcal{L}^{1,0}_{\underline{a}}}{(\partial q^{1,1})^2}\, u^1_i = 0\,,\\
\frac{\partial B_{i\,a\,j\,b}}{\partial q^{k\,\underline{d}}} &= \int du\, dv\, \frac{\partial^3 \mathcal{L}^{2,2}_0}{(\partial q^{1,1})^2\, q^{1,0\,\underline{d}}}\, \varepsilon_{b\,a}\, u^1_k u^1_{(i}u^{-1}_{j)}\\
&= \int du\, dv\, \partial^{0,2}\, \frac{\partial^2 \mathcal{L}^{1,0}_{\underline{d}}}{(\partial q^{1,1})^2}\, \varepsilon_{b\,a}\, u^1_k u^1_{(i}u^{-1}_{j)} = 0\,.
\end{aligned} \tag{6.15}$$

The same conclusions about splitting into a sum of separate actions can be made for the terms including fermionic fields. Thus the superfield action (6.7) actually amounts to a sum of superfield actions for $q^{1,1}$ and $\hat{q}^{\,1,0\,\underline{a}}$.

Action for $\hat{q}^{\,i\,\underline{a}}$ *and* $\hat{q}^{\,\underline{i}\,a}$. In the general case the initial action of the multiplets $\hat{q}^{\,i\,\underline{a}}$ and $\hat{q}^{\,\underline{i}\,a}$ can be written in the analytic subspace (2.7) in the form

$$S = \int \mu^{-2,-2}\,\mathcal{L}^{2,2}_{II}\,(q^{0,1\,\underline{i}}, q^{1,0\,\underline{a}}, f^{1,1}, g^{1,1}, \theta^{1,0\,\underline{i}}, \theta^{0,1\,\underline{a}}, u, v)\,. \tag{6.16}$$

As discussed in Subsec. 3.2, using the constraints (3.11), (3.12), we can unfold the Largangian in (6.16) in such a way that it involves only the superfunctions $q^{1,0\,\underline{a}}$ and $q^{0,1\,\underline{i}}$

$$\begin{aligned}\mathcal{L}^{2,2} &= \mathcal{L}^{2,2}_0(q^{1,0\,\underline{a}}, q^{0,1\,\underline{i}}, u, v) + \mathcal{L}^{-1,-1}_{\underline{i}\,\underline{a}}(q^{1,0\,\underline{a}}, q^{0,1\,\underline{i}}, u, v)\,D^{2,0}q^{0,1\,\underline{i}}\,D^{0,2}q^{1,0\,\underline{a}} \\ &+ \mathcal{L}^{1,0}_{\underline{a}}(q^{1,0\,\underline{a}}, q^{0,1\,\underline{i}}, u, v)\,D^{0,2}q^{1,0\,\underline{a}} + \mathcal{L}^{0,1}_{\underline{i}}(q^{1,0\,\underline{a}}, q^{0,1\,\underline{i}}, u, v)\,D^{2,0}q^{0,1\,\underline{i}}\,.\end{aligned} \tag{6.17}$$

To find the constraints on the superpotentials in (6.17), we once again demand that the variation of (6.17) under the supersymmetry transformation is a sum of total harmonic derivatives of arbitrary functions with the proper harmonic $U(1)$ charges:

$$\delta\,\mathcal{L}^{2,2} = D^{2,0}\,(\varepsilon^{-1,0\,\underline{i}}\,A^{1,2}_{\underline{i}} + \varepsilon^{0,-1\,\underline{a}}\,B^{0,3}_{\underline{a}}) + D^{0,2}\,(\varepsilon^{-1,0\,\underline{i}}\,C^{3,0}_{\underline{i}} + \varepsilon^{0,-1\,\underline{a}}\,D^{2,1}_{\underline{a}})\,. \tag{6.18}$$

The functions in the r. h. s. of (6.18) depend on the same arguments as the Lagrangian in (6.16). Comparing the coefficients of $\varepsilon^{1,0\,\underline{i}}$ and $\varepsilon^{0,1\,\underline{a}}$ in both sides of (6.18), we find that the functions A and D have the following structure

$$A^{1,2}_{\underline{i}} = -f^{1,1}(\mathcal{L}^{0,1}_{\underline{i}} + \mathcal{L}^{-1,-1}_{\underline{i}\,\underline{a}}D^{2,0}q^{0,1\,\underline{i}}),\ D^{2,1}_{\underline{a}} = -g^{1,1}(\mathcal{L}^{1,0}_{\underline{a}} + \mathcal{L}^{-1,-1}_{\underline{i}\,\underline{a}}D^{0,2}q^{1,0\,\underline{a}}). \tag{6.19}$$

Then, it is easy to demonstrate that the most general structure of the Grassmann functions B and C in (6.18), compatible with the Lorentz covariance and dimensional considerations, is given by

$$B^{0,3}_{\underline{a}} = g^{1,1}\,(b^{-1,2}_{\underline{a}} + b^{-3,1}_{\underline{i}\,\underline{a}}D^{2,0}q^{0,1\,\underline{i}}),\ C^{3,0}_{\underline{i}} = f^{1,1}\,(c^{2,-1}_{\underline{i}} + c^{1,-3}_{\underline{i}\,\underline{a}}D^{0,2}q^{1,0\,\underline{a}}), \tag{6.20}$$

where the functions b's and c's depend on $q^{1,0\,\underline{a}}$, $q^{0,1\,\underline{i}}$ and harmonics u, v. The set of constraints which follows from (6.18) contains these functions

$$\begin{aligned}&\frac{\partial\mathcal{L}^{2,2}_0}{\partial q^{1,0\,\underline{a}}} = \partial^{0,2}\mathcal{L}^{1,0}_{\underline{a}} - \partial^{2,0}b^{-1,2}_{\underline{a}}\,, \quad \frac{\partial\mathcal{L}^{2,2}_0}{\partial q^{0,1\,\underline{i}}} = \partial^{2,0}\mathcal{L}^{0,1}_{\underline{i}} - \partial^{0,2}c^{2,-1}_{\underline{i}}\,, \\ &\frac{\partial\mathcal{L}^{0,1}_{\underline{i}}}{\partial q^{1,0\,\underline{a}}} = \partial^{0,2}\mathcal{L}^{-1,-1}_{\underline{i}\,\underline{a}} - \frac{\partial b^{-1,2}_{\underline{a}}}{\partial q^{0,1\,\underline{i}}} - \partial^{2,0}b^{-3,1}_{\underline{i}\,\underline{a}}\,, \\ &\frac{\partial\mathcal{L}^{1,0}_{\underline{a}}}{\partial q^{0,1\,\underline{i}}} = \partial^{2,0}\mathcal{L}^{-1,-1}_{\underline{i}\,\underline{a}} - \frac{\partial c^{2,-1}_{\underline{i}}}{\partial q^{1,0\,\underline{a}}} - \partial^{0,2}c^{1,-3}_{\underline{i}\,\underline{a}}\,.\end{aligned} \tag{6.21}$$

The presence of these additional harmonic functions is the difference between this system of constraints and the unique constraint (3.23) for the case of the previous pair of multiplets. However, a closer inspection of the harmonic charge structure of functions b and c leads to the conclusion that these functions can be eliminated from the constraints (6.21) after a proper redefinition of the superfield potentials $\mathcal{L}$'s. Thus, without loss of generality, one can set all these functions equal to zero in (6.21)

$$\frac{\partial \mathcal{L}_0^{2,2}}{\partial q^{1,0\,\underline{a}}} = \partial^{0,2}\mathcal{L}_{\underline{a}}^{1,0}\,, \quad \frac{\partial \mathcal{L}_{\underline{a}}^{1,0}}{\partial q^{0,1\,\underline{i}}} = \partial^{2,0}\mathcal{L}_{\underline{i}\,\underline{a}}^{-1,-1}\,,$$
$$\frac{\partial \mathcal{L}_0^{2,2}}{\partial q^{0,1\,\underline{i}}} = \partial^{2,0}\mathcal{L}_{\underline{i}}^{0,1}\,, \quad \frac{\partial \mathcal{L}_{\underline{i}}^{0,1}}{\partial q^{1,0\,\underline{a}}} = \partial^{0,2}\mathcal{L}_{\underline{i}\,\underline{a}}^{-1,-1}\,. \tag{6.22}$$

As in the previous case, we now need to descend to the components. The parts of the component action involving the physical and auxiliary bosonic fields are obtained in the standard way. After substituting the component expansion of $q^{1,0\,\underline{a}}$, Eq. (3.14), and $q^{0,1\,\underline{i}}$, Eq. (3.15), into (6.17) and integrating over θ's, one finds that:

i) The auxiliary boson part of the action consists of the two pieces

$$S^1_{\rm auxb} = \frac{1}{8}\int d^2x\; G(q^{i\,\underline{a}}, q^{\underline{i}\,a})\; F^a_{\underline{i}}\, F^{\underline{i}}_a\,, \quad S^2_{\rm auxb} = \frac{1}{8}\int d^2x\; \tilde{G}(q^{i\,\underline{a}}, q^{\underline{i}\,a})\; F^k_{\underline{b}}\, F^{\underline{b}}_k, \tag{6.23}$$

where the scalar functions $G(q)$, $\tilde{G}(q)$ are defined by

$$G(q^{i\,\underline{a}}, q^{\underline{i}\,a}) = \int du\, dv\; g(q,\tilde{q},u)\,, \quad g(q,\tilde{q},u) = \left.\frac{\partial \mathcal{L}^{1,0\,\underline{a}}}{\partial q^{1,0\,\underline{a}}}\right|_{\theta=0}\,, \tag{6.24}$$

$$\tilde{G}(q^{i\,\underline{a}}, q^{\underline{i}\,a}) = \int du\, dv\; \tilde{g}(q,\tilde{q},v)\,, \quad \tilde{g}(q,\tilde{q},v) = \left.\frac{\partial \mathcal{L}^{0,1\,\underline{k}}}{\partial q^{0,1\,\underline{k}}}\right|_{\theta=0}\,. \tag{6.25}$$

Here we denoted

$$q \sim \left. q^{1,0\,\underline{a}}\right|_{\theta=0}\,, \quad \tilde{q} \sim \left. q^{0,1\,\underline{k}}\right|_{\theta=0}\,. \tag{6.26}$$

The following remark is to the point here. From a straightforward calculation making use of the constraints (6.22), one finds that the off-diagonal bilinear terms of the auxiliary fields coming from the multiplets $\hat{q}^{\,1,0\,\underline{a}}$ and $\hat{q}^{\,0,1\,\underline{i}}$ are cancelled among themselves. Thus, the total action for the auxiliary bosons part is reduced to a sum of two terms

$$S_{\rm auxb} = S^1_{\rm auxb} + S^2_{\rm auxb}\,. \tag{6.27}$$

The form of each term is the same as in the case of the corresponding single multiplet. As a difference, the functions $G(q,\tilde{q})$, $\tilde{G}(q,\tilde{q})$ can now depend on two different sets of physical bosonic fields, those from $q^{1,0\,\underline{a}}$ and $q^{0,1\,\underline{i}}$. We shall return to this issue later.

ii) After integrating over θ's, the physical boson part takes the form

$$S_{\rm phb} = \int d^2x\, du\, dv \Big\{ \mathcal{L}^{1,0}_{\underline{a}}\, \partial_{++}\partial_{--} q^{i\,\underline{a}}\, u^{-1}_i + \frac{\partial \mathcal{L}^{1,0}_{\underline{a}}}{\partial q^{1,0\,\underline{b}}}\, \partial_{++} q^{k\,\underline{b}}\, \partial_{--} q^{i\,\underline{a}}\, u^1_i u^{-1}_k$$
$$+ \mathcal{L}^{0,1}_{\underline{i}}\, \partial_{++}\partial_{--} q^{\underline{i}\,a}\, v^{-1}_a + \frac{\partial \mathcal{L}^{0,1}_{\underline{k}}}{\partial q^{0,1\,\underline{i}}}\, \partial_{++} q^{\underline{k}\,b}\, \partial_{--} q^{\underline{i}\,a}\, v^1_b v^{-1}_a$$
$$- \frac{\partial^2 \mathcal{L}^{2,2}_0}{\partial q^{1,0\,\underline{a}}\, \partial q^{0,1\,\underline{k}}}\, \partial_{++} q^{i\,\underline{a}}\, \partial_{--} q^{\underline{k}\,b}\, u^{-1}_i v^{-1}_b - \mathcal{L}^{-1,-1}_{\underline{k}\,\underline{a}}\, \partial_{++} q^{\underline{k}\,b}\, \partial_{--} q^{i\,\underline{a}}\, u^1_i v^1_b \Big\}. \tag{6.28}$$

As in the previous case, à priori it includes mixed terms with x-derivatives.

Nevertheless, the constraints (6.22) can be used to show that the off-diagonal terms do not contribute. Thus (6.28) is diagonalized

$$S_{\rm phb} = S^1_{\rm phb} + S^2_{\rm phb}, \tag{6.29}$$

where

$$S^1_{\rm phb} = \frac{1}{2}\int d^2x\ \left\{ G_{i\,\underline{a}\,j\,\underline{b}}\,(q^{i\,\underline{a}}, q^{\underline{i}\,a})\ + 2\, B_{i\,\underline{a}\,j\,\underline{b}}\,(q^{i\,\underline{a}}, q^{\underline{i}\,a}) \right\}\, \partial_{++} q^{i\,\underline{a}}\, \partial_{--} q^{j\,\underline{b}}\,, \tag{6.30}$$

$$S^2_{\rm phb} = \frac{1}{2}\int d^2x\ \left\{ \tilde{G}_{\underline{i}\,a\,\underline{j}\,b}\,(q^{i\,\underline{a}}, q^{\underline{i}\,a})\ + 2\, \tilde{B}_{\underline{i}\,a\,\underline{j}\,b}\,(q^{i\,\underline{a}}, q^{\underline{i}\,a}) \right\}\, \partial_{++} q^{\underline{i}\,a}\, \partial_{--} q^{\underline{j}\,b} \tag{6.31}$$

and

$$G_{i\,\underline{a}\,j\,\underline{b}}(q) = \varepsilon_{i\,j}\, \varepsilon_{\underline{a}\,\underline{b}}\, G(q), \quad B_{i\,\underline{a}\,j\,\underline{b}}\,(q) = \int du\, dv\ \varepsilon_{\underline{b}\,\underline{a}}\, u^1_{(i} u^{-1}_{j)}\, g(q, \tilde{q}, u), \tag{6.32}$$

$$\tilde{G}_{\underline{i}\,a\,\underline{j}b}\,(q) = \varepsilon_{\underline{i}\,\underline{j}}\, \varepsilon_{a\,b}\, \tilde{G}(q), \quad \tilde{B}_{\underline{i}\,a\,\underline{j}\,b}\,(q) = \int du\, dv\ \varepsilon_{\underline{i}\,\underline{j}}\, v^1_{(a} v^{-1}_{b)}\, \tilde{g}(q, \tilde{q}, v). \tag{6.33}$$

We see that at the component level the sigma-model action of the pair of multiplets (6.17) is also reduced to a sum of two independent sigma-model actions. A difference of these actions from the original sigma-model actions for each multiplet is the presence of mixed dependence on both sets of physical bosons $q^{i\,\underline{a}}$ and $q^{\underline{i}\,a}$ in each metric function G and $\tilde{G}$ and in the torsion potentials. Let us now demonstrate that the supersymmetry constraints actually require both the metric functions and torsion potentials to depend only on their «own» types of the physical bosons.

We show this for the function G and the corresponding torsion term B. Let us consider the derivative

$$\frac{\partial G(q)}{\partial q^{\underline{i}\,a}} = \int du\,dv\,\frac{\partial^2 \mathcal{L}^{1,0\,\underline{a}}}{\partial q^{1,0\,\underline{a}}\,\partial q^{0,1\,\underline{i}}}\,v^1_a\,. \tag{6.34}$$

Using the constraint

$$\frac{\partial \mathcal{L}^{1,0}_{\underline{a}}}{\partial q^{0,1\,\underline{i}}} = \partial^{2,0}\mathcal{L}^{-1,-1}_{\underline{i}\,\underline{a}}\,, \tag{6.35}$$

we find that, after integrating by parts with respect to the harmonic derivative $\partial^{2,0}$, the r. h. s. of (6.34) vanishes, hence $G(q)$ does not depend on $q^{\underline{i}\,a}$.

Analogously, for the torsion potential $B_{i\,\underline{a}\,j\,\underline{b}}$ one finds

$$\frac{\partial}{\partial q^{\underline{k}\,c}} B_{i\,\underline{a}\,j\,\underline{b}}\,(q,\tilde{q}) = \int du\,dv\,\varepsilon_{\underline{b}\,\underline{a}}\,u^1_{(i}u^{-1}_{j)}\,\frac{\partial^2 \mathcal{L}^{1,0\,\underline{d}}}{\partial q^{1,0\,\underline{d}}\,\partial q^{0,1\,\underline{k}}}\,v^1_c\,. \tag{6.36}$$

Integrating by parts with respect to the harmonic derivative $\partial^{0,2}$ ($v^1_a = \partial^{0,2}v^{-1}_a$) and using the constraint

$$\frac{\partial \mathcal{L}^{2,2}_0}{\partial q^{1,0\,\underline{a}}} = \partial^{0,2}\mathcal{L}^{1,0}_{\underline{a}}\,, \tag{6.37}$$

we immediately find that (6.36) is vanishing.

Once again, repeating the same analysis for the fermionic terms, one can be convinced that the similar splitting into a sum of independent actions takes place for these terms as well. Hence, this phenomenon persists at the full superfield level for the multiplets $\hat{q}^{\,\underline{i}\,a}$ and $\hat{q}^{\,i\,\underline{a}}$, like in the previously considered case of the multiplets $\hat{q}^{\,i\,a}$ and $\hat{q}^{\,i\,\underline{a}}$.

Adding more multiplets. As the last topic of this Subsection, let us briefly discuss the case when the Lagrangian $\mathcal{L}^{2,2}$ is originally allowed one to depend on three different types of the twisted multiplet, say, on the following triple of superfields

$$(\,q^{1,1}\,, \quad \hat{q}^{\,1,0\,\underline{a}}\,, \quad \hat{q}^{\,0,1\,\underline{i}}\,)\,. \tag{6.38}$$

Firstly we note that the corresponding Lagrangian should have the same structure as in (6.17), because the inclusion of the analytic superfield $q^{1,1}$ as an additional functional parameter in the superpotentials in (6.17) is harmless for its form. The requirement that the action for the triple (6.38) is invariant under the supersymmetry transformations leads to the system of constraints which looks the same as in the case of pair of the multiplets $\hat{q}^{\,1,0\,\underline{a}}$ and $\hat{q}^{\,0,1\,\underline{i}}$, i.e. are given by Eqs. (6.22). The straightforward calculation of the component action with making use of the constraints (6.22) leads to the following conclusions about its structure:

i) The auxiliary boson part of the action is reduced to a sum of three pieces, and each of these pieces corresponds to the auxiliary boson part of the relevant separate multiplet.

ii) As in the previous cases, the physical boson part of the action contains some off-diagonal terms with x-derivatives.

An inspection of these mixed terms shows that their structure is similar to that we met in the two previous cases. Although there appear some extra pieces arising, e. g., from the action of the pair of $(q^{1,1}, \hat{q}^{\,0,1\,\underline{i}})$, the physical boson part can be fully diagonalized as before, by using the constraints (6.22). The result of this procedure can be schematically written as the splitting of the action into a sum of the three pieces for the separate multiplets

$$S(q^{1,1}\,,\ \hat{q}^{\,1,0\,\underline{a}}\,,\ \hat{q}^{\,0,1\,\underline{i}}) = S(q^{1,1}) + S(\hat{q}^{\,1,0\,\underline{a}}) + S(\hat{q}^{\,0,1\,\underline{i}}) \tag{6.39}$$

where the metric and torsion terms can still depend on all three sets of physical bosons. However, using the constraints once again, it is easy to demonstrate that both scalar functions and torsion potentials in every piece can bear dependence only on the physical bosons of its «own» multiplet. The proof of separation of the fermionic terms follows the same routine.

In a similar way one can prove the separation property for any number of non-equivalent multiplets.

6.2. Potential Terms for $\hat{q}^{\,1,0\,\underline{a}}$ and $\hat{q}^{\,0,1\,\underline{i}}$. In Sec. 5 we constructed the potential terms for separate twisted multiplets. Here we show the existence of mixed mass terms which involve different types of multiplets. These terms are in fact of the same form as those given in [11, 12] and can be constructed only for multiplets belonging to the same «self-dual» pair. A new finding is the general form of the relevant scalar potential arising after elimination of the auxiliary fields. This potential and the accompanying Yukawa-type fermionic terms are the only mixed interaction of twisted multiplets of different types compatible with $\mathcal{N} = (4, 4)$ supersymmetry.

The candidate mixed mass-terms can be written in the analytic superspace in the following form

$$S^M = M \int \mu^{-2,-2}\, C^{0,0}_{\underline{i}\,\underline{k}\,\underline{a}\,\underline{b}}\, \theta^{1,0\,\underline{k}}\, \theta^{0,1\,\underline{b}}\, q^{1,0\,\underline{a}}\, q^{0,1\,\underline{i}}\,, \tag{6.40}$$

$$S^M_1 = M_1 \int \mu^{-2,-2}\, C^{0,0}_{\underline{i}\,\underline{k}\,\underline{a}\,\underline{b}}\, \theta^{1,0\,\underline{k}}\, \theta^{0,1\,\underline{b}}\, q^{1,1}\, \tilde{q}^{\,\underline{i}\,\underline{a}}\,, \tag{6.41}$$

$$S^M_2 = M_2 \int \mu^{-2,-2}\, C^{-1,0}_{\underline{k}\,\underline{a}\,\underline{b}}\, \theta^{1,0\,\underline{k}}\, \theta^{0,1\,\underline{b}}\, q^{1,1}\, q^{1,0\,\underline{a}}\,, \tag{6.42}$$

etc. All terms of this kind with higher powers of the involved superfields can be shown to vanish because of the corresponding harmonic constraints. As for the terms (6.40)–(6.42), only those given in (6.40) and (6.41) can respect $\mathcal{N}=(4,4)$ supersymmetry for nonvanishing harmonic constants C. The term (6.42) and any other similar term involving superfields from different «self-dual» pairs (e. g. from $\hat{q}^{\,1,0\,\underline{a}}$ and $\hat{q}^{\,\underline{i}\,\underline{a}}$) can easily be shown to vanish as a consequence of the requirement of supersymmetry.

Without loss of generality, let us restrict our consideration to the mass term (6.40). Computing the supersymmetry variation of the action (6.40), it is easy to find that (6.40) is invariant provided the harmonic constants C satisfy the following conditions

$$\partial^{2,0}C^{0,0}_{\underline{i}\,\underline{a}\,\underline{k}\,\underline{b}}=0\,,\quad \partial^{0,2}C^{0,0}_{\underline{i}\,\underline{a}\,\underline{k}\,\underline{b}}=0\,,\quad C^{0,0}_{\underline{i}\,\underline{a}\,\underline{k}\,\underline{b}}=\varepsilon_{\underline{i}\,\underline{k}}\,\varepsilon_{\underline{a}\,\underline{b}}\,. \tag{6.43}$$

After performing the integration over Grassmann and harmonic variables in (6.40), one finds the *off-shell* component form of this term:

$$S^{M}=-\frac{M}{4}\int d^2x\,\left\{\,q^{\underline{k}\,b}\,F_{\underline{k}\,b}+q^{i\,\underline{a}}\,F_{i\,\underline{a}}\,\right\}. \tag{6.44}$$

For sake of simplicity, we shall consider only bosonic part of the full component on-shell action. After eliminating the auxiliary fields $F_{\underline{k}\,b}$ and $F_{i\,\underline{a}}$ in the sum $S^{\rm bos}_{(a)}+S^{\rm bos}_{(b)}+S^{M}$, where $S^{\rm bos}_{(a)}$ and $S^{\rm bos}_{(b)}$ are bosonic parts of the component sigma-model actions for $\hat{q}^{\,1,0\,\underline{a}}$ and $\hat{q}^{\,0,1\,\underline{i}}$ ($S^{\rm bos}_{(a)}$ is given by Eq. (3.33) and $S^{\rm bos}_{(b)}$ has a similar form), the induced *on-shell* scalar potential term reads

$$S^{\rm pot}=\frac{M^2}{8}\int d^2x\,\left\{G^{-1}\,q^{\underline{k}\,b}\,q_{\underline{k}\,b}+\tilde{G}^{-1}\,q^{i\,\underline{a}}\,q_{i\,\underline{a}}\right\}. \tag{6.45}$$

Here $G=G(q^{i\,\underline{a}})$ and $\tilde{G}=\tilde{G}(q^{\underline{k}\,b})$ are the bosonic scalar metrics of the $\hat{q}^{\,1,0\,\underline{a}}$ and $\hat{q}^{\,0,1\,\underline{i}}$ multiplets (they are defined by Eqs. (3.34), (3.36) and by similar ones for $q^{\underline{k}\,b}$). Thus we see that in the general interaction case corresponding to nontrivial metric functions, the potential (6.45) contain mixed couplings of two different twisted multiplets. It is easy to restore the fermionic terms as well. We see that (6.45) yields mass terms for the involved fields even in the case of constant functions G and $\tilde{G}$, i. e., if one starts from the free kinetic actions of the twisted multiplets considered. This is a difference from similar superfield terms (5.1), (5.7) for single multiplets.

The most general *off-shell* mass term for the pair of multiplets ($\hat{q}^{\,1,0\,\underline{a}}$, $\hat{q}^{\,0,1\,\underline{i}}$) can be written as a sum of the following three pieces

$$S^{M}_{(1+2)}=S^{M}+S^{m}_{(a)}+S^{m}_{(b)}\,, \tag{6.46}$$

where

$$S^{m}_{(b)}=-\frac{m_{(b)}}{4}\int d^2x\,C^{k\,\underline{b}}\,F_{k\,\underline{b}}\,, \tag{6.47}$$

and two other terms in (6.46) are given by the expressions (6.44) and (5.5). After eliminating the auxiliary fields $F_{\underline{i}\,a}$ and $F_{k\,\underline{b}}$ in the sum $S^{\rm bos}_{(a)} + S^{\rm bos}_{(b)} + S^{M}_{(1+2)}$, the most general *on-shell* potential part of the action is obtained in the form

$$
\begin{aligned}
S^{\rm Pot} \;\; &= \;\; \frac{1}{8}\int d^2x\,\Big\{ G^{-1}\,(m^2_{(a)}\,C^{\underline{k}\,b}\,C_{\underline{k}\,b} + 2\,m_{(a)}\,M\,C^{\underline{k}\,b}\,q_{\underline{k}\,b} + M^2 q^{\underline{k}\,b}\,q_{\underline{k}\,b}) \\
&+ \;\; \tilde{G}^{-1}\,(m^2_{(b)}\,C^{i\,\underline{a}}\,C_{i\,\underline{a}} + 2\,m_{(b)}\,M\,C^{i\,\underline{a}}\,q_{i\,\underline{a}} + M^2 q^{i\,\underline{a}}\,q_{i\,\underline{a}})\Big\}\,. \qquad (6.48)
\end{aligned}
$$

CONCLUSIONS

In this paper we extended our previous analysis of the manifestly $\mathcal{N} = (4,4)$ supersymmetric off-shell description of the twisted multiplet $q^{1,1}$ in the $SU(2)\times SU(2)$ HSS [15] to the case of other three types of such a multiplet, which differ in assignments of their component fields with respect to the full *R*-symmetry group $SO(4)_L \times SO(4)_R$ of $\mathcal{N} = (4,4)$ 2D Poincaré superalgebra. We constructed off-shell superfield actions for each of these new multiplets in the analytic subspace of the $SU(2)\times SU(2)$ HSS and, as an example, discussed the special case of superconformally invariant $\mathcal{N} = (4,4)$ superextension of the $SU(2)\times U(1)$ group manifold WZNW sigma model associated with one of these multiplets (represented by the analytic superfunction $q^{1,0\,\underline{a}}$). Since the Lagrangians of these alternative twisted multiplets are expressed in terms of the harmonic analytic superfunctions having nonstandard transformation properties under $\mathcal{N} = (4,4)$ supersymmetry, the requirement that the corresponding actions are supersymmetric leads to certain constraints on the structure of the Lagrangians. Using these constraints, we were able to show that the bosonic target geometries of sigma models for the new multiplets are of the same sort as in the case of the $q^{1,1}$ multiplet considered in [15]. We also discussed massive extensions of general sigma-model actions of the multiplets $\hat{q}^{\,1,0\,\underline{a}}$ and $\hat{q}^{\,i\,\underline{a}}$ and, as an example, presented a massive deformation of the conformal WZNW action of the $\hat{q}^{\,1,0\,\underline{a}}$ multiplet. Like the sigma-model actions, the mass terms for the new multiplets reveal the same structure as those for the $q^{1,1}$ multiplet.

The basic new findings of our study are related to the analysis of the options when two or more multiplets of different sort are allowed to interact with each other via the sigma-model- or/and mass term-type analytic Lagrangians. We have found that $\mathcal{N} = (4,4)$ supersymmetry requires the general sigma-model action of any pair of such multipelts to reduce to a sum of sigma-model actions of separate multiplets, and this phenomenon persists in the cases when a larger number of different multiplets is involved into the game. The only possibility to arrange mutual interactions of the twisted multiplets of different types is via the appropriate mixed mass terms. The latter are bilinear in the multuplets belonging to the same «self-dual» pair which is characterized by the property that the $SU(2)$ assignments of the physical and auxiliary bosonic fields of the involved multiplets are complementary to each other. The multiplets belonging to different such pairs, can interact with each other neither via sigma-model type actions nor via mass terms. For a «self-dual» pair of twisted multiplet we have given the most general form of the scalar bosonic potential which arises as a result of eliminating auxiliary fields in the sum of general sigma-model actions of these multiplets and three possible mass terms, including the mixed one.

One of the possible directions of extending the study undertaken in this paper is to couple the considered models to conformal $\mathcal{N} = (4, 4)$ supergravity in the $SU(2) \times SU(2)$ HSS formulation of Ref. [18]. On this way one can hope to discover new off-shell versions of Poincaré $\mathcal{N} = (4, 4)$ supergravity, with the new types of twisted multiplet as superconformal compensators. One more interesting task is to study a possible effect of incorporating the additional twisted multiplets into more general HSS sigma models with noncommuting left and right quaternionic structures on the target space [17]. More technical work, which is now under way [33], is to repeat the analysis of the present paper in terms of $\mathcal{N} = (2, 2)$ superfields. The $\mathcal{N} = (2, 2)$ superspace language is used in many studies of $\mathcal{N} = (2, 2)$ and $\mathcal{N} = (4, 4)$ supersymmetric sigma models with torsion, and it is capable to make some proofs and observations of the present paper more tractable and clear.

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APPENDIX

Analytic superspace integration measure

$$\int d^2\theta^{1,0}\, d^2\theta^{0,1} = \frac{1}{16}\, \varepsilon^{\underline{k}\,\underline{n}}\, \varepsilon^{\underline{c}\,\underline{d}}\, \frac{\partial}{\partial\theta^{1,0\,\underline{k}}}\, \frac{\partial}{\partial\theta^{1,0\,\underline{n}}}\, \frac{\partial}{\partial\theta^{0,1\,\underline{c}}}\, \frac{\partial}{\partial\theta^{0,1\,\underline{d}}}\,, \tag{A.1}$$

$$\int d^2\theta^{1,0}\, d^2\theta^{0,1}\, (\theta^{1,0})^2(\theta^{0,1})^2 = 1\,. \tag{A.2}$$

Complete solution to the constraints (3.11) – (3.13)

$$\begin{aligned} g^{1,1} &= -\beta^{i\,a} u^1_i v^1_a + \theta^{1,0\,\underline{i}} F^a_{\underline{i}} v^1_a + 2i\theta^{0,1\,\underline{a}} \partial_{--} q^i_{\underline{a}} u^1_i \\ &\quad - 2i\theta^{1,0\,\underline{i}}\theta^{0,1\,\underline{a}} \partial_{--}\alpha_{\underline{i}\,\underline{a}} + i(\theta^{1,0})^2 \partial_{++}\beta^{i\,a} u^{-1}_i v^1_a + i(\theta^{0,1})^2 \partial_{--}\beta^{i\,a} u^1_i v^{-1}_a \\ &\quad - i\theta^{1,0\,\underline{i}} (\theta^{0,1})^2 \partial_{--} F^a_{\underline{i}} v^{-1}_a + 2(\theta^{1,0})^2 \theta^{0,1\,\underline{a}} \partial_{++}\partial_{--} q^i_{\underline{a}} u^{-1}_i \\ &\quad + (\theta^{1,0})^2 (\theta^{0,1})^2 \partial_{++}\partial_{--}\beta^{i\,a} u^{-1}_i v^{-1}_a \,, \end{aligned} \tag{A.3}$$

$$\begin{aligned} f^{1,1} &= -\rho^{i\,a} u^1_i v^1_a + 2i\theta^{1,0\,\underline{i}} \partial_{++} q^a_{\underline{i}} v^1_a - \theta^{0,1\,\underline{a}} F^i_{\underline{a}} u^1_i \\ &\quad + 2i\theta^{1,0\,\underline{i}}\theta^{0,1\,\underline{a}} \partial_{++}\gamma_{\underline{i}\,\underline{a}} + i(\theta^{1,0})^2 \partial_{++}\rho^{i\,a} u^{-1}_i v^1_a + i(\theta^{0,1})^2 \partial_{--}\rho^{i\,a} u^1_i v^{-1}_a \\ &\quad + 2\theta^{1,0\,\underline{i}} (\theta^{0,1})^2 \partial_{++}\partial_{--} q^a_{\underline{i}} v^{-1}_a + i(\theta^{1,0})^2 \theta^{0,1\,\underline{a}} \partial_{++} F^i_{\underline{a}} u^{-1}_i \\ &\quad + (\theta^{1,0})^2 (\theta^{0,1})^2 \partial_{++}\partial_{--}\rho^{i\,a} u^{-1}_i v^{-1}_a \,, \end{aligned} \tag{A.4}$$

$$\begin{aligned} h^{0,1\underline{i}} &= \xi^{\underline{i}\,a} v^1_a + \theta^{1,0\,\underline{i}} F^{i\,a} u^{-1}_i v^1_a + 2i\theta^{0,1\,\underline{a}} \partial_{--} q^{\underline{i}}_{\underline{a}} - i(\theta^{0,1})^2 \partial_{--}\xi^{\underline{i}\,a} v^{-1}_a \\ &\quad + 2i\theta^{1,0\,\underline{i}}\theta^{0,1\,\underline{a}} \partial_{--}\psi^i_{\underline{a}} u^{-1}_i - i\theta^{1,0\,\underline{i}} (\theta^{0,1})^2 \partial_{--} F^{i\,a} u^{-1}_i v^{-1}_a \,, \end{aligned} \tag{A.5}$$

$$\begin{aligned} f^{1,0\underline{a}} &= \psi^{i\,\underline{a}} u^1_i + 2i\theta^{1,0\,\underline{i}} \partial_{++} q^{\underline{a}}_{\underline{i}} - \theta^{0,1\,\underline{a}} F^{i\,a} u^1_i v^{-1}_a - i(\theta^{1,0})^2 \partial_{++}\psi^{i\,\underline{a}} u^{-1}_i \\ &\quad - 2i\theta^{1,0\,\underline{i}}\theta^{0,1\,\underline{a}} \partial_{++}\xi^a_{\underline{i}} v^{-1}_a + i(\theta^{1,0})^2 \theta^{0,1\,\underline{a}} \partial_{++} F^{i\,a} u^{-1}_i v^{-1}_a \,, \end{aligned} \tag{A.6}$$

$$\begin{aligned} t^{1,1} &= F^{i\,a} u^1_i v^1_a - 2i\theta^{1,0\,\underline{i}} \partial_{++}\xi^a_{\underline{i}} v^1_a + 2i\theta^{0,1\,\underline{a}} \partial_{--}\psi^i_{\underline{a}} u^1_i \\ &\quad - i(\theta^{1,0})^2 \partial_{++} F^{i\,a} u^{-1}_i v^1_a - i(\theta^{0,1})^2 \partial_{--} F^{i\,a} u^1_i v^{-1}_a + 4\theta^{1,0\,\underline{i}}\theta^{0,1\,\underline{a}} \partial_{++}\partial_{--} q_{\underline{i}\,\underline{a}} \\ &\quad - 2\theta^{1,0\,\underline{i}} (\theta^{0,1})^2 \partial_{++}\partial_{--}\xi^a_{\underline{i}} v^{-1}_a + 2(\theta^{1,0})^2 \theta^{0,1\,\underline{a}} \partial_{++}\partial_{--}\psi^i_{\underline{a}} u^{-1}_i \\ &\quad - (\theta^{1,0})^2 (\theta^{0,1})^2 \partial_{++}\partial_{--} F^{i\,a} u^{-1}_i v^{-1}_a . \end{aligned} \tag{A.7}$$

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