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THE b-ADIC DIAPHONY OF AN ARBITRARY (t, m, s)-NET

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b-ичная диафония на произвольной (t, m, s)-сетке

Получены оценки распределения одного класса s-мерных равномерно распределенных сеток.

Представлен обзор исследований в области равномерно распределенных последовательностей и сеток. Определены дискрепанс и диафония, которые являются мерами отклонения распределения данной последовательности по отношению к идеальному распределению. Оценки для этих мер зависят от основания b счетной системы, в которой построены соответствующие последовательности и сетки, и основываются на ортонормированных функциональных системах.

Рассмотрена (t, m, s)-сетка в s-мерном единичном кубе, которая содержит b^m-точек. При этом b^t-точек попадают в параллелепипед объемом b^{t-m}, $0 \leqslant t \leqslant m$, где m — целое.

Доказана оценка для b-ичной диафонии, которая зависит от основания b, числа точек, размерности, но не зависит от способа конструирования сетки.

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The b-Adic Diaphony of an Arbitrary (t, m, s)-Net

The paper considers the possibility to estimate the distribution of class s-dimensional uniformly distributed nets.

A brief survey of the investigations in the area of uniformly distributed sequences and nets is made. The discrepancy and the diaphony are defined, which are measures for deviation of the distribution of a given sequence from an ideal distribution. The estimations of these measures depend on the base b of the number system in which the corresponding sequences and nets are constructed, and are based on orthonormal functional systems.

The (t, m, s)-net, containing b^m points, such that b^t points are contained in parallelepiped with volume b^{t-m}, $0 \leqslant t \leqslant m$, in s-dimensional unit cube is considered.

An estimation of the b-adic diaphony is proved, which depends on the base b, the number of the points, the dimension, but does not depend on the way of the construction of the net.

The investigation has been performed at the Laboratory of Information Technologies, JINR.

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INTRODUCTION

Let $\xi = (x_j)_{j\geq 1}$ be a sequence in $[0,1)$ and J is an arbitrary subinterval of $[0,1)$.

For an arbitrary integer $N \geq 1$ we separate the initial part $\{x_0, \ldots, x_{N-1}\}$ of the sequence ξ. We denote with $A_N(\xi; J) = |\{j : 0 \leq j \leq N-1, \{x_j\} \in J\}|$, where $\{x_j\}$ is the fractional part of x_j.

Definition 1. *The sequence ξ is uniformly distributed mod 1 if for every subinterval $J \subseteq [0,1)$ the equality*

$$\lim_{N\to\infty} \frac{A_N(\xi; J)}{N} = |J|$$

is hold, where $|J|$ is the length of J.

The beginning of the theory of uniformly distributed sequences is connected to the problem of a declination of heavenly body form his orbit by means of attraction of other heavenly bodies. Weyl [33] is considered as an originator of the theory.

The theory of uniformly distributed sequences investigates the possibility to prove criteria for uniform distribution of sequences and to construct uniformly distributed sequences from one side and define measures for uniform distribution from another one. The investigations of uniformly distributed sequences are directly connected to the orthonormal functional systems. The first criteria for uniform distribution of sequences are proved by Weyl [33] and are based on the trigonometric functional system.

The construction of uniformly distributed sequences and nets, depending on the number system, brings to the necessity of new investigation means, corresponding to the number system. Using the dyadic number system, Sloss and Blyth [29] have proved the necessary and sufficient conditions for uniform distribution, which are based on the Walsh [3] functional system. Hellekalek and Leeb [11] have defined a measure for uniform distribution on the base of the Walsh functions. This measure is called dyadic diaphony.

There are sequences and nets constructed to b-adic number system and generalized number system. An apparatus corresponding to the number system for investigation of these sequences and nets is worked and developed by the author and Grozdanov in [6–8, 28].

Definition 2. *Let $N \geq 1$ be an arbitrary fixed integer and $\xi_N = \{x_0, \dots x_{N-1}\}$ is a net of real numbers in $[0,1)$. The quantity $D(\xi_N)$ defined by the equality*

$$D(\xi_N) = \sup_{J \subseteq [0,1)} \left| \frac{A(\xi_N; J)}{N} - |J| \right|$$

is called a discrepancy of the given net.

Remark. Let $\xi = (x_j)_{j \geq 0}$ be a sequence of real numbers. For each integer $N \geq 1$ the discrepancy $D_N(\xi)$ of the sequence ξ is defined as the discrepancy of the initial part $\{x_0, \dots, x_{N-1}\}$ containing the first N members of the sequence ξ.

The importance of investigation of uniformly distributed sequences and nets is conditioned by their applications. The uniformly distributed sequences and nets are used for numerical, Monte Carlo and quasi-Monte Carlo integrations. The Koksma–Hlawka [10,13] inequality shows that the error depends on the variation of the function and the discrepancy of the net. A set of articles is devoted to the constructions of sequences and nets with low discrepancy. In this direction Niederreiter [18–21], Faure [2–4], Tezuka [30,31] and many other authors have been working. The so-called (t,m,s)-nets (see Definition 4) and (t,s)-sequences are used in the last few years. The newest investigations of Pillichshammer and Dick [22–26] are related to construction of (t,m,s)-nets and estimations of the discrepancy, the L_2-discrepancy and the diaphony of (t,m,s)-nets.

Up to now, the studies of the (t,m,s)-nets and their measures are related to a concrete construction of the (t,m,s)-net, though this construction is very general. When we talk about the estimations of measures for the distribution of the (t,m,s)-nets, the investigations show that these estimations do not depend on the concrete construction of the (t,m,s)-net. The definition of (t,m,s)-net and the definitions of the measures confirm this statement. The result presented in this paper is based only on the definition of (t,m,s)-net and does not depend on the construction of the net.

Section 1 gives a review of basic definitions and a survey of some necessary results. In Section 2 the result and its proof are given. Section 3 proposes a brief discussion and issues for future investigations.

1. SURVEY

In this section some basic definitions and results are given necessary for the formulation and the proof of the result in the present paper.

Let $b \geq 2$ and $s \geq 1$ be fixed integers and $[0,1)^s$ is a s-dimensional unit cube. We denote $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

Definition 3. *An elementary interval to base b is defined to be an interval of the form*

$$\prod_{i=1}^{s}\left[\frac{a_i}{b^{g_i}},\frac{a_i+1}{b^{g_i}}\right)$$

with integers $g_i, a_i \geq 0$ and $a_i = 0, 1, \ldots, b^{g_i}-1$ for $1 \leq i \leq s$.

Definition 4. *Let $0 \leq t \leq m$ be integers. A (t, m, s)-net to base b is a set of b^m points in $[0,1)^s$ having the property that in every elementary interval to base b of volume b^{t-m} there are exactly b^t points of the set.*

Definition 5. (i) *Let $\omega = \exp\left(\frac{2\pi\mathbf{i}}{b}\right)$. The Rademacher functions to base b are defined by*

$$r_0(x) = \omega^a, \quad \textit{for} \quad \frac{a}{b} \leq x < \frac{a+1}{b}, \quad a = 0, 1, \ldots, b-1,$$

and for $k \geq 1$ by

$$r_k(x+1) = r_k(x) = r_0(b^k x).$$

(ii) *The functions of Walsh to base b is defined as follows:*

$$w_0(x) = 1 \quad \textit{for each} \quad x \in [0,1),$$

and if $k \geq 1$ has a b-adic representation $k = k_g b^{\alpha_g} + k_{g-1} b^{\alpha_{g-1}} + \ldots + k_0 b^{\alpha_0}$, where $\alpha_g > \alpha_{g-1} > \ldots > \alpha_0$ and $k_j \in \{1, 2, \ldots, b-1\}$ for $0 \leq j \leq g$, then the kth Walsh function to base b is defined as

$$w_k(x) = r_{\alpha_g}^{k_g}(x) r_{\alpha_{g-1}}^{k_{g-1}}(x) \ldots r_{\alpha_0}^{k_0}(x) \quad \textit{for each} \quad x \in [0,1).$$

The system $\mathcal{W}(b) = \{w_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^{s} w_{k_i}(x_i), \quad \mathbf{k} = (k_1, k_2, \ldots, k_s) \in \mathbf{N}_0^s,$ $\mathbf{x} = (x_1, x_2, \ldots, x_s) \in [0,1)^s\}$ is called the Walsh functional system to base b. This system is defined by Chrestenson [1].

When $b = 2$ from the system $\mathcal{W}(b)$ the original system of Walsh $\mathcal{W}(2)$ is obtained.

Let $N \geq 1$ be an arbitrary fixed integer.

Definition 6. *The b-adic diaphony of the first N elements of the sequence $\xi = (\mathbf{x}_j)_{j \geq 0}$ in $[0,1)^s$ is defined as*

$$F_N(\mathcal{W}(b), \xi) = \left(\frac{1}{(b+1)^s - 1} \sum_{\mathbf{k} \in \mathbf{N}_0^s, \mathbf{k} \neq \mathbf{0}} \rho(\mathbf{k}) \left| \frac{1}{N} \sum_{j=0}^{N-1} w_{\mathbf{k}}(\mathbf{x}_j) \right|^2 \right)^{\frac{1}{2}},$$

where for vector $\mathbf{k} = (k_1, k_2, \ldots, k_s) \in \mathbf{N}_0^s,\quad \rho(\mathbf{k}) = \prod_{i=1}^{s} \rho(k_i)$ *and for every* $k \geq 0$

$$\rho(k) = \begin{cases} b^{-2g}, & \text{if } b^g \leq k < b^{g+1}, \quad g \geq 0, \quad g \in \mathbf{Z} \\ 1, & \text{if } k = 0. \end{cases}$$

When $b = 2$ from Definition 6 the definition of the dyadic diaphony of Hellekalek and Leeb [11] is obtained.

The definition of b-adic diaphony is given by Grozdanov and Stoilova [6]. The generalizations of this definition could be found in the PhD thesis of Stoilova and in [5,9].

The results obtained by investigation of the applications of the (t, m, s)-nets for numerical, Monte Carlo and quasi-Monte Carlo integrations are necessary for us to the proof of the statement. Such a result is the next Lemma.

Lemma 1. *Let* $N = b^m$ *and* $\xi_N = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\}$ *be a* (t, m, s)*-net to base* b*. If* $\mathbf{k} = (k_1, k_2, \ldots, k_s) \in \mathbf{N}_0^s,\quad \mathbf{k} \neq \mathbf{0}$ *is such that there exist integers* $g_i \geq 0$ *with* $k_i < b^{g_i}$ *for* $1 \leq i \leq s$ *and* $g_1 + g_2 + \ldots + g_s \leq m - t$*, then*

$$\sum_{j=0}^{N-1} w_{\mathbf{k}}(\mathbf{x}_j) = 0.$$

The proof of this Lemma could be found in [15] and [17].

In [25] an estimation of the dyadic diaphony of digital (t, m, s)-net is presented. But this estimation depends on the construction of the net.

Theorem 1. *Let* $\xi_{2^m} = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{2^m-1}\}$ *be a digital* (t, m, s)*-net over* $\mathbf{Z}_2$*, with* $t < m$ *and with regular generator matrices* $C_1, C_2, \ldots, C_s$*. Then we have*

$$F_{2^m}^2(\mathcal{W}(2), \xi_{2^m}) \leq c(s) 2^{2t} \frac{(m-t)^{s-1}}{2^{2m}},$$

where $c(s) > 0$ *depends only on the dimension* s*.*

Theorem 1 of Dick and Pillichshammer is proved with arbitrary generator matrices. They do not use concrete form of the matrices, but these matrices are used in the proof of the theorem. This engenders the question for the possibility to escape of the dependence on the construction.

2. STATEMENT

Theorem 2. *Let $N = b^m$ and ξ_N be a (t, m, s)-net to base b with $t \leq m$. Then for the b-adic diaphony $F(\mathcal{W}(b); \xi_N)$ of the net ξ_N we have*

$$F(\mathcal{W}(b); \xi_N) \leq C(b, s) b^t \frac{(m-t)^{\frac{s-1}{2}}}{b^m},$$

where the constant $C(b, s) > 0$ depends on the dimension s and on the base of the number system b.

Proof. Using Lemma 1 we have

$$[NF(\mathcal{W}(b); \xi_N)]^2 = \frac{1}{(b+1)^s - 1} \sum_{\mathbf{k} \in \mathbf{N}_0^s} \rho(\mathbf{k}) \left| \sum_{j=0}^{N-1} w_{\mathbf{k}}(\mathbf{x}_j) \right|^2 =$$

$$\frac{1}{(b+1)^s - 1} \sum_{r=1}^{s} \binom{s}{r} \sum_{\substack{g_1, \ldots, g_r = 1, \\ g_1 + \ldots + g_r \leq m - t}}^{\infty} \times$$

$$\sum_{k_1 = b^{g_1 - 1}}^{b^{g_1} - 1} \cdots \sum_{k_r = b^{g_r - 1}}^{b^{g_r} - 1} b^{-2(g_1 + \ldots + g_r + r)} \left| \sum_{j=0}^{N-1} w_{\mathbf{k}}(\mathbf{x}_j) \right|^2 +$$

$$+ \frac{1}{(b+1)^s - 1} \sum_{r=1}^{s} \binom{s}{r} \sum_{\substack{g_1, \ldots, g_r = 0, \\ g_1 + \ldots + g_r \geq m - t - r}}^{\infty} \times$$

$$\sum_{k_1 = b^{g_1}}^{b^{g_1+1} - 1} \cdots \sum_{k_r = b^{g_r}}^{b^{g_r+1} - 1} b^{-2(g_1 + \ldots + g_r)} \left| \sum_{j=0}^{N-1} w_{\mathbf{k}}(\mathbf{x}_j) \right|^2 \leq$$

$$\leq \frac{b^{2m}}{(b+1)^s - 1} \sum_{r=1}^{s} \binom{s}{r} (b-1)^r \sum_{\substack{g_1, \ldots, g_r = 0, \\ g_1 + \ldots + g_r \geq m - t - r}}^{\infty} b^{-2(g_1 + \ldots + g_r)} \leq$$

$$\leq \frac{b^{2m}}{(b+1)^s - 1} \sum_{r=1}^{s} \binom{s}{r} (b-1)^r \sum_{h = m-t-r}^{\infty} \sum_{\substack{g_1, \ldots, g_r = 0, \\ g_1 + \ldots + g_r = h}}^{\infty} b^{-2h} \leq$$

$$\leq \frac{b^{2m}}{(b+1)^s - 1} \sum_{r=1}^{s} \binom{s}{r} (b-1)^r \sum_{h = m-t-r}^{\infty} b^{-2h} h^{r-1} \leq$$

$$\leq \frac{b^{2m}}{(b+1)^s-1}\sum_{r=1}^{s}\binom{s}{r}(b-1)^r b^{-2(m-t-r)}(m-t)^{r-1}\sum_{h=1}^{\infty} b^{-2h}h^{s-1} \leq$$

$$\leq \frac{b^{2t}(m-t)^{s-1}}{(b+1)^s-1}c_1(b)\sum_{r=0}^{s}\binom{s}{r}(b-1)^r b^{2r} = \frac{b^{2t}(m-t)^{s-1}}{(b+1)^s-1}c_1(b)(1+(b-1)b^2)^s,$$

whence the proof of Theorem 2 is completed.

3. DISCUSSION

The proven result shows that the order of the b-adic diaphony of the (t, m, s)-nets does not depend on the construction. The concrete construction exercises an influence on the constant. From the point of view of the using of (t, m, s)-nets for Monte Carlo and quasi-Monte Carlo integrations is clear that the value of the constant is important. Furthermore, the Monte Carlo and quasi-Monte Carlo methods with uniformly distributed sequences and nets are used in the computer graphics. Investigations in this direction were made by Keller [12]. This engenders the necessity of the search for such a construction so that the constant is the smallest one.

Another interesting question is to find a connection between b-adic diaphony and the error of the integration.

The question for general lower estimation of the b-adic diaphony of an arbitrary net takes an essential place. The similar estimation of the dyadic diaphony is made by Dick and Pillichshammer.

The solving of these and others problems is a subject of future research.

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REFERENCES

1. *Chrestenson H. E.* A class of generalized Walsh functions // Pacific J. Math. 1955. V. 5. P. 17.

2. *Faure H.* Discrépance de suites associees a un system de numeration (en dimention un) // Bull. Soc. Math. France. 1987. V. 109. P. 273.

3. *Faure H.* Variations on $(0, s)$-sequences // J. of Complexity. 2001. V. 17. P. 741.

4. *Faure H., Chaix H.* Discrépance et diaphonie des suites de Van der Corput généralisées // C. R. Acad. Sci. Paris. 1990. V. 310, Serie I. P. 315.

5. *Grozdanov V., Nikolova E., Stoilova S.* Generalized b-adic diaphony // Comp. Ren. Akad. Bul. Sci. 2003. V. 56, No 4. P. 23.

6. *Grozdanov V., Stoilova S.* On the theory of b-adic diaphony // Comp. Ren. Akad. Bul. Sci. 2001. V. 54, No 3. P. 31.

7. *Grozdanov V., Stoilova S.* Multiplicative system and uniform distribution of sequences // Comp. Ren. Akad. Bul. Sci. 2001. V. 54, No 5. P. 9.

8. *Grozdanov V., Stoilova S.* Haar-type functions and uniform distribution of sequences // Comp. Ren. Akad. Bul. Sci. 2001. V. 54, No 12. P. 23.

9. *Grozdanov V., Stoilova S.* The general diaphony // Comp. Ren. Akad. Bul. Sci. 2004. V. 57, No 1. P. 13.

10. *Hlawka E.* Zur angenäherten Berechnung mehrfacher Integrate // Monatsh. Math. 1962. V. 66. P. 140.

11. *Hellekalek P., Leeb H.* Dyadic diaphony // Acta Arith. V. LXXX. No 2. P. 187.

12. *Keller A., Kollig T.* Efficient multidimensional sampling // Eurographics. 2002. V. 21. No 3.

13. *Koksma J. F.* Diophantische Approximationen. Berlin: Springer, 1936.

14. *Kuipers A., Niederreiter H.* Uniform distribution of sequences. New York: Wiley, 1974.

15. *Larcher G., Niederreiter H., Schmid W.* Digital nets and sequences constructed over finite rings and their application to quasi-Monte Carlo integration // Monatsh. Math. 1996. V. 121. P. 231.

16. *Larcher G., Pirsic G.* Base change problems for generalized Walsh series and multivariate numerical integration // Pac. J. Math. 1999. V. 189, No 1. P. 75.

17. *Larcher G., Traunfellner C.* On the numerical integration of Walsh series by number-theoretic methods // Math. Comp. 1994. V. 63, No 207. P. 277.

18. *Niederreiter H.* Application of diophantine approximation to numerical integration // «Diophantine Approximation and Its Application» / Ed. C. F. Osgoot. 1988.

19. *Niederreiter H.* Point sets and sequences with small discrepancy // Monatsh. Math. 1988. V. 104. P. 273.

20. *Niederreiter H.* Low-discrepancy and low-dispersion sequences // J. Number Theory. 1988 V. 30.P. 51.

21. *Niederreiter H.* Low-discrepancy point sets obtained by digital constructions over finite fields // Czechoslovak Math. J. 1992. V. 42. P. 143.

22. *Pillichshammer F.* Improbed upper bounds for the star discrepancy of digital nets in dimension 3 // Acta Arithmetica. 2003. V. 108.2. P. 167.

23. *Pillichshammer F., Dick J.* On the mean square weighted L_2 discrepancy of randomized digital (t, m, s)-nets over $\mathbf{Z}_2$. 2003 (to be published).

24. *Pillichshammer F., Dick J.* Diaphony, discrepancy, spectral test and worst-case error. 2004 (to be published).

25. *Pillichshammer F., Dick J.* Dyadic diaphony of digital nets over $\mathbf{Z}_2$. 2004 (to be published).

26. *Pillichshammer F., Dick J., Kuo F., Sloan I. H.* Construction algorithms for polynomial lattice rules for multivariate integration. 2004 (to be published).

27. *Rademacher H.* Einige Sätze über Reihen von allgemeinen Orthogonalfunctionen // Math. Ann. 1922. V. 87. P. 112.

28. *Stoilova S.* Applications of Price functions and Haar-type functions to the numerical integration. Preprint IMI BAN. Sofia, 2003.

29. *Sloss B. G., Blyth W. F.* Walsh functions and uniform distribution mod 1 // Tohoku Math. J. 1993. V. 45. P. 555.

30. *Tezuka S.* A generalization of Faure sequences and its efficient implementation. Research report IBM RT0105. 1994. P. 1.

31. *Tezuka S.* Uniform random numbers: Theory and practice. Boston: Kluwer Academic Publishers, 1995.

32. *Walsh J. L.* A closed set of normal orthogonal function // Amer. J. Math. 1923. V. 45. P. 5.

33. *Weyl H.* Über die Gleichverteilung von Zahlen mod. Eins. // Math. Ann. 1916. V. 77. P. 313.

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