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ON THE b-ADIC DIAPHONY OF THE GENERALIZED ZAREMBA NET

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b-ичная диафония обобщенной сетки Зарембы

В работе определен широкий класс двумерных сеток при основе b-счетной системы ($b \geqslant 2$, целое). Вычислен модуль суммы Уолша для этих сеток. Доказаны оценки для b-ичной диафонии обобщенной сетки Зарембы и получен ее точный порядок.

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On the b-adic Diaphony of the Generalized Zaremba Net

In this paper a very broad class of two-dimensional nets is defined. This class generalizes the classes of the Roth and Zaremba nets in the b-adic number system. For this class of nets, the module of the Walsh sum in a base b is calculated. Estimations of the b-adic diaphony of the generalized Zaremba net are proved. The exact order and the exact constant in this order of the b-adic diaphony of the generalized Zaremba net are found.

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INTRODUCTION

Let $s \geqslant 1$ be an arbitrary integer and $[0,1)^s$ is s-dimensional unit cube. For fixed $N \geqslant 0$ let $\xi_N = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\}$ be a net in $[0,1)^s$. Let J be an arbitrary parallelepiped in $[0,1)^s$, defined by $J = \prod_{j=1}^{s}[u_j, v_j)$, where for every $j, \quad 1 \leqslant j \leqslant s, \quad 0 \leqslant u_j < v_j < 1$. We signify by $A(\xi_N; J)$ the number of the points of the net ξ_N belonging to J.

Definition 1. *The net ξ_N is called uniformly distributed in $[0,1)^s$ if for every $J \subseteq [0,1)^s$ the equality*

$$\lim_{N \to \infty} \frac{A(\xi_N; J)}{N} = V(J),$$

where $V(J)$ denotes the volume of J, holds.

From this definition it is impossible to compare distribution of two uniformly distributed nets. For that reason the measures of the distribution are defined.

Let $b \geqslant 2$ be fixed integer and $\omega = \exp(2\pi \mathbf{i}/b)$

Definition 2. *(i) The Rademacher functions $\{\phi_k(x)\}_{k \geqslant 0}, x \in [0,1)$ to the base b are defined by*

$$\phi_0(x) = \omega^a, \ \ \textit{for} \ \ \frac{a}{b} \leq x < \frac{a+1}{b}, \quad a = 0, 1, \ldots, b-1$$

and for $k \geqslant 1$ by

$$\phi_k(x+1) = \phi_k(x) = \phi_0(b^k x).$$

(ii) The Walsh [10] function to the base b is defined as follows:

$$\psi_0(x) = 1 \ \ \textit{for each} \ \ x \in [0,1);$$

and if $k \geqslant 1$ has a b-adic representation $k = k_g b^{\alpha_g} + k_{g-1} b^{\alpha_{g-1}} + \ldots + k_0 b^{\alpha_0}$, where $\alpha_g > \alpha_{g-1} > \ldots > \alpha_0$ and $k_j \in \{1, 2, \ldots, b-1\}$ for $0 \leqslant j \leqslant g$, then the k-th Walsh function to the base b is defined as

$$\psi_k(x) = \phi_{\alpha_g}^{k_g}(x) \phi_{\alpha_{g-1}}^{k_{g-1}}(x) \ldots \phi_{\alpha_0}^{k_0}(x) \ \ \textit{for each} \ \ x \in [0,1).$$

The system $\psi(b) = \{\psi_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^{s} \psi_{k_i}(x_i), \quad \mathbf{k} = (k_1, k_2, \ldots, k_s) \in \mathbf{N}_0^s$, $\mathbf{x} = (x_1, x_2, \ldots, x_s) \in [0,1)^s\}$ is called the Walsh functional system to the base b. This system is defined by Chrestenson [1].

The measure for uniform distribution of nets based on the Walsh functional system to the base b is called b-adic diaphony and it is defined in the next definition.

Definition 3. *The b-adic diaphony $F(\psi(b);\xi_N)$ of the net ξ_N in $[0,1)^s$ is defined as*

$$F(\psi(b);\xi_N)=\left(\frac{1}{(b+1)^s-1}\sum_{\mathbf{k}\neq\mathbf{0}}\rho(\mathbf{k})\left|\frac{1}{N}\sum_{i=0}^{N-1}\psi_{\mathbf{k}}(\mathbf{x}_i)\right|^2\right)^{1/2},$$

where for vector $\mathbf{k}=(k_1,\ldots,k_s)$ with non-negative integer coordinates $k_1,\ldots,k_s$, $\rho(\mathbf{k})=\prod_{j=1}^{s}\rho(k_j)$ and for integer $k\geqslant 0$

$$\rho(k)=\begin{cases} b^{-2g}, & \text{if } \ b^g\leqslant k<b^{g+1}, \quad g\geqslant 0, \quad g\in\mathbf{Z},\\ 1, & \text{if } \ k=0.\end{cases}$$

Definition 3 is a generalization of the definition of the dyadic diaphony by Hellekalek and Leeb [5].

In quasi-Monte-Carlo methods the uniformly distributed nets are used for research images. In this sense it is important to be able to estimate distribution of the nets. The b-adic diaphony gives the possibility for the estimation of the distribution of the nets. Very often in quasi-Monte-Carlo methods the Roth [6] net is used.

In Sec. 1 the definition of a very broad class of two-dimensional nets is defined. For this class nets the b-adic diaphony is estimated. Sec. 2 gives the necessary preliminary results for the proofs. In Sec. 3 the proof is demonstrated.

1. STATEMENTS OF THE RESULTS

Let in the set $G(b)=\{0,1,\ldots,b-1\}$ introduce the operations: for every $m,n\in G(b)$ we define $m\oplus n=m+n(\text{mod} \ \ b)$ and

$$m\ominus n=\begin{cases} m-n, & \text{if } \ m\geqslant n,\\ b+m-n, & \text{if } \ m<n.\end{cases}$$

Let the real $x,y\in[0,1)$ have b-adic representations respectively $x=\sum_{j=0}^{\infty}x_jb^{-j-1}$ and $y=\sum_{j=0}^{\infty}y_jb^{-j-1}$. We define $x\dot{+}y=\sum_{j=0}^{\infty}(x_j\oplus y_j)b^{-j-1}$ and $x\dot{-}y=\sum_{j=0}^{\infty}(x_j\ominus y_j)b^{-j-1}$.

Let $i=\sum_{j=0}^{\infty} a_j(i)b^j$ be the b-adic expansion of i and let $\zeta_b(i)=\sum_{j=0}^{\infty} a_j(i)b^{-j-1}$ be the i-th element of the Van der Corput-Halton [3] sequence in the base b.

Let $\nu > 0$ be an arbitrary integer. Let for $0 \leqslant j \leqslant \nu-1$ $\mu_j \equiv \alpha j+\beta(\mathrm{mod}\ b)$ with fixed $\alpha, \beta \in \{0, 1, \ldots, b-1\}$ and $\mu = 0.\mu_0\mu_1 \ldots \mu_{\nu-1}$. For $0 \leqslant i \leqslant b^\nu - 1$ we put $\zeta'_b(i) = \zeta_b(i) \oplus \mu$ and $\eta_{b,\nu}(i) = i/b^\nu$.

Let $\eta_{b,\nu} = \{\eta_{b,\nu}(i) : 0 \leqslant i \leqslant b^\nu - 1\}$ and $\zeta'_{b,\nu} = \{\zeta'_b(i) : 0 \leqslant i \leqslant b^\nu - 1\}$.

Definition 4. *For each positive integer ν, the generalized Zaremba net in the base b, composed of b^ν points, is defined as*

$$Z_{b,\nu}^{\alpha,\beta} = \{(\eta_{b,\nu}(i), \zeta'_b(i)) : 0 \leqslant i \leqslant b^\nu - 1\}.$$

When $\alpha = 1$ and $\beta = 0$, the net $Z_{b,\nu}^{1,0}$ is introduced by Warnock [11]. When $b = 2$, $\alpha = 1$ and $\beta = 1$, the net $Z_{2,\nu}^{1,1}$ is the original Zaremba [4] net. If $\alpha = 0$ and $\beta = 0$, then $Z_{b,\nu}^{0,0}$ is the original net of Roth $R_{b,\nu}$.

In the paper we obtain an order $\mathcal{O}\left(\frac{\sqrt{\log b^\nu}}{b^\nu}\right)$ of the b-adic daphony of the generalized Zaremba net. The following results hold.

Theorem 1. *For each integer $\nu > 0$, the inequations*

$$\frac{b^2-1}{b+2}\frac{\nu}{b^{2\nu}} - \frac{b}{b+2}\frac{1}{b^{2\nu}} + \frac{b}{b+2}\frac{1}{b^{3\nu}} \leqslant F^2(\mathcal{W}(b); Z_{b,\nu}^{\alpha,\beta}) \leqslant$$

$$\leqslant \frac{b^2-1}{b+2}\frac{\nu}{b^{2\nu}} + \frac{2(b+1)}{(b+2)}\frac{1}{b^{2\nu}} - \frac{b}{b+2}\frac{1}{b^{3\nu}}$$

hold.

Corollary 1. *The following equation holds*

$$\lim_{\nu\to\infty} \frac{b^\nu F(\mathcal{W}(b); Z_{b,\nu}^{\alpha,\beta})}{\sqrt{\log b^\nu}} = \sqrt{\frac{b^2-1}{(b+2)\log b}}.$$

2. PRELIMINARY RESULTS

Lemma 1. *Let $\nu > 0$ be an arbitrary integer and g_1, $0 \leqslant g_1 \leqslant \nu - 1$, is a fixed integer. For an arbitrary integer k_1, $b^{g_1} \leqslant k_1 < b^{g_1+1}$, we will use the representation*

$$k_1 = \alpha_{g_1} b^{g_1} + \alpha_{g_1-1} b^{g_1-1} + \ldots + \alpha_m b^m, \tag{1}$$

where $0 \leqslant m \leqslant g_1$ and for $m \leqslant j \leqslant g_1$, $\alpha_j \in \{0, 1, \ldots, b-1\}, \alpha_m, \alpha_{g_1} \neq 0$. We define the integer k_1^* as

$$k_1^* = \overline{\alpha}_m b^{\nu-1-m} + \overline{\alpha}_{m-1} b^{\nu-2-m} + \ldots + \overline{\alpha}_{g_1} b^{\nu-1-g_1},$$

where for every j, $m \leqslant j \leqslant g_1$ $\alpha_j \oplus \overline{\alpha}_j = 0$.

Then for every integer g_2, $0 \leqslant g_2 \leqslant \nu - 1$, k_2, $b^{g_2} \leqslant k_2 < b^{g_2+1}$, *the equation*

$$\left|\sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i))\psi_{k_2}(\zeta'_{b,\nu}(i))\right| = \begin{cases} b^\nu, & \textit{if } k_2 = k_1^*, \\ 0, & \textit{if } k_2 \neq k_1^* \end{cases}$$

holds.

Proof. For an arbitrary integer i, $0 \leqslant i < b^\nu$, we use the representation $i = \sum_{j=0}^{\nu-1} i_j b^j$. Then $\zeta'_{b,\nu}(i) = \sum_{j=0}^{\nu-1} (i_j \oplus \mu_j) b^{-j-1}$. For integer k_1 in the form (1), we have

$$\psi_{k_1}(\eta_{b,\nu}(i)) = \prod_{j=m}^{g_1} \omega^{\alpha_j i_{\nu-1-j}}. \tag{2}$$

Let $k_2 = k_1^*$. Then

$$\psi_{k_2}(\zeta'_{b,\nu}(i)) = \prod_{j=m}^{g_1} \omega^{\overline{\alpha}_j (i_{\nu-1-j} + \mu_{\nu-1-j})}. \tag{3}$$

From (2) and (3) for every i, $0 \leqslant i < b^\nu$, $\psi_{k_1}(\eta_{b,\nu}(i))\psi_{k_2}(\zeta'_{b,\nu}(i)) = \prod_{j=m}^{g_1} \omega^{\overline{\alpha}_j \mu_{\nu-1-j}}$ and

$$\left|\sum_{i=0}^{b^{\nu-1}} \psi_{k_1}(\eta_{b,\nu}(i))\psi_{k_2}(\zeta'_{b,\nu}(i))\right| = \left|\prod_{j=m}^{g_1} \omega^{\overline{\alpha}_j \mu_{\nu-1-j}} \sum_{i=0}^{b^{\nu-1}} 1\right| = b^\nu.$$

Let, now, $k_2 \neq k_1^*$. We use the representation $k_1 = \sum_{j=0}^{\nu-1} \alpha_j b^j$, where for $0 \leqslant j \leqslant m-1$ and $g_1 + 1 \leqslant j \leqslant \nu - 1$ $\alpha_j = 0$, for $m \leqslant j \leqslant g_1$ $\alpha_j \in \{0, 1, \ldots, b-1\}$, $\alpha_m, \alpha_{g_1} \neq 0$ and $k_2 = \sum_{j=0}^{\nu-1} \beta_j b^j$, where $\beta_j \in \{0, 1, \ldots, b-1\}$. Then

$$\sum_{i=0}^{b^{\nu-1}} \psi_{k_1}(\eta_{b,\nu}(i))\psi_{k_2}(\zeta'_{b,\nu}(i)) = \sum_{i_0=0}^{b-1} \cdots \sum_{i_{\nu-1}=0}^{b-1} \prod_{n=0}^{\nu-1} \omega^{\alpha_n i_{\nu-1-n}} \prod_{t=0}^{\nu-1} \omega^{\beta_t (i_t + \mu_t)} =$$

$$= \prod_{t=0}^{\nu-1} \omega^{\beta_t \mu_t} \prod_{n=0}^{\nu-1} \sum_{i_n=0}^{b-1} \omega^{(\alpha_{\nu-1-n} + \beta_n) i_n}. \tag{4}$$

The condition $k_2 \neq k_1^*$ shows that any $\delta,\ \ 0 \leqslant \delta \leqslant \nu-1$, exists such that $\beta_\delta \neq \overline{\alpha}_{\nu-1-\delta}$. Then we have $\sum\limits_{i_\delta=0}^{b-1} \omega^{(\alpha_{\nu-1-\delta}+\beta_\delta)i_\delta} = 0$ and from (4) we obtain

$$\sum_{i=0}^{b^{\nu-1}} \psi_{k_1}(\eta_{b,\nu}(i))\psi_{k_2}(\zeta'_{b,\nu}(i)) = 0.$$

Lemma 2. *Let $\nu > 0$ be an arbitrary integer and the fixed integers g_1 and g_2 satisfy the conditions*

$$0 \leqslant g_1 \leqslant \nu-1 < \nu \leqslant g_2.$$

An arbitrary integer $k_1,\ \ b^{g_1} \leqslant k_1 < b^{g_1+1}$, has the representation in the form

$$k_1 = \sum_{j=0}^{g_1} \alpha_j b^j = \sum_{j=0}^{\nu-1} \alpha_j b^j, \tag{5}$$

where for $j,\ \ 0 \leqslant j \leqslant g_1,\ \ \alpha_j \in \{0,1,\ldots,b-1\},\ \ \alpha_{g_1} \neq 0$ and for every $j,\ \ g_1+1 \leqslant j \leqslant \nu-1,\ \ \alpha_j = 0$.

We define the integer k^* thus, so $b^{g_2} \leqslant k^* < b^{g_2+1}$ and we represent it in the form

$$k^* = \sum_{j=0}^{\nu-1} \overline{\alpha}_{\nu-1-j} b^j + \sum_{j=\nu}^{g_2} \beta_j b^j,$$

where for $j,\ \ 0 \leqslant j \leqslant \nu-1,\ \ \overline{\alpha}_{\nu-1-j} \oplus \alpha_j = 0$ and for every $j,\ \ \nu \leqslant j \leqslant g_2,\ \ \beta_j \in \{0,1,\ldots,b-1\},\ \ \beta_{g_2} \neq 0$.

Then for every $k_2,\ \ b^{g_2} \leqslant k_2 < b^{g_2+1}$, the equality

$$\left| \sum_{i=0}^{b^{\nu}-1} \psi_{k_1}(\eta_{b,\nu}(i))\psi_{k_2}(\zeta'_{b,\nu}(i)) \right| = \begin{cases} b^\nu, & \textit{if}\ \ k_2 = k^* \\ 0, & \textit{if}\ \ k_2 \neq k^*. \end{cases}$$

holds.

Proof. An arbitrary integer i, $0 \leqslant i < b^\nu$, has the representation $i = \sum\limits_{j=0}^{\nu-1} i_j b^j$. Then $\eta_{b,\nu}(i) = \sum\limits_{j=0}^{\nu-1} i_{\nu-1-j} b^{-j-1}$ and $\zeta'_{b,\nu}(i) = \sum\limits_{j=0}^{\nu-1} (i_j \oplus \mu_j) b^{-j-1}$. For integer k_1 in the form (5) we have

$$\psi_{k_1}(\eta_{b,\nu}(i)) = \prod_{j=0}^{g_1} \omega^{\alpha_j i_{\nu-1-j}} = \prod_{j=0}^{\nu-1} \omega^{\alpha_j i_{\nu-1-j}} = \prod_{j=0}^{\nu-1} \omega^{\alpha_{\nu-1-j} i_j}. \tag{6}$$

Let $k_2 = k^*$. We obtain

$$\psi_{k_2}(\zeta'_{b,\nu}(i)) = \prod_{j=0}^{\nu-1} \omega^{\overline{\alpha}_{\nu-1-j}(i_j+\mu_j)}. \tag{7}$$

From (6) and (7) we obtain

$$\left|\sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i))\psi_{k_2}(\zeta'_{b,\nu}(i))\right| = \left|\sum_{i=0}^{b^\nu-1}\prod_{j=0}^{\nu-1} \omega^{\alpha_{\nu-1-j}i_j} \prod_{j=0}^{\nu-1} \omega^{\overline{\alpha}_{\nu-1-j}(i_j+\mu_j)}\right| =$$

$$= \left|\prod_{j=0}^{\nu-1} \omega^{\overline{\alpha}_{\nu-1-j}+\mu_j} \sum_{i=0}^{b^\nu-1} 1\right| = b^\nu.$$

Let, now, $k_2 \neq k^*$. This shows that an index t, $0 \leqslant t \leqslant \nu-1$, exists such that $\overline{\alpha}_{\nu-1-t} \neq \alpha_t$. Let $\overline{\alpha}_{\nu-1-t} = \beta_t$. Then we have

$$\psi_{k_1}(\eta_{b,\nu}(i))\psi_{k_2}(\zeta'_{b,\nu}(i)) = \prod_{j=0}^{t-1} \omega^{\overline{\alpha}_{\nu-1-j}\mu_j} \prod_{j=t+1}^{\nu-1} \omega^{\overline{\alpha}_{\nu-1-j}\mu_j}\omega^{\beta_t\mu_t}\omega^{(\alpha_t+\beta_t)i_t}.$$

From here we obtain

$$\sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i))\psi_{k_2}(\zeta'_{b,\nu}(i)) =$$

$$= \prod_{j=0}^{t-1} \omega^{\overline{\alpha}_{\nu-1-j}\mu_j} \prod_{j=t+1}^{\nu-1} \omega^{\overline{\alpha}_{\nu-1-j}\mu_j}\omega^{\beta_t\mu_t} \sum_{i_0=0}^{b-1} 1 \dots \sum_{i_t=0}^{b-1} \omega^{(\alpha_t+\beta_t)i_t} \dots \sum_{i_{\nu-1}=0}^{b-1} 1 = 0$$

because for $\alpha_t \neq \beta_t$, $\sum_{i_t=0}^{b-1} \omega^{(\alpha_t+\beta_t)i_t} = 0$.

Lemma 3. *Let $\nu > 0$ be an arbitrary integer and the fixed integers g_1 and g_2 satisfy the conditions*

$$0 \leqslant g_2 \leqslant \nu - 1 < \nu \leqslant g_1.$$

An arbitrary integer k_2, $b^{g_2} \leqslant k_2 < b^{g_2+1}$, has the representation in the form

$$k_2 = \sum_{j=0}^{g_2} \beta_j b^j = \sum_{j=0}^{\nu-1} \beta_j b^j,$$

where for j, $0 \leqslant j \leqslant g_2$, $\beta_j \in \{0, 1, \dots, b-1\}$, $\beta_{g_2} \neq 0$ and for every j, $g_2 + 1 \leqslant j \leqslant \nu - 1$, $\beta_j = 0$.

We define the integer k^{**} thus, so $b^{g_1} \leqslant k^{**} < b^{g_1+1}$ and we represent it in the form

$$k^{**} = \sum_{j=0}^{\nu-1} \overline{\beta}_{\nu-1-j} b^j + \sum_{j=\nu}^{g_1} \alpha_j b^j,$$

where for j, $0 \leqslant j \leqslant \nu-1$, $\overline{\beta}_{\nu-1-j} \oplus \beta_j = 0$ and for every j, $\nu \leqslant j \leqslant g_1$, $\alpha_j \in \{0, 1, \ldots, b-1\}$, $\alpha_{g_1} \neq 0$.

Then for every k_1, $b^{g_1} \leqslant k_1 < b^{g_1+1}$, the equality

$$\left| \sum_{i=0}^{b^\nu - 1} \psi_{k_1}(\eta_{b,\nu}(i)) \psi_{k_2}(\zeta'_{b,\nu}(i)) \right| = \begin{cases} b^\nu, & \text{if } k_1 = k^{**} \\ 0, & \text{if } k_1 \neq k^{**}. \end{cases}$$

holds.

The proof of Lemma 3 is similar to the proof of Lemma 2.

For an arbitrary integer $k \geqslant 0$, we define the function $\delta_{b^\nu}(k)$ as

$$\delta_{b^\nu}(k) = \begin{cases} 1, & \text{if } k \equiv 0 (mod \;\; b^\nu) \\ 0, & \text{if } k \not\equiv 0 (mod \;\; b^\nu). \end{cases}$$

Lemma 4. *For every integer $k \geqslant 0$, the equality*

$$\sum_{i=0}^{b^\nu - 1} \psi_k \left(\frac{i}{b^\nu} \right) = b^\nu \delta_{b^\nu}(k)$$

holds.

Proof. Let an arbitrary i, $0 \leqslant i < b^\nu$, have the representation in the form

$$i = i_0 b^{\nu-1} + i_1 b^{\nu-2} + \ldots + i_{\nu-2} b + i_{\nu-1},$$

where $i_{(j)} \in \{0, 1, \ldots, b-1\}$, $0 \leqslant j \leqslant \nu-1$. Then

$$\frac{i}{b^\nu} = 0.i_0 i_1 \ldots i_{\nu-2} i_{\nu-1}. \tag{8}$$

The number k has the representation

$$k = \sum_{j=0}^{\infty} k_j b^j, \quad k_j \in \{0, 1, \ldots, b-1\}. \tag{9}$$

From Definition 2, (8) and (9) we have

$$\psi_k \left(\frac{i}{b^\nu} \right) = \prod_{j=0}^{\infty} \omega^{i_j k_j}. \tag{10}$$

Let $k \equiv 0(\bmod \ b_\nu)$. Then $\delta_{b^\nu}(k) = 1$ and from (9) we obtain

$$k = \sum_{j=\nu}^{\infty} k_j b^j. \qquad (11)$$

From (10) we obtain

$$\psi_k\left(\frac{i}{b^\nu}\right) = \prod_{j=0}^{\nu-1} \omega^{i_j k_j} \prod_{j=\nu}^{\infty} \omega^{i_j k_j}. \qquad (12)$$

From (8) we have for $j \geqslant \nu \quad i_j = 0$ and from (11) we have for every $j, \quad 0 \leqslant j \leqslant \nu - 1, \quad k_j = 0$.

Hence from (12) we obtain

$$\sum_{i=0}^{b^\nu-1} \psi_k\left(\frac{i}{b^\nu}\right) = \sum_{i=0}^{b^\nu-1} 1 = b^\nu,$$

whence the statement of Lemma 4 is proved in the case of $k \equiv 0(\bmod \ b_\nu)$.

Let $k \not\equiv 0(\bmod \ b_\nu)$. Then from (9) follows the existence of the index $t, \quad 0 \leqslant t \leqslant \nu - 1$, such that $k_t \neq 0$. Then from (9) and (10) we obtain

$$\sum_{i=0}^{b^\nu-1} \psi_k\left(\frac{i}{b^\nu}\right) = \sum_{i_0=0}^{b-1} \omega^{i_0 k_0} \ldots \sum_{i_t=0}^{b-1} \omega^{i_t k_t} \ldots \sum_{i_{\nu-1}=0}^{b-1} \omega^{i_{\nu-1} k_{\nu-1}} = 0$$

because for $k_t \neq 0, \quad \sum_{i_t=0}^{b-1} \omega^{i_t k_t} = 0.$

From the definition of the function $\delta_{b^\nu}(k)$ when $k \not\equiv 0(\bmod \ b_\nu)$ the statement of Lemma 4 is obtained.

We define the sets $A(g, \alpha) = \{k : k = k_g b^g + k_{g-1} b^{g-1} + \ldots + k_\alpha b^\alpha, \alpha \leqslant j \leqslant g, k_j \in \{0, 1, \ldots b-1\}, k_\alpha, k_g \neq 0\}$. It is obvious

$$|A(g, \alpha)| = \begin{cases} (b-1)^2 b^{g-1-\alpha}, & \textit{if} \ \ 0 \leqslant \alpha \leqslant g-1, \\ b-1, & \textit{if} \ \ \alpha = g. \end{cases}$$

Lemma 5. *The b-adic diaphony $F(\psi(b); \eta_{b,\nu})$ and $F(\psi(b); \zeta'_{b,\nu})$ of the nets $\eta_{b,\nu}$ and $\zeta'_{b,\nu}$ satisfy respectively the equalities*

$$F(\psi(b); \eta_{b,\nu}) = \frac{1}{b^\nu}, \quad F(\psi(b); \zeta'_{b,\nu}) = \frac{1}{b^\nu}.$$

Proof. From definition 3 we obtain

$$b^{2\nu} F^2(\psi(b); \eta_{b,\nu}) = \frac{1}{b} \sum_{g=0}^{\infty} b^{-2g} \sum_{k=b^g}^{b^{g+1}-1} \left| \sum_{i=0}^{b^\nu-1} \psi_k(\eta_{b,\nu}(i)) \right|^2 =$$

$$= \frac{1}{b}\sum_{g=0}^{\infty} b^{-2g} \sum_{\alpha=0}^{g} \sum_{k\in A(g,\alpha)} \left| \sum_{i=0}^{b^\nu-1} \psi_k(\eta_{b,\nu}(i)) \right|^2 .$$

Using Lemma 4, we obtain

$$b^{2\nu} F^2(\psi(b);\eta_{b,\nu}) = \frac{1}{b}\sum_{g=0}^{\infty} b^{-2g} \sum_{\alpha=0}^{g} \sum_{k\in A(g,\alpha)} \left| \sum_{i=0}^{b^\nu-1} \psi_k(\eta_{b,\nu}(i)) \right|^2$$

$$= \frac{1}{b}\sum_{g=0}^{\infty} b^{-2g} \sum_{\alpha=0}^{g} \sum_{k\in A(g,\alpha)} b^{2\nu}\delta_{b^\nu}(k).$$

Then

$$F^2(\psi(b);\eta_{b,\nu}) = \frac{1}{b}\sum_{g=0}^{\nu-1} b^{-2g}\sum_{\alpha=0}^{g} \sum_{k\in A(g,\alpha)} \delta_{b^\nu}(k) + \frac{1}{b}\sum_{g=\nu}^{\infty} b^{-2g}\sum_{\alpha=0}^{\nu-1} \sum_{k\in A(g,\alpha)} \delta_{b^\nu}(k) +$$

$$+\frac{1}{b}\sum_{g=\nu}^{\infty} b^{-2g} \sum_{\alpha=\nu}^{g} \sum_{k\in A(g,\alpha)} \delta_{b^\nu}(k) = \frac{1}{b}\sum_{g=\nu}^{\infty} b^{-2g} \sum_{\alpha=\nu}^{g} \sum_{k\in A(g,\alpha)} 1 = \frac{1}{b}\frac{1}{b^{2\nu}} \sum_{k\in A(\nu,\nu)} 1$$

$$+\frac{1}{b}\sum_{g=\nu+1}^{\infty} b^{-2g} \sum_{\alpha=\nu}^{g-1} \sum_{k\in A(g,\alpha)} 1 + \frac{1}{b}\sum_{g=\nu+1}^{\infty} b^{-2g} \sum_{k\in A(g,g)} 1$$

$$= \frac{b-1}{b}\frac{1}{b^{2\nu}} + \frac{(b-1)^2}{b^2}\sum_{g=\nu+1}^{\infty} b^{-g} \sum_{\alpha=\nu}^{g-1} b^{-\alpha} + \frac{b-1}{b}\sum_{g=\nu+1}^{\infty} b^{-2g}$$

$$= \frac{b-1}{b}\frac{1}{b^{2\nu}} + \frac{(b-1)^2}{b^2}\sum_{g=\nu+1}^{\infty} b^{-g}\left[\frac{b}{b-1}\frac{1}{b^\nu}\left(1-\frac{b^\nu}{b^g}\right)\right] + \frac{1}{b(b+1)}\frac{1}{b^{2\nu}}$$

$$= \frac{b-1}{b}\frac{1}{b^{2\nu}} + \frac{b-1}{b}\frac{1}{b^\nu}\sum_{g=\nu+1}^{\infty} b^{-g} - \frac{b-1}{b}\sum_{g=\nu+1}^{\infty} b^{-2g} + \frac{1}{b(b+1)}\frac{1}{b^{2\nu}} = \frac{1}{b^{2\nu}}.$$

3. PROOF OF THEOREM

From the definition of the b-adic diaphony we have the following:

$$F^2(\psi(b);Z_{b,\nu}^{\alpha,\beta}) = \frac{1}{b(b+2)}\sum_{\mathbf{k}\neq\mathbf{0}} \rho(\mathbf{k}) \left| \frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i))\psi_{k_2}(\zeta'_{b,\nu}(i)) \right|^2 =$$

$$= \frac{1}{b(b+2)} \sum_{k=1}^{\infty} \rho(k) \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} \psi_k(\eta_{b,\nu}(i)) \right|^2 +$$

$$+ \frac{1}{b(b+2)} \sum_{k=1}^{\infty} \rho(k) \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} \psi_k(\zeta'_{b,\nu}(i)) \right|^2 +$$

$$+ \frac{1}{b(b+2)} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \rho(k_1,k_2) \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i)) \psi_{k_2}(\zeta'_{b,\nu}(i)) \right|^2 =$$

$$= \frac{1}{b+2} F^2(\psi(b);\eta_{b,\nu}) + \frac{1}{b+2} F^2(\psi(b);\zeta'_{b,\nu}) +$$

$$+ \frac{1}{b(b+2)} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \rho(k_1,k_2) \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i)) \psi_{k_2}(\zeta'_{b,\nu}(i)) \right|^2 . \qquad (13)$$

For the sum in (13) we have the representation

$$\frac{1}{b(b+2)} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \rho(k_1,k_2) \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i)) \psi_{k_2}(\zeta'_{b,\nu}(i)) \right| =$$

$$= \frac{1}{b(b+2)} \sum_{g_1=0}^{\nu-1} b^{-2g_1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=0}^{\nu-1} b^{-2g_2} \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i)) \psi_{k_2}(\zeta'_{b}(i)) \right|^2 +$$

$$+ \frac{1}{b(b+2)} \sum_{g_1=0}^{\nu-1} b^{-2g_1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=\nu}^{\infty} b^{-2g_2} \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i)) \psi_{k_2}(\zeta'_{b,\nu}(i)) \right|^2 +$$

$$+ \frac{1}{b(b+2)} \sum_{g_1=\nu}^{\infty} b^{-2g_1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=0}^{\nu-1} b^{-2g_2} \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i)) \psi_{k_2}(\zeta'_{b,\nu}(i)) \right|^2 +$$

$$+ \frac{1}{b(b+2)} \sum_{g_1=\nu}^{\infty} b^{-2g_1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=\nu}^{\infty} b^{-2g_2} \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i)) \psi_{k_2}(\zeta'_{b,\nu}(i)) \right|^2 =$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4. \qquad (14)$$

For the sum Σ_1 using the sets $A(g,\alpha)$ we have

$$\Sigma_1 = \frac{1}{b(b+2)} \sum_{g_1=0}^{\nu-1} b^{-2g_1} \sum_{m=0}^{g_1} \sum_{k_1 \in A(g_1,m)} \sum_{g_2=0}^{\nu-1} b^{-2g_2} \times$$

$$\times \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i))\psi_{k_2}(\zeta'_{b,\nu}(i)) \right|^2 =$$

$$= \frac{1}{b(b+2)} \sum_{g_1=0}^{\nu-1} b^{-2g_1} \sum_{k_1 \in A(g_1,g_1)} \sum_{g_2=0}^{\nu-1} b^{-2g_2} \times$$

$$\times \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i))\psi_{k_2}(\zeta'_{b,\nu}(i)) \right|^2 +$$

$$+ \frac{1}{b(b+2)} \sum_{g_1=1}^{\nu-1} b^{-2g_1} \sum_{m=0}^{g_1-1} \sum_{k_1 \in A(g_1,m)} \sum_{g_2=0}^{\nu-1} b^{-2g_2} \times$$

$$\times \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i))\psi_{k_2}(\zeta'_{b,\nu}(i)) \right|^2 . \qquad (15)$$

From Lemma 1 and (15) we obtain

$$\Sigma_1 = \frac{1}{b(b+2)} \sum_{g_1=0}^{\nu-1} b^{-2g_1} b^{-2(\nu-1-g_1)} \sum_{k_1 \in A(g_1,g_1)} 1 +$$

$$+ \frac{1}{b(b+2)} \sum_{g_1=1}^{\nu-1} b^{-2g_1} \sum_{m=0}^{g_1-1} b^{-2(\nu-1-m)} \sum_{k_1 \in A(g_1,m)} 1 =$$

$$= \frac{b}{(b+2)b^{2\nu}} \sum_{g_1=0}^{\nu-1} \sum_{k_1 \in A(g_1,g_1)} 1 +$$

$$+ \frac{b}{(b+2)b^{2\nu}} \sum_{g_1=1}^{\nu-1} b^{-2g_1} \sum_{m=0}^{g_1-1} b^{2m} \sum_{k_1 \in A(g_1,m)} 1 =$$

$$= \frac{b(b-1)}{b+2} \frac{\nu}{b^{2\nu}} + \frac{(b-1)^2}{(b+2)b^{2\nu}} \sum_{g_1=1}^{\nu-1} b^{-g_1} \sum_{m=0}^{g_1-1} b^m =$$

$$= \frac{b^2-1}{b+2} \frac{\nu}{b^{2\nu}} - \frac{b}{b+2} \frac{1}{b^{2\nu}} + \frac{b}{b+2} \frac{1}{b^{3\nu}}. \qquad (16)$$

From Lemma 2 for the sum Σ_2 we obtain

$$\Sigma_2 = \frac{b-1}{b(b+2)} \frac{1}{b^\nu} \sum_{g_1=0}^{\nu-1} b^{-2g_1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=\nu}^{\infty} b^{-g_2} = \frac{1}{b+2} \frac{1}{b^{2\nu}} \sum_{g_1=0}^{\nu-1} b^{-2g_1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} 1 =$$

$$= \frac{b-1}{b+2}\frac{1}{b^{2\nu}}\sum_{g_1=0}^{\nu-1} b^{-g_1} = \frac{b}{b+2}\frac{1}{b^{2\nu}} - \frac{b}{b+2}\frac{1}{b^{3\nu}}. \quad (17)$$

By analogy, from Lemma 3 for the sum Σ_3 we obtain

$$\Sigma_3 = \frac{b}{b+2}\frac{1}{b^{2\nu}} - \frac{b}{b+2}\frac{1}{b^{3\nu}}. \quad (18)$$

Using the trivial estimation $\left|\sum_{i=0}^{b^\nu-1} \psi_{k_1}(\eta_{b,\nu}(i))\psi_{k_2}(\zeta'_b(i))\right| \leqslant b^\nu$, we obtain the estimation

$$\Sigma_4 \leq \frac{b}{b+2}\frac{1}{b^{2\nu}}. \quad (19)$$

From (13), (14), (16), (17), (18), (19) and Lemma 5 we obtain

$$F^2(\psi(b); Z_{b,\nu}^{\alpha,\beta}) \leqslant \frac{b^2-1}{b+2}\frac{\nu}{b^{2\nu}} + \frac{2(b+1)}{b+2}\frac{1}{b^{2\nu}} - \frac{b}{b+2}\frac{1}{b^{3\nu}}.$$

From (13) and (14) we have

$$F^2(\psi(b); Z_{b,\nu}^{\alpha,\beta}) \geqslant \Sigma_1.$$

From (16) we have

$$F^2(\psi(b); Z_{b,\nu}^{\alpha,\beta}) \geqslant \frac{b^2-1}{b+2}\frac{\nu}{b^{2\nu}} - \frac{b}{b+2}\frac{1}{b^{2\nu}} + \frac{b}{b+2}\frac{1}{b^{3\nu}},$$

so that Theorem 1 is completely proved.

Corollary 1 is obtained directly from Theorem 1.

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