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A. P. Bakulev[1], A. I. Karanikas[2], N. G. Stefanis[3]

ANALYTICITY PROPERTIES OF THREE-POINT FUNCTIONS IN QCD BEYOND LEADING ORDER

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[1] E-mail: bakulev@thsun.jinr.ru
[2] University of Athens, Department of Physics, Nuclear and Particle Physics Section, Panepistimiopolis, GR-15771 Athens, Greece;
E-mail: akaranik@cc.uoa.gr
[3] Institut für Theoretische Physik II, Ruhr-Universität Bochum, D-44780 Bochum, Germany;
E-mail: stefanis@tp2.ruhr-uni-bochum.de

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Аналитические свойства трехточечных функций в КХД
в нелидирующем порядке

Обсуждается удаление нефизических сингулярностей из пертурбативной (факторизуемой) части формфактора пиона, являющейся классическим примером 3-точечной функции в КХД. Изучаются различные процедуры «аналитизации» в смысле Ширкова и Соловцова в сравнении со стандартной теорией возмущений КХД. Показано, что требование аналитичности формфактора *как целого*, предложенное ранее Караникасом и Стефанисом, делает ИК-конечной не только сильную константу связи и ее степени, но также и потенциально большие логарифмы масштаба факторизации (появляющиеся впервые в следующем за ведущим порядке теории возмущений КХД). Предложенная схема аналитизации обобщает аналитическую теорию возмущений Ширкова–Соловцова на дробные степени сильной константы связи и уменьшает зависимость адронных КХД-характеристик от выбора ренормализационной схемы и масштаба, в том числе и от масштаба факторизации.

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Analyticity Properties of Three-Point Functions in QCD Beyond Leading Order

The removal of unphysical singularities in the perturbatively calculable part of the pion form factor — a classic example of a three-point function in QCD — is discussed. Different «analytization» procedures in the sense of Shirkov and Solovtsov are examined in comparison with standard QCD perturbation theory. We show that demanding the analyticity of the partonic amplitude as a *whole*, as proposed before by Karanikas and Stefanis, one can make infrared finite not only the strong running coupling and its powers, but also cure potentially large logarithms (that first appear in the next-to-leading order) containing the factorization scale and modifying the discontinuity across the cut along the negative real axis. The scheme used here generalizes the Analytic Perturbation Theory of Shirkov and Solovtsov to non-integer powers of the strong coupling and diminishes the dependence of QCD hadronic quantities on all perturbative scheme and scale-setting parameters, including the factorization scale.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

The phenomenology of QCD exclusive processes depends in a crucial way on the analytic properties of hadronic (hard) scattering amplitudes as functions of the strong running coupling. A perturbatively calculable short-distance part of the reaction amplitude at the parton level is isolated either by subtraction or by factorization. To get a quantitative interpretation of such quantities in practice and compare them with experimental data, one has to get rid of the artificial Landau singularity at $Q^2 = \Lambda^2_{\rm QCD}$ ($\Lambda_{\rm QCD} \equiv \Lambda$ in the following), where Q^2 is the large mass scale in the process. A proposal to solve this problem (in the spacelike region) without introducing exogenous infrared (IR) regulators, like an effective, or a dynamically generated, gluon mass [1] (see, for instance, [2–10] for such applications), was made by Shirkov and Solovtsov (SS) [11–13], based on general principles of local Quantum Field Theory. This theoretical framework — termed Analytic Perturbation Theory (APT) — was further expanded beyond the one-loop level of two-point functions to define an analytic* coupling and its powers in the timelike region [14–21], embracing previous attempts [22–27] in this direction**.

However, first applications [30, 31] of this sort of approach to three-point functions, beyond the leading order of QCD perturbation theory, have made it clear that, ultimately, there must be an extension of this formalism from the level of the running coupling and its powers to the level of amplitudes. The reason is that in three-point functions at the next-to-leading order (NLO) level, and beyond, logarithms of a distinct scale (serving as the factorization or evolution scale) appear that though they do not change the nature of the Landau pole, they affect the discontinuity across the cut along the negative real axis $-\infty < Q^2 < 0$. On account of factorization, we expect that this effect should be small, of the order of a few percent, because any change caused by the variation of the factorization scale should be of the next higher order. However, to achieve a high-precision theoretical prediction, one should reduce this uncertainty, lifting the limitations imposed by the lack of knowledge about uncalculated higher-order corrections. To encompass such logarithmic terms in the «analytization» procedure, one should demand the analyticity of the partonic amplitude

*The term «analyticity» is used here as a synonym for «spectrality» and «causality» [12].

**A somewhat different approach was reviewed recently in [28]; see also [29].

as a *whole* [32, 33] and calculate the dispersive image of the coupling (or of its powers) *in conjunction* with these logarithms. This Karanikas–Stefanis (KS) «analytization» scheme effectively amounts to the generalization of APT to non-integer powers of the running coupling: Fractional APT (FAPT), as we shall show below.

In this work we expand the Shirkov–Solovtsov «analytization» approach to include the dispersive images of such terms, using as a case study the pion form factor at NLO in the $\overline{\text{MS}}$ scheme with various renormalization-scale settings and also in the α_V-scheme [34]. To this end, we contrast the KS «analytization» with the *naive* [30, 31] and the *maximal* [35] «analytization» procedures and work out their key mutual differences as they first appear in NLO, while a fully-fledged analysis of FAPT is given in an accompanying paper [36]. We argue that augmenting the $\overline{\text{MS}}$ scheme with the KS «analytization» prescription provides an optimized method to calculate perturbatively higher-order corrections to partonic «observables» in QCD because it practically eliminates all scheme and scale-setting ambiguities owing to the renormalization and factorization scales. It is worth emphasizing at this point that the focus of Ref. [32] was on the calculation of power corrections to the pion's electromagnetic form factor. Such contributions are outside the scope of the present investigation.

The plan of this paper is as follows. In Sec. 2 we review the convolution formalism for the calculation of the short-distance part of the pion form factor within perturbative QCD at NLO. In Sec. 3 we discuss the Shirkov–Solovtsov type «analytization» procedures [11,30–33,35] and work out their mutual differences, focusing on the KS «analytization» and its properties. This discussion extends and generalizes the original KS analysis that covered only the LO of the perturbative expansion of the pion form factor and ignoring evolution. Section 4 contains the results for the factorized pion form factor in different schemes and with different scale settings, employing the KS «analytization» in comparison with those based on APT and also standard QCD perturbation theory in NLO. Our conclusions with a summary of our main results are presented in Sec. 5. Some important technical details are collected in three appendices.

2. FACTORIZABLE PART OF THE PION FORM FACTOR AT NLO IN STANDARD QCD PERTURBATION THEORY

The leading-twist factorizable part of the electromagnetic pion form factor can be expressed as a convolution in the form [37,38]

$$F_\pi^{\text{Fact}}(Q^2;\mu_{\text{R}}^2) = \Phi_\pi^*(x,\mu_{\text{F}}^2) \underset{x}{\otimes} T_{\text{H}}(x,y,Q^2;\mu_{\text{F}}^2,\mu_{\text{R}}^2) \underset{y}{\otimes} \Phi_\pi(y,\mu_{\text{F}}^2), \tag{2.1}$$

where $\otimes$ denotes the usual convolution symbol ($A(z)\underset{z}{\otimes}B(z) \equiv \int_0^1 dz A(z)B(z)$) over the longitudinal momentum fraction variable x (y) and μ_{F} represents the factorization scale at which the separation between the long- (small transverse momentum) and short-distance (large transverse momentum) dynamics takes place, with $\mu_{\rm R}$ labelling the renormalization (coupling constant) scale. The nonperturbative input is encoded in the pion distribution amplitude (DA) $\Phi_\pi(y, \mu_{\rm F}^2)$, whereas the short-distance interactions are represented by the hard-scattering amplitude $T_{\rm H}(x, y, Q^2; \mu_{\rm F}^2, \mu_{\rm R}^2)$. This is the amplitude for a collinear valence quark–antiquark pair with total momentum P struck by a virtual photon with momentum q, satisfying $q^2 = -Q^2$, to end up again in a configuration of a parallel valence quark–antiquark pair with momentum $P' = P + q$. It can be calculated perturbatively in the form of a power-series expansion in the QCD coupling, the latter to be evaluated at the reference scale of renormalization $\mu_{\rm R}^2$:

$$T_{\rm H}^{\rm NLO}(x, y, Q^2; \mu_{\rm F}^2, \mu_{\rm R}^2) = \alpha_s\left(\mu_{\rm R}^2\right) T_{\rm H}^{(0)}(x, y, Q^2) + \\ + \frac{\alpha_s^2\left(\mu_{\rm R}^2\right)}{4\pi} T_{\rm H}^{(1)}(x, y, Q^2; \mu_{\rm F}^2, \mu_{\rm R}^2). \quad (2.2)$$

The leading-order (LO) contribution to $T_{\rm H}(x, y, Q^2; \mu_{\rm F}^2)$ reads

$$T_{\rm H}^{(0)}(x, y, Q^2) = \frac{N_{\rm T}}{Q^2}\frac{1}{\overline{x}\,\overline{y}} \equiv \frac{1}{Q^2} t_{\rm H}^{(0)}(x, y), \quad (2.3)$$

where

$$N_{\rm T} = \frac{2\pi C_{\rm F}}{C_{\rm A}} = \frac{8\pi}{9}, \quad (2.4)$$

$C_{\rm F} = \left(N_{\rm c}^2 - 1\right)/2N_{\rm c} = 4/3$, $C_{\rm A} = N_{\rm c} = 3$ are the color factors of $SU(3)_{\rm c}$, and the notation $\bar{z} \equiv 1 - z$ has been used. The usual color decomposition of the NLO correction [39] — marked by self-explainable labels — is given by (omitting the variables x and y)

$$Q^2 T_{\rm H}^{(1)}\left(Q^2; \mu_{\rm F}^2, \mu_{\rm R}^2\right) = C_{\rm F}\, t_{\rm H}^{(1,{\rm F})}\left(\frac{\mu_{\rm F}^2}{Q^2}\right) + b_0\, t_{\rm H}^{(1,\beta)}\left(\frac{\mu_{\rm R}^2}{Q^2}\right) + C_{\rm G}\, t_{\rm H}^{(1,{\rm G})}, \quad (2.5)$$

where $C_{\rm G} = (C_{\rm F} - C_{\rm A}/2)$ and b_0 is the first coefficient of the β function, see Appendix A, Eq. (A1). Here we explicitly factorized out a trivial $1/Q^2$ dependence and used for the coefficients in front of each factor the notation $t_{\rm H}$ with appropriate superscripts.

With reference to the application of the Brodsky–Lepage–Mackenzie (BLM) [40] scale setting in fixing the renormalization point later on, we single out the b_0-proportional (i. e., the N_f-dependent) term, given by

$$t_{\rm H}^{(1,\beta)}\left(x, y; \frac{\mu_{\rm R}^2}{Q^2}\right) = t_{{\rm H},1}^{(1,\beta)}(x, y) + t_{{\rm H},2}^{(1,\beta)}\left(x, y; \frac{\mu_{\rm R}^2}{Q^2}\right) \quad (2.6a)$$

with

$$t_{\mathrm{H},1}^{(1,\beta)}(x,y) = t_{\mathrm{H}}^{(0)}(x,y)\left[\frac{5}{3} - \ln(\overline{x}\,\overline{y})\right], \tag{2.6b}$$

$$t_{\mathrm{H},2}^{(1,\beta)}\left(x,y;\frac{\mu_{\mathrm{R}}^2}{Q^2}\right) = t_{\mathrm{H}}^{(0)}(x,y)\ln\frac{\mu_{\mathrm{R}}^2}{Q^2}, \tag{2.6c}$$

and present the color singlet part of t_{H} in the form

$$t_{\mathrm{H}}^{(1,\mathrm{F})}\left(x,y;\frac{\mu_{\mathrm{F}}^2}{Q^2}\right) = t_{\mathrm{H},1}^{(1,\mathrm{F})}(x,y) + t_{\mathrm{H},2}^{(1,\mathrm{F})}\left(x,y;\frac{\mu_{\mathrm{F}}^2}{Q^2}\right), \tag{2.7a}$$

$$t_{\mathrm{H},2}^{(1,\mathrm{F})}\left(x,y;\frac{\mu_{\mathrm{F}}^2}{Q^2}\right) = t_{\mathrm{H}}^{(0)}(x,y)\left[2\Big(3+\ln(\overline{x}\,\overline{y})\Big)\ln\frac{Q^2}{\mu_{\mathrm{F}}^2}\right]. \tag{2.7b}$$

Explicit expressions for $t_{\mathrm{H},1}^{(1,\mathrm{F})}(x,y)$ and for the color non-singlet part, $t_{\mathrm{H}}^{(1,\mathrm{G})}(x,y)$, cf. Eq. (2.5), are supplied in Appendix B (see Eqs. (B1), (B2)).

The scaled hard-scattering amplitude, Eq. (2.2), truncated at the NLO and evaluated at the renormalization scale $\mu_{\mathrm{R}}^2 = \lambda_{\mathrm{R}} Q^2$, reads

$$\begin{aligned} Q^2 T_{\mathrm{H}}^{\mathrm{NLO}}\left(x,y,Q^2;\mu_{\mathrm{F}}^2,\lambda_{\mathrm{R}}Q^2\right) &= \alpha_s\left(\lambda_{\mathrm{R}}Q^2\right)\, t_{\mathrm{H}}^{(0)}(x,y) + \frac{\alpha_s^2\left(\lambda_{\mathrm{R}}Q^2\right)}{4\pi}\times \\ \times\, C_{\mathrm{F}}\, t_{\mathrm{H},2}^{(1,\mathrm{F})}\left(x,y;\frac{\mu_{\mathrm{F}}^2}{Q^2}\right) &+ \frac{\alpha_s^2\left(\lambda_{\mathrm{R}}Q^2\right)}{4\pi}\left\{b_0\, t_{\mathrm{H}}^{(1,\beta)}(x,y;\lambda_{\mathrm{R}}) + t_{\mathrm{H}}^{(\mathrm{FG})}(x,y)\right\}, \end{aligned} \tag{2.8}$$

where we have introduced the shorthand notation

$$t_{\mathrm{H}}^{(\mathrm{FG})}(x,y) \equiv C_{\mathrm{F}}\, t_{\mathrm{H},1}^{(1,\mathrm{F})}(x,y) + C_{\mathrm{G}}\, t_{\mathrm{H}}^{(1,\mathrm{G})}(x,y). \tag{2.9}$$

To calculate the factorizable part of the pion form factor, one has to convolute this expression with the pion DA for each hadron in the initial and final states. In leading twist 2, the pion DA at the normalization scale $\mu_0^2 \approx 1$ GeV^2 is given by

$$\varphi_\pi(x,\mu_0^2) = 6x(1-x)\left[1 + a_2(\mu_0^2)C_2^{3/2}(2x-1) + a_4(\mu_0^2)C_4^{3/2}(2x-1) + \ldots\right], \tag{2.10}$$

with all nonperturbative information being encapsulated in the Gegenbauer coefficients a_n. In this analysis we use those coefficients determined before by Bakulev, Mikhailov, and Stefanis (BMS) in [41] with the aid of QCD sum rules with nonlocal condensates:

$$a_2^{\mathrm{BMS}} = 0.20, \quad a_4^{\mathrm{BMS}} = -0.14, \quad a_n^{\mathrm{BMS}} = 0, \quad n > 4, \tag{2.11}$$

where the vacuum quark virtuality $\lambda_q^2 = 0.4$ GeV^2 has been used. This set of values was found [42,43] to be consistent at the 1σ level with the high-precision

CLEO data [44] on the pion–photon transition form factor, with all other model DAs being outside — at least — the 2σ error ellipse (see [45] for the latest compilation of models in comparison with the CLEO and CELLO [46] data). Notice that the particular parameterization (shape) of the pion DA chosen is irrelevant for the considerations to follow.

3. ANALYTICITY OF PARTONIC AMPLITUDES BEYOND LO

3.1. Analytic Running Coupling in QCD. The main stumbling block in applying fixed-order perturbation theory at low momenta Q^2 is the non-physical Landau singularity of the running strong coupling at $Q^2 = \Lambda^2$, which entails the appearance of IR renormalons in the perturbative expansion. To ensure the analyticity of the coupling in the infrared, one can follow different strategies [11, 25, 27, 47–51] all based on the basic assumption that the *physical* coupling should stay IR finite and analytic in the whole momentum range, though its precise value at $Q^2 = 0$ is still a matter of debate [11, 21, 28, 52, 53]. Imposing the analyticity of the coupling in the sense of Shirkov and Solovtsov [11], we replace the strong running coupling and its powers by their analytic versions:

$$\left[\alpha_s^{(n)}\left(Q^2\right)^m\right]^{\text{an}} \equiv \mathcal{A}_m^{(n)}\left(Q^2\right) \quad \text{with} \quad \left[f\left(Q^2\right)\right]^{\text{an}} = \frac{1}{\pi}\int_0^\infty \frac{\textbf{Im}\left[f(-\sigma)\right]}{\sigma + Q^2 - i\epsilon}\, d\sigma, \tag{3.1}$$

where the loop order is explicitly indicated by the superscript n in parenthesis and

$$\mathcal{A}_1^{(1)}\left(Q^2\right) = \frac{4\pi}{b_0}\left[\frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2}\right] \equiv \bar{\alpha}_s\left(Q^2\right), \tag{3.2}$$

with the last step connecting to the SS notation [11], and $\alpha_s(0) = 4\pi/b_0$. The two-loop running coupling in standard QCD perturbation theory can be expressed [19] in terms of the Lambert function W_{-1} to read

$$\alpha_s^{(2)}\left(Q^2\right) = -\frac{4\pi}{b_0 c_1}\left[1 + W_{-1}\left(-\frac{1}{c_1 e}\left(\frac{\Lambda^2}{Q^2}\right)^{1/c_1}\right)\right]^{-1}. \tag{3.3}$$

For some more explanations we refer the interested reader to [42], Appendix C, Eqs. (C15) and (C20). Then, the analytic image of the kth power of the coupling [14] is obtained from the dispersion relation

$$\mathcal{A}_k^{(2)}\left(Q^2\right) = \frac{1}{\pi}\int_0^\infty d\sigma \frac{\rho_k^{(2)}(\sigma)}{\sigma + Q^2 - i\epsilon} \tag{3.4}$$

with the spectral density

$$\rho_k^{(2)}(t) = \left(\frac{4\pi}{b_0 c_1}\right)^k \mathbf{Im}\left(-\frac{1}{1+W_1(z(t))}\right)^k . \tag{3.5}$$

In the numerical calculations below, we use an approximate form suggested in [35]:

$$\mathcal{A}_1^{(2,\text{fit})}\left(Q^2\right) = \frac{4\pi}{b_0}\left\{\frac{1}{\ell\left[\ln\left(Q^2/\Lambda_{21}^2\right),\, c_{21}^{\text{fit}}\right]} + \frac{1}{1-\exp(\ell\left[\ln\left(Q^2/\Lambda_{21}^2\right),\, c_{21}^{\text{fit}}\right])}\right\}; \tag{3.6}$$

$$\mathcal{A}_2^{(2,\text{fit})}\left(Q^2\right) = \\ = \left(\frac{4\pi}{b_0}\right)^2\left\{\frac{1}{\ell\left[\ln\left(Q^2/\Lambda_{22}^2\right), c_{22}^{\text{fit}}\right]^2} - \frac{\exp(\ell\left[\ln\left(Q^2/\Lambda_{22}^2\right), c_{22}^{\text{fit}}\right])}{\left[1-\exp(\ell\left[\ln\left(Q^2/\Lambda_{22}^2\right), c_{22}^{\text{fit}}\right])\right]^2}\right\}, \tag{3.7}$$

where the values of the fit parameters are listed in the Table and

$$\ell[L,c] \equiv L + c\,\ln\sqrt{L^2+4\pi^2}. \tag{3.8}$$

Parameters entering Eqs. (3.6) and (3.7) for the value $\Lambda_{\mathbf{QCD}}^{N_f=3} = 400$ MeV

Parameters	c_{21}^{fit}	Λ_{21}	c_{22}^{fit}	Λ_{22}
Values	-1.015	67 MeV	-1.544	34.5 MeV

3.2. «Analytization» Procedures. Let us now see how analyticity can be implemented on the parton-level pion form factor in NLO accuracy of perturbative QCD. We discuss three «analytization» procedures*:

- *Naive* «analytization» [30, 31, 35]

$$\left[Q^2 T_{\text{H}}\left(x,y,Q^2;\mu_{\text{F}}^2,\lambda_{\text{R}}Q^2\right)\right]_{\text{SS}}^{\text{naive-an}} = \mathcal{A}_1^{(2)}(\lambda_{\text{R}}Q^2)\,t_{\text{H}}^{(0)}(x,y) + \\ + \frac{\left(\mathcal{A}_1^{(2)}(\lambda_{\text{R}}Q^2)\right)^2}{4\pi}\left[b_0\,t_{\text{H}}^{(1,\beta)}(x,y;\lambda_{\text{R}}) + t_{\text{H}}^{(\text{FG})}(x,y) + \right. \\ \left. + C_{\text{F}}\,t_{\text{H},2}^{(1,\text{F})}\left(x,y;\frac{\mu_{\text{F}}^2}{Q^2}\right)\right]. \tag{3.9}$$

*One should not worry about the factor $1/Q^2$ because under «analytization» it reproduces itself, i.e., $\left[\left[f\left(Q^2\right)\right]^{\text{an}}/Q^2\right]^{\text{an}} = \left[f\left(Q^2\right)\right]^{\text{an}}/Q^2$.

- *Maximal* «analytization» [35]

$$\left[Q^2 T_{\rm H}\left(x,y,Q^2;\mu_{\rm F}^2,\lambda_{\rm R}Q^2\right)\right]_{\rm SS}^{\text{max-an}} = \mathcal{A}_1^{(2)}(\lambda_{\rm R}Q^2)\, t_{\rm H}^{(0)}(x,y)+ \\ + \frac{\mathcal{A}_2^{(2)}(\lambda_{\rm R}Q^2)}{4\pi}\left[b_0\, t_{\rm H}^{(1,\beta)}(x,y;\lambda_{\rm R}) + t_{\rm H}^{(\rm FG)}(x,y)+ \right. \\ \left. + C_{\rm F}\, t_{\rm H,2}^{(1,\rm F)}\left(x,y;\frac{\mu_{\rm F}^2}{Q^2}\right)\right]. \quad (3.10)$$

- *Amplitude* «analytization» proposed by Karanikas and Stefanis in [32, 33].

The first method replaces α_s and its powers by the Shirkov–Solovtsov analytic coupling [11] and its powers, whereas the second one uses for the powers of α_s their own analytic images, transforming this way the power-series expansion in $[\alpha_s\left(Q^2\right)]^n$ in a functional expansion in terms of the functions $\mathcal{A}_n\left(Q^2\right)$ [13, 16]. Imposing analyticity in the sense of Karanikas–Stefanis [32], differs from the previous two approaches in that it demands the *whole* partonic amplitude has the correct analytical behavior as a function of Q^2. This entails the «analytization» of terms of the form $\left(\alpha_s^{(n)}\left(Q^2\right)\right)^m \ln(Q^2/\mu_{\rm F}^2)$, which appear in exclusive amplitudes at NLO of QCD perturbation theory and contain an additional scale, $\mu_{\rm F}^2$. There are, in principle, two possibilities how to proceed any further. One option is provided by setting $\mu_{\rm F}^2 \simeq Q^2$ and then face the problem of «analytization» of terms like $\left[\alpha_s\left(Q^2\right)/\alpha_s(\mu_0^2)\right]^\eta$, where $\eta = \gamma_n^{(0)}/(2b_0)$ is a fractional number, as discussed in [36]. Another possibility is to fix the factorization scale $\mu_{\rm F}^2$ at some value and then to redefine the original Shirkov–Solovtsov «analytization» procedure in order to take the dispersive image of the coupling (or of its powers) together with these logarithmic terms. This second route is followed in the present work. It is important to note that the KS «analytization» procedure reduces in LO of fixed-order perturbation theory to the *maximal* one, as shown in [32], provided evolution effects of the pion distribution amplitudes are ignored.

Applying now this generalized «analytization» concept, we get

$$\left[Q^2 T_{\rm H}\left(x,y,Q^2;\mu_{\rm F}^2,\lambda_{\rm R}Q^2\right)\right]_{\rm KS}^{\rm an} = \mathcal{A}_1^{(2)}(\lambda_{\rm R}Q^2)\, t_{\rm H}^{(0)}(x,y)+ \\ + \frac{\mathcal{A}_2^{(2)}(\lambda_{\rm R}Q^2)}{4\pi}\left(b_0\, t_{\rm H}^{(1,\beta)}(x,y;\lambda_{\rm R}) + t_{\rm H}^{(\rm FG)}(x,y)\right)+ \\ + \left[\frac{\left(\alpha_s^{(2)}(\lambda_{\rm R}Q^2)\right)^2}{4\pi} C_{\rm F}\, t_{\rm H}^{(0)}(x,y)\left(6+2\ln(\overline{x}\,\overline{y})\right)\ln\frac{Q^2}{\mu_{\rm F}^2}\right]_{\rm KS}^{\rm an}. \quad (3.11)$$

In order to have the same scale argument in the logarithmic term as in the running coupling, we substitute $\ln(Q^2/\mu_{\rm F}^2) = \ln(\lambda_{\rm R} Q^2/\Lambda^2) - \ln(\lambda_{\rm R}\mu_{\rm F}^2/\Lambda^2)$ to obtain

$$\begin{aligned}
\left[Q^2 T_{\rm H}\left(x,y,Q^2;\mu_{\rm F}^2,\lambda_{\rm R}Q^2\right)\right]_{\rm KS}^{\rm an} &= \mathcal{A}_1^{(2)}(\lambda_{\rm R}Q^2)\, t_{\rm H}^{(0)}(x,y) + \\
&+ \frac{\mathcal{A}_2^{(2)}(\lambda_{\rm R}Q^2)}{4\pi}\left[b_0\, t_{\rm H}^{(1,\beta)}(x,y;\lambda_{\rm R}) + t_{\rm H}^{(\rm FG)}(x,y) - \right. \\
&\left. - C_{\rm F}\, t_{\rm H}^{(0)}(x,y)\,(6+2\ln(\bar{x}\,\bar{y}))\ln\frac{\lambda_{\rm R}\mu_{\rm F}^2}{\Lambda^2}\right] + \\
&+ \left[\frac{\left(\alpha_s^{(2)}(\lambda_{\rm R}Q^2)\right)^2}{4\pi} C_{\rm F}\, t_{\rm H}^{(0)}(x,y)\,(6+2\ln(\bar{x}\,\bar{y}))\ln\frac{\lambda_{\rm R}Q^2}{\Lambda^2}\right]_{\rm KS}^{\rm an} . \qquad (3.12)
\end{aligned}$$

Finally, we arrive at

$$\begin{aligned}
&\left[Q^2 T_{\rm H}(x,y,Q^2;\mu_{\rm F}^2,\lambda_{\rm R}Q^2)\right]_{\rm KS}^{\rm an} = \mathcal{A}_1^{(2)}(\lambda_{\rm R}Q^2)\, t_{\rm H}^{(0)}(x,y) + \\
&+ \frac{\mathcal{A}_2^{(2)}(\lambda_{\rm R}Q^2)}{4\pi}\left[b_0\, t_{\rm H}^{(1,\beta)}(x,y;\lambda_{\rm R}) + t_{\rm H}^{(\rm FG)}(x,y) + C_{\rm F}\, t_{\rm H,2}^{(1,\rm F)}\left(x,y;\frac{\mu_{\rm F}^2}{Q^2}\right)\right] + \\
&+ \frac{\Delta_2^{(2)}\left(\lambda_{\rm R}Q^2\right)}{4\pi}\left[C_{\rm F}\, t_{\rm H}^{(0)}(x,y)\,\left(6+2\ln(\bar{x}\,\bar{y})\right)\right], \qquad (3.13)
\end{aligned}$$

where the deviation from Eq. (3.10) is encoded in the term

$$\Delta_2^{(2)}\left(Q^2\right) \equiv \mathcal{L}_2^{(2)}\left(Q^2\right) - \mathcal{A}_2^{(2)}\left(Q^2\right)\,\ln\left[Q^2/\Lambda^2\right] \qquad (3.14)$$

with

$$\mathcal{L}_2^{(2)}\left(Q^2\right) \equiv \left[\left(\alpha_s^{(2)}\left(Q^2\right)\right)^2 \ln\left(\frac{Q^2}{\Lambda^2}\right)\right]_{\rm KS}^{\rm an} = \frac{4\pi}{b_0}\left[\frac{\left(\alpha_s^{(2)}\left(Q^2\right)\right)^2}{\alpha_s^{(1)}\left(Q^2\right)}\right]_{\rm KS}^{\rm an} . \qquad (3.15)$$

It is important to distinguish between the two contributions in Eq. (3.14). The first amounts to the «analytization» of the product of the coupling with a logarithm, or equivalently of fractional powers of the coupling, as shown in [36]. The second bears an additional logarithmic dependence on the momentum scale Q^2 relative to the expression obtained with the *maximal* «analytization» procedure. The subscript KS in the last equation signifies that this expression should be analyticized according to the KS prescription. To obtain a clearer idea of its meaning and demonstrate its essence, the «analytization» is performed in three incremental steps. First, a simplified version of this expression is considered, which results by provisionally replacing the two-loop coupling in the numerator by its one-loop counterpart. Then, the ratio of the couplings after «analytization» reduces to (dash-dotted line in Fig. 1, a)

$$\mathcal{L}_2^{(1)\,\text{approx}}\left(Q^2\right) = \frac{4\pi}{b_0}\,\mathcal{A}_1^{(1)}\left(Q^2\right). \tag{3.16}$$

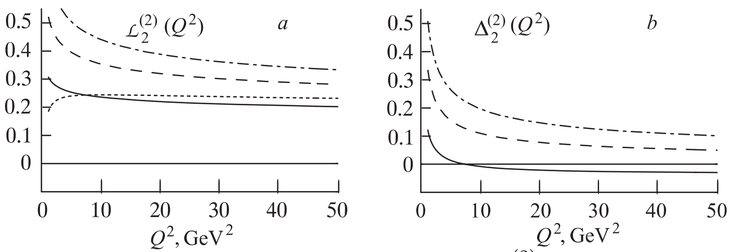


Fig. 1. a) Results for the analyticized logarithmic term, $\mathcal{L}_2^{(2)}\left(Q^2\right)$, using three different «analytization» procedures: the one-loop approximate KS «analytization», $\mathcal{L}_2^{(1)\,\text{approx}}\left(Q^2\right)$ (dash-dotted line), the two-loop approximate KS «analytization», $\mathcal{L}_2^{(2)\,\text{approx}}\left(Q^2\right)$ (dashed line), and the exact two-loop BMKS «analytization» $\mathcal{L}_2^{(2)}\left(Q^2\right)$ (solid line). For comparison, we also show here the corresponding *maximal* «analytization» curve (dotted line). b) Results are shown for the corresponding analyticized contributions $\Delta_2^{(2)}\left(Q^2\right)$, using the same three «analytization» procedures for $\mathcal{L}_2^{(2)}\left(Q^2\right)$ as in panel a

Second, we discuss an analogous situation, in which the one-loop coupling in the denominator is (inconsistently) traded for its two-loop counterpart. In this case, the ratio of the couplings after «analytization» becomes (dashed line in Fig. 1, a)

$$\mathcal{L}_2^{(2)\,\text{approx}}\left(Q^2\right) = \frac{4\pi}{b_0}\,\mathcal{A}_1^{(2)}\left(Q^2\right). \tag{3.17}$$

Finally, we provide the exact result for the KS «analytization» of expression (3.15) (solid line in Fig. 1, a), with the derivation presented in Appendix C, while more general expressions are given in [36]:

$$\mathcal{L}_2^{(2)}\left(Q^2\right) = \frac{4\pi}{b_0}\left[\mathcal{A}_1^{(2)}\left(Q^2\right) + c_1\,\frac{4\pi}{b_0}\,f_{\mathcal{L}}\left(Q^2\right)\right], \tag{3.18}$$

where

$$f_{\mathcal{L}}\left(Q^2\right) = \sum_{n\geq 0}\left[\psi(2)\zeta(-n-1) - \frac{d\zeta(-n-1)}{dn}\right]\frac{\left[-\ln\left(Q^2/\Lambda^2\right)\right]^n}{\Gamma(n+1)} \tag{3.19}$$

and $\zeta(z)$ is the Riemann zeta-function. Equation (3.14) is illustrated in Fig. 1, b for the different expressions of $\mathcal{L}_2\left(Q^2\right)$ given by Eqs. (3.16), (3.17), and (3.18), using the same line designations as in Fig. 1, a. Let us close this discussion by commenting that in the region where there are experimental data available [54,55] (i. e., well below 10 GeV^2), Eq. (3.14) is governed by $\mathcal{L}_2^{(2)}\left(Q^2\right)$, which entails a small enhancement of the hard-scattering amplitude for $Q^2 \leqslant 7.25$ GeV^2.

4. FACTORIZED PION FORM FACTOR AT NLO — STANDARD AND ANALYTICIZED

The calculation of the factorized pion form factor proceeds in terms of Eq. (2.1) and involves the convolution of expression (3.10) for the *maximal* «analytization» case, or expression (3.13) for the KS «analytization» case with the pion DA for which we employ in both cases the BMS parameterization [41], as discussed in Sec. 2. On that basis, we can obtain the scaled, factorized part of the pion form factor, $Q^2 F_\pi^{\text{Fact}}\left(Q^2; \mu_{\text{R}}^2 = \lambda_{\text{R}} Q^2\right)$, using Eq. (2.8) and the following set of substitutions*:

$$t_{\text{H}}^{(0)}(x,y) \quad \to \quad 8\pi\, f_\pi^2\, (1 + a_2 + a_4)^2\,, \tag{4.1}$$

$$-t_{\text{H}}^{(0)}(x,y) \ln \overline{xy} \quad \to \quad 8\pi f_\pi^2\, (1 + a_2 + a_4)\, [3 + (43/6) a_2 + (136/15) a_4]\,, \tag{4.2}$$

$$\begin{aligned} t_{\text{H}}^{(\text{FG})}(x,y) \quad \to \quad & 8\pi\, f_\pi^2 \big[-15.67 - a_2\, (21.52 - 6.22 a_2) \\ & - a_4\, (7.37 - 37.40\, a_2 - 33.61\, a_4) \big]. \end{aligned} \tag{4.3}$$

Notice that evolving the BMS pion DA from the initial scale μ_0^2 to the scale μ_{F}^2 at the NLO level will generate higher Gegenbauer harmonics of the form $x\,\overline{x} C_{2n}^{3/2}(2x-1)$ with $n \geqslant 3$. However, we have shown in [35] (see also [56]) that for the calculation of the pion form factor it is actually sufficient to restrict ourselves to the LO evolution and neglect NLO evolution effects. Hence, for our purposes in the present analysis, we set

$$a_{2n}(\mu_{\text{F}}^2) = a_{2n}(\mu_0^2) \left[\frac{\alpha_s(\mu_{\text{F}}^2)}{\alpha_s(\mu_0^2)} \right]^{\gamma_n^{(0)}/(2b_0)}. \tag{4.4}$$

The lowest-order anomalous dimensions can be represented in closed form by

$$\gamma_n^{(0)} = 2 C_{\text{F}} \left[4 S_1(n+1) - 3 - \frac{2}{(n+1)(n+2)} \right] \tag{4.5}$$

*Here, we write for the sake of brevity $a_2 = a_2^{\text{BMS}}(\mu_{\text{F}}^2)$ and $a_4 = a_4^{\text{BMS}}(\mu_{\text{F}}^2)$ and use the values given in Eq. (2.11).

with $S_1(n+1) = \sum_{i=1}^{n+1} 1/i = \psi(n+2) - \psi(1)$, while the function $\psi(z)$ is defined as $\psi(z) = d \ln \Gamma(z)/dz$.

Following the master plan for «analytization», exposed in the previous section, we obtain the following expressions for the factorized pion form factor:

- *Naive* «analytization» [30, 31, 35]:

$$\begin{aligned}\left[F_\pi^{\text{Fact}}(Q^2;\lambda_{\text{R}}Q^2)\right]_{\text{NaivAn}} &= \mathcal{A}_1^{(2)}(\lambda_{\text{R}}Q^2)\mathcal{F}_\pi^{\text{LO}}\left(Q^2\right)\\ &+ \frac{1}{\pi}\left[\mathcal{A}_1^{(2)}(\lambda_{\text{R}}Q^2)\right]^2\mathcal{F}_\pi^{\text{NLO}}\left(Q^2,\mu_{\text{F}}^2;\lambda_{\text{R}}\right).\end{aligned} \quad (4.6)$$

- *Maximal* «analytization» [35]:

$$\begin{aligned}\left[F_\pi^{\text{Fact}}(Q^2;\lambda_{\text{R}}Q^2)\right]_{\text{MaxAn}} &= \mathcal{A}_1^{(2)}(\lambda_{\text{R}}Q^2)\,\mathcal{F}_\pi^{\text{LO}}\left(Q^2\right)\\ &+ \frac{1}{\pi}\,\mathcal{A}_2^{(2)}(\lambda_{\text{R}}Q^2)\,\mathcal{F}_\pi^{\text{NLO}}\left(Q^2,\mu_{\text{F}}^2;\lambda_{\text{R}}\right).\end{aligned} \quad (4.7)$$

- KS *amplitude* «analytization» (this work) — cf. Eqs. (3.13) and (3.15):

$$\begin{aligned}\left[F_\pi^{\text{Fact}}(Q^2;\lambda_{\text{R}}Q^2)\right]_{\text{KS}} &= \mathcal{A}_1^{(2)}(\lambda_{\text{R}}Q^2)\,\mathcal{F}_\pi^{\text{LO}}\left(Q^2\right)\\ &+ \frac{1}{\pi}\,\mathcal{A}_2^{(2)}(\lambda_{\text{R}}Q^2)\,\mathcal{F}_\pi^{\text{NLO}}\left(Q^2,\mu_{\text{F}}^2;\lambda_{\text{R}}\right)\\ &+ \frac{\Delta_2^{(2)}\left(\lambda_{\text{R}}Q^2\right)}{\pi}\,\Delta_{\text{F}}\mathcal{F}_\pi^{\text{NLO}}\left(Q^2\right).\end{aligned} \quad (4.8)$$

Here we use the following notations:

$$\mathcal{F}_\pi^{\text{LO}}\left(Q^2\right) = \frac{8\,\pi f_\pi^2}{Q^2}\left(1+a_2+a_4\right)^2, \quad (4.9)$$

$$\begin{aligned}\mathcal{F}_\pi^{\text{NLO}}\left(Q^2,\mu_{\text{F}}^2;\lambda_{\text{R}}\right) = \frac{2\pi f_\pi^2}{Q^2}\Big[&b_0\left(1+a_2+a_4\right)^2\left(\ln\lambda_{\text{R}} - \ln\lambda_{\text{BLM}}(a_2,a_4)\right) - \\ &- 15.67 - a_2\left(21.52 - 6.22a_2\right) - a_4\left(7.37 - 37.40a_2 - 33.61a_4\right)\Big] + \\ &+ \Delta_{\text{F}}\mathcal{F}_\pi^{\text{NLO}}\left(Q^2\right)\ln\frac{Q^2}{\mu_{\text{F}}^2}\end{aligned} \quad (4.10)$$

and we explicitly display the contribution due to $t_{\text{H},2}^{(1,\text{F})}\left(x,y;\mu_{\text{F}}^2/Q^2\right)$, see Eq. (2.7b):

$$\Delta_{\text{F}}\mathcal{F}_\pi^{\text{NLO}}(Q^2) = -\frac{2\,\pi f_\pi^2}{Q^2}C_{\text{F}}\left(1+a_2+a_4\right)\left[(25/3)a_2 + (182/15)a_4\right]. \quad (4.11)$$

In order to make our formulas more compact, we implement the BLM scale:

$$\lambda_{\rm BLM}(a_2, a_4) = \exp\left[-\frac{5}{3} - \frac{3 + (43/6)a_2 + (136/15)a_4}{1 + a_2 + a_4}\right]. \qquad (4.12)$$

The «analytization» augmented perturbation theory works very well. This is illustrated by the results in Figs. 2, 3, and 4. The first of these figures compares

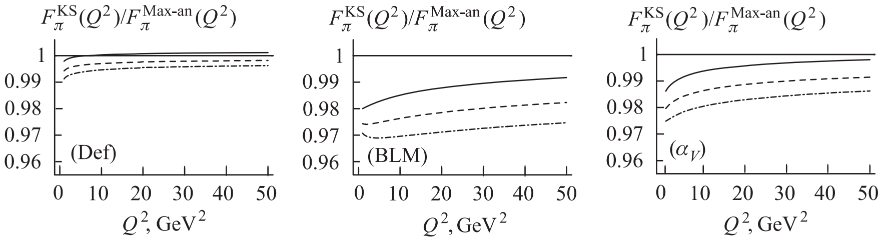
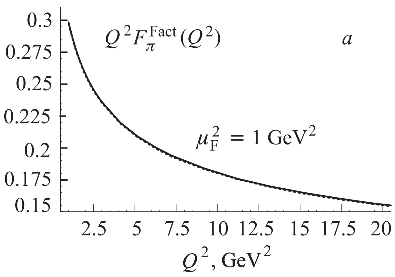


Fig. 2. Results for the ratio of the factorized pion form factors, using two different «analytization» procedures: KS «analytization» and «maximal analytization», $F_\pi^{\rm KS}\left(Q^2\right)/F_\pi^{\text{Max-an}}\left(Q^2\right)$. The designations are: dash-dotted line — one-loop approximation of the KS logarithmic term ($\mathcal{L}_2^{(1)\,\text{approx}}\left(Q^2\right)$); dashed line — two-loop approximation ($\mathcal{L}_2^{(2)\,\text{approx}}\left(Q^2\right)$); solid line — exact two-loop KS «analytization» ($\mathcal{L}_2^{(2)}\left(Q^2\right)$). Left panel: default scale setting ($\lambda = 1$); middle panel: BLM scale setting; right panel: α_V scheme. The factorization scale $\mu_{\rm F}^2$ is set equal to 5.76 GeV^2 [57]

the specific issues of the KS «analytization» procedure relative to those of the *maximal* one for the ratio of the corresponding factorized form factors. A few words are in order here. One sees that using the default $\overline{\rm MS}$ scheme, the KS «analytization» procedure yields a result almost coincident with that provided by the *maximal* one. On the other hand, in the BLM scheme and also in the α_V scheme, the KS prediction is smaller by a few percent. Moreover, one observes by comparison with Fig. 11, right panel in Ref. [35] that the BLM prediction, which in the *maximal* procedure was the largest one, becomes in the case of the KS prescription comparable with the prediction of the default scheme. As a result, the inherent theoretical uncertainties due to the involved perturbative parameters, defining a renormalization scheme and scale setting, are further reduced. A second important feature of the KS procedure is that the dependence of $F_\pi^{\rm Fact}(Q^2)$ on the factorization scale is almost diminished, as indicated in Fig. 3. Indeed, varying the factorization scale from 1 GeV^2 to 10 GeV^2, the form factor changes by a mere 1.5 percent. Even setting the factorization scale to the theoretical value of 50 GeV^2, the induced variation in the form-factor magnitude reaches just the level of about 2.5 percent. In the case of the *maximal* «analytization» procedure, the dependence on the factorization scale is also a mild one, but the corresponding variation is, in round terms, two times larger.



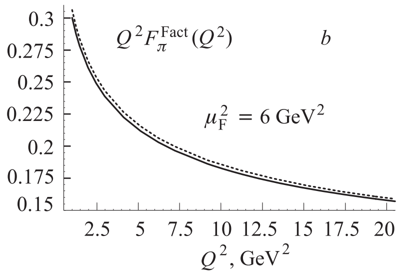
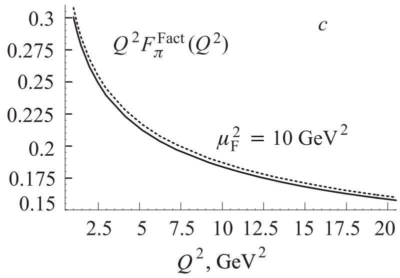


Fig. 3. Results for the factorized pion form factors, using two different «analytization» procedures: KS «analytization» (solid line) and «maximal analytization» (dotted line) for different values of the factorization scale: *a*) $\mu_{\rm F}^2 = 1$ GeV^2, *b*) $\mu_{\rm F}^2 = 6$ GeV^2, *c*) $\mu_{\rm F}^2 = 10$ GeV^2. For all a, b, c we show results for the BLM scale setting



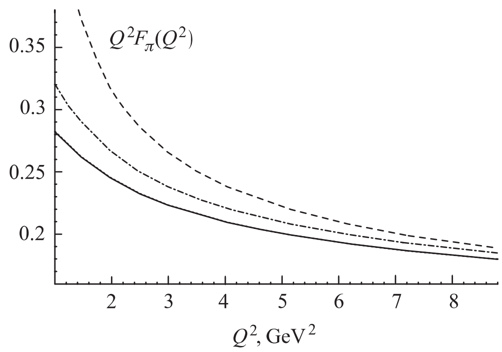


Fig. 4. Results for the factorized pion form factor, scaled with Q^2, and assuming the default scale setting ($\mu_{\rm R}^2 = Q^2$) in standard perturbation theory and APT. The latter is implemented in terms of two different «analytization» procedures: *naive* «analytization» and *maximal* «analytization». The designations are: dashed line — standard perturbation theory; dash-dotted line — *naive* APT; solid line — *maximal* APT. The prediction obtained with the KS «analytization» is too close to that found with the *maximal* one to differentiate these curves graphically. The factorization scale $\mu_{\rm F}^2$ is set equal to 5.76 GeV^2

The fourth figure demonstrates the impact of «analytization» on the factorized pion's electromagnetic form factor, using various «analytization» prescriptions. The dashed line denotes the prediction obtained with standard QCD perturbation theory in the $\overline{\text{MS}}$ scheme and applying the default scale setting $\mu_{\text{R}}^2 = Q^2$. The *naive* «analytization» prediction is represented by the dash-dotted line and the analogous one for the *maximal* «analytization» by the solid line below it. The result of the calculation according to the KS «analytization» practically coincides with that of the *maximal* one. This behavior is also reflected in Fig. 2, where we see that the differences among the three «analytization» procedures are of the order of a few percent in the whole Q^2 range considered.

Note that as regards the whole pion form factor, i. e., taking into account also the soft part, the differences would be further reduced. For full details the reader is referred to [35].

5. SUMMARY AND CONCLUSIONS

We have discussed different «analytization» procedures to ensure the analyticity of the factorized electromagnetic pion form factor at NLO of QCD perturbation theory. The main features and relative merits of each «analytization» concept following from the presented analysis are:

- The *naive* «analytization» [30, 31] retains the power-series expansion of perturbative QCD, but replaces $(\alpha_s^{(n)}(Q^2))^m$ by $(\mathcal{A}_1^{(n)}(Q^2))^m$. As it was shown in [30,31], this reduces the value of the NLO correction, though the sensitivity to the renormalization scheme adopted and the renormalization scale-setting chosen is still substantial, resulting into a rather strong variation of the form-factor predictions [35]. Moreover, this procedure does not respect nonlinear relations of the coupling because these correspond to different dispersive images.

- The *maximal* «analytization» [35] trades the power-series expansion for a functional non-power-series expansion in terms of $\mathcal{A}_m^{(n)}(Q^2)$ [11, 16, 17], minimizing the variation of the form-factor predictions owing to the renormalization scheme and scale setting. It is, however, insufficient to cure logarithms of the momentum scale multiplying the running coupling. Such terms modify the spectral density, i. e., the discontinuity across the cut along the negative real axis and have therefore to be taken into account.

- Applying the «analytization» procedure at the level of the partonic amplitude itself [32,33], bears all advantages of the *maximal* «analytization» plus a further reduced dependence on the perturbative scales — especially the dependence on the factorization scale. This has been verified by explicit calculation. We have employed the $\overline{\text{MS}}$ scheme with various scale settings and also the α_V scheme. In addition, we have varied the factorization scale in the range $1 - 10$ GeV^2. While the predictions for the factorized pion form factor, calculated with the *maximal* procedure, were affected by this variation on the level of 3%, their counterparts, derived with the KS prescription, were influenced by less than 1%. Though the KS method does not really «gain up» relative to the *maximal* «analytization» procedure with respect to the factorized pion form factor, as one observes from Fig. 4, it is able to further improve the perturbative treatment because it extends the notion of analyticity to non-integer powers of the strong running coupling — FAPT. Such powers become relevant when one has to calculate the analytic image of powers of the strong coupling in combination with logarithms, the latter first appearing at NLO of fixed-order perturbation theory, or in terms of evolution factors [36]. Hence, the KS «analytization» requirement treats all logarithms that have a non-zero spectral density, and hence modify the discontinuity across the cut along the negative real axis, on the same footing and irrespective of their source being in the running coupling (and its powers), or logarithms entailed by ERBL or DGLAP evolution.

In conclusion, the KS «analytization» enables the variation of the factorization scale and the choice of various renormalization schemes and scale settings, including the BLM one, with undiminished quality of the theoretical predictions from scheme (scale) to scheme (scale), virtually eliminating the dependence on such parameters and upgrading the $\overline{\text{MS}}$ scheme to an optimized factorization and renormalization scheme. From a broader perspective one may interpret these findings as indicating that the analyticity of the partonic three-point function is as important and fundamental as the underlying symmetries of the theory and should be preserved together with them in the maximal possible way.

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APENDIX A
QCD β FUNCTION AT NLO

The first coefficients of the β function are

$$b_0 = \frac{11}{3}\, C_{\text{A}} - \frac{4}{3}\, T_{\text{R}} N_f \,, \qquad b_1 = \frac{34}{3}\, C_{\text{A}}^2 - \left(4C_{\text{F}} + \frac{20}{3}\, C_{\text{A}}\right) T_{\text{R}} N_f \,. \quad \text{(A1)}$$

Here, $T_{\rm R} = 1/2$ and N_f denotes the number of flavors, whereas the expansion of the β function in the NLO approximation is given by

$$\beta(\alpha_s(\mu^2)) = -\alpha_s(\mu^2)\left[b_0\left(\frac{\alpha_s(\mu^2)}{4\pi}\right) + b_1\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2\right]. \qquad \text{(A2)}$$

APENDIX B
NLO CORRECTION TO THE PION FORM FACTOR

Here we present the detailed expressions for the color decomposition of the NLO correction to the hard amplitude $T_{\rm H}$, which describe the factorized part of the pion form factor [35, 39] (see Eqs. (2.5)–(2.7b)):

$$t_{\rm H,1}^{(1,\rm F)}(x,y) = \frac{N_{\rm T}}{\overline{x}\,\overline{y}}\left[-\frac{28}{3} + \left(6-\frac{1}{x}\right)\ln\overline{x} + \left(6-\frac{1}{y}\right)\ln\overline{y} + \ln^2(\overline{x}\,\overline{y})\right], \qquad \text{(B1)}$$

$$t_{\rm H}^{(1,\rm G)}(x,y) = \frac{2N_{\rm T}}{\overline{x}\,\overline{y}}\Bigg[-\frac{10}{3} + \ln\left(\frac{\overline{x}}{x}\right)\ln\left(\frac{y}{\overline{y}}\right) -$$
$$-4\left(\frac{\ln\overline{x}}{x} + \frac{\ln\overline{y}}{y}\right) - \tilde{H}(x,y) - R(x,y)\Bigg]. \qquad \text{(B2)}$$

The functions $\tilde{H}(x,y)$ and $R(x,y)$ are defined by

$$\tilde{H}(x,y) = \left[\mathbf{Li}\left(\frac{\overline{y}}{x}\right) + \mathbf{Li}\left(\frac{\overline{x}}{y}\right) + \mathbf{Li}\left(\frac{xy}{\overline{x}\,\overline{y}}\right) - \mathbf{Li}\left(\frac{x}{\overline{y}}\right) - \mathbf{Li}\left(\frac{y}{\overline{x}}\right) - \mathbf{Li}\left(\frac{\overline{x}\,\overline{y}}{xy}\right)\right] \qquad \text{(B3)}$$

and

$$R(x,y) = \frac{1}{(x-y)^2}\Big[(2xy-x-y)(\ln x + \ln y) - (y\overline{y}^2 + x\overline{x}^2)(1-x-y)\tilde{H}(x,\overline{y}) -$$
$$-2\left(xy^2 + y^2 - 5xy + y + 2x^2\right)\frac{\ln\overline{y}}{y} - 2\left(yx^2 + x^2 - 5xy + x + 2y^2\right)\frac{\ln\overline{x}}{x}\Big]. \qquad \text{(B4)}$$

APPENDIX C
«ANALYTIZATION» OF POWERS OF THE COUPLING MULTIPLIED BY LOGARITHMS

We present here the derivation of $\mathcal{L}_2^{(2)}\left(Q^2\right)$, done in collaboration with S. Mikhailov. To this end, let us first introduce

$$a_s\left(Q^2\right) \equiv \frac{b_0}{4\pi}\,\alpha_s\left(Q^2\right). \tag{C1}$$

For this quantity we can write a renormalization group solution in the form

$$\left[a_s^{(2)}\left(Q^2\right)\right]^2 \ln\left(\frac{Q^2}{\Lambda^2}\right) = a_s^{(2)}\left(Q^2\right) + \left[a_s^{(2)}\left(Q^2\right)\right]^2 c_1 \ln\left[\frac{a_s^{(2)}\left(Q^2\right)}{1+c_1 a_s^{(2)}\left(Q^2\right)}\right]. \tag{C2}$$

Expanding the expression $\ln[1+c_1 a_s^{(2)}\left(Q^2\right)]$ and retaining terms up to order a_s^2, we find

$$\left[a_s^{(2)}\left(Q^2\right)\right]^2 \ln\left(\frac{Q^2}{\Lambda^2}\right) = a_s^{(2)}\left(Q^2\right) + \left[a_s^{(2)}\left(Q^2\right)\right]^2 c_1 \ln\left[a_s^{(2)}\left(Q^2\right)\right]. \tag{C3}$$

To get rid of the logarithm, we use the following trick:

$$\left[a_s^{(2)}\left(Q^2\right)\right]^2 \ln\left(\frac{Q^2}{\Lambda^2}\right) = a_s^{(2)}\left(Q^2\right) + c_1 \frac{d}{d\varepsilon}\left[a_s^{(2)}\left(Q^2\right)\right]^{2+\varepsilon}\Big|_{\varepsilon=0} \tag{C4}$$

and return to the original coupling to obtain

$$\left[\alpha_s^{(2)}\left(Q^2\right)\right]^2 \ln\left(\frac{Q^2}{\Lambda^2}\right) = \frac{4\pi}{b_0}\,\alpha_s^{(2)}\left(Q^2\right) + c_1 \frac{4\pi}{b_0}\frac{d}{d\varepsilon}\left[\alpha_s^{(2)}\left(Q^2\right)\right]^{2+\varepsilon}\Big|_{\varepsilon=0}. \tag{C5}$$

Now we can proceed with the «analytization» of the term $[\alpha_s^{(2)}\left(Q^2\right)]^2 \ln\left(Q^2\right)$, giving rise to analytic expressions for non-integer powers of the coupling, i.e.,

$$\left\{\left[\alpha_s^{(2)}\left(Q^2\right)\right]^2 \ln\left(\frac{Q^2}{\Lambda^2}\right)\right\}_{\text{an}} = \frac{4\pi}{b_0}\,\mathcal{A}_1^{(2)}\left(Q^2\right) + c_1\left[\frac{d}{d\varepsilon}\,\mathcal{A}_{2+\varepsilon}^{(2)}\left(Q^2\right)\right]_{\varepsilon=0}. \tag{C6}$$

Using the representation [36]

$$\left(\frac{b_0}{4\pi}\right)^2 \mathcal{A}_\nu^{(2)}\left(Q^2\right) = \frac{-1}{\Gamma(\nu)}\sum_{n\geq 0}\zeta(1-\nu-n)\frac{\left[-\ln\left(Q^2/\Lambda^2\right)\right]^n}{\Gamma(n+1)} \tag{C7}$$

and performing the differentiation, we finally obtain

$$\left\{\left[\alpha_s^{(2)}\left(Q^2\right)\right]^2 \ln\left(\frac{Q^2}{\Lambda^2}\right)\right\}_{\text{an}} = \frac{4\pi}{b_0}\left[\mathcal{A}_1^{(2)}\left(Q^2\right) + c_1\frac{4\pi}{b_0}\,f_{\mathcal{L}}\left(Q^2\right)\right], \tag{C8}$$

with $f_{\mathcal{L}}\left(Q^2\right)$ being defined in Eq. (3.19).

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Издательский отдел Объединенного института ядерных исследований
141980, г. Дубна, Московская обл., ул. Жолио-Кюри, 6.
E-mail: publish@pds.jinr.ru
www.jinr.ru/publish/