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ON THE HAMILTONIAN REDUCTION OF GEODESIC MOTION ON $SU(3)$ TO $SU(3)/SU(2)$

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Гердт В. и др. E2-2006-3

Гамильтонова редукция геодезического движения на $SU(3)$ к $SU(3)/SU(2)$

Редуцированная гамильтонова система на $T^*(SU(3)/SU(2))$ получена из риманова геодезического движения на групповом многообразии $SU(3)$, параметризованном обобщенными углами Эйлера и наделенном канонической биинвариантной метрикой. Наши вычисления показывают, что метрика, определяемая редуцированным гамильтоновым потоком на пространстве орбит $SU(3)/SU(2) \simeq \mathbb{S}^5$, неизометрична и даже геодезически неэквивалентна стандартной римановой метрике на 5-сфере $\mathbb{S}^5$, вложенной в $\mathbb{R}^6$.

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On the Hamiltonian Reduction of Geodesic Motion on $SU(3)$ to $SU(3)/SU(2)$

The reduced Hamiltonian system on $T^*(SU(3)/SU(2))$ is derived from a Riemannian geodesic motion on the $SU(3)$ group manifold parameterized by the generalized Euler angles and endowed with a bi-invariant metric. Our calculations show that the metric defined by the derived reduced Hamiltonian flow on the orbit space $SU(3)/SU(2) \simeq \mathbb{S}^5$ is not isometric or even geodesically equivalent to the standard Riemannian metric on the five-sphere $\mathbb{S}^5$ embedded into $\mathbb{R}^6$.

The investigation has been performed at the Laboratory of Information Technologies, JINR.

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1. INTRODUCTION

Symmetry plays a central role in our pursuit of a better understanding of nature. Through the preservation or artful breaking of symmetry, powerful models have been developed which describe the fundamental forces and which have, so far, withstood all tests. Indeed, any endeavour to go beyond this standard model also has, at its heart, an appropriate symmetry argument.

An immediate consequence of symmetry is that it permits for a reduction in the relevant degrees of freedom needed to describe a given problem. In a gauge theory this reduction implies that not all the degrees of freedom present in the formulation of the theory correspond to physical degrees of freedom. So, for example, in Quantum Electrodynamics, with its $U(1)$ gauge symmetry, the potential A_μ, which naively has four degrees of freedom, describes the photon, which has just two physical degrees of freedom. Understanding how this type of reduction should best take place and how the process of quantizing a system interacts with the symmetry, has driven many of the important advances in our understanding of gauge theories [1].

In many cases, the reduction to the true physical degrees of freedom in a field theory has been fruitfully studied through simpler, finite-dimensional systems. In particular, coset spaces of the form G/H, where G and H are finite-dimensional Lie groups, have provided much insight [2] into how global and topological properties of these configuration spaces can be encoded into the quantization process via generalized notions of reduction to the true degrees of freedom [3].

In all investigations to date, specific details on dynamical aspects of the reduction to G/H have been restricted to groups for which manageable parameterizations of the group elements exist. Essentially this has restricted attention to groups directly related to the rotation group and its covering, $SU(2)$. However, recently there has been much progress in finding suitable parameterizations for the higher-dimensional unitary groups [4–7] and particularly for the group $SU(3)$ [8, 9]. These advances open the door to detailed investigations of dynamics on spaces such as the five-sphere, $\mathbb{S}^5$, now viewed as the reduction from $SU(3)$ to $SU(3)/SU(2)$. By exploiting our concrete description of this reduction we shall see a new phenomenon for this system: different metric structures emerge depending on whether the five-sphere is viewed as the coset space or via its natural embedding in six-dimensional Euclidean space. This is, to the best of our knowledge, the first explicit example of this metric property of reduction.

The plan of the paper is as follows. We will conclude this introduction with a brief summary of the classical Hamiltonian reduction procedure. Then, in Sec. 2, we will see how this procedure is applied to the group $SU(2)$. This section does not contain any new results, but fixes notation and introduces themes that will prepare us for the reduction on the configuration space $SU(3)$ which will be presented in detail in Sec. 3. Then, in Sec. 4 we will investigate the possible Riemannian structures that arise on the quotient space $\mathbb{S}^5$ and discuss the possible metric and geodesic correspondences. In Appendix we will collect together the details of our consistent parameterization of $SU(3)$.

1.1. Hamiltonian Reduction. Consider the special class of Lagrangian systems whose configuration space is a compact matrix Lie group G. This means that the state of a system at fixed time $t = 0$ is characterized by an element of the Lie group $g(0) \in G$ and the evolution is described by the curve $g(t)$ on the group manifold [10, 11]. The «free evolution» on the semi-simple group G is, by definition, the Riemannian geodesic motion on the group manifold with respect to the so-called Cartan–Killing metric [12, 13]

$$\mathrm{d}s^2_G = \kappa \, \mathrm{Tr} \left(g^{-1}\mathrm{d}g \otimes g^{-1}\mathrm{d}g \right) ,$$

where κ is a normalization factor. The geodesics are given by the extremal curves of the action functional

$$S[g] = \frac{\kappa}{2} \int_0^T \mathrm{dt}\, \mathrm{Tr} \left(g^{-1}\dot{g}\, g^{-1}\dot{g} \right) . \tag{1.1}$$

This action is invariant under the continuous left translation

$$g(t) \;\to\; g(\varepsilon)\, g(t), \qquad \varepsilon = (\,\varepsilon_1\,, \varepsilon_2\,, \dots\,, \varepsilon_{\dim G}),$$

and therefore the system possesses the integrals of motion $\mathcal{I}_1\,, \mathcal{I}_2\,, \dots\,, \mathcal{I}_{\dim G}$. The existence of these integrals of motion allows us to reduce the number of degrees of freedom of the system using the well-known method of Hamiltonian reduction [10, 11].

For a generic Hamiltonian system defined on T^*M with symmetry associated to the Lie group G action, the level set of the corresponding integrals of motion

$$M_c = \mathcal{I}^{-1}(c), \tag{1.2}$$

where c is a set of arbitrary real constants $c = (c_1\,, \dots\,, c_{\dim G})$, determines the reduced Hamiltonian system on the *reduced phase space* $F_c \subset M_c$. The subset F_c is described by the isotropy group, G_c, of the integrals level set M_c

$$F_c = M_c / G_c .$$

Here we are interested in a special case when the manifold M is itself a group manifold and the symmetry transformation are group translations. Now the level set M_c is a subset of the trivial cotangent bundle T^*G which can be identified with the product of the group G and its algebra, $G \times g$. The level set given by the integrals $\mathcal{I}_1 = c_1, \mathcal{I}_2 = c_2, \ldots, \mathcal{I}_N = c_N$, $N \leqslant \dim G$, defines the isotropy group $G_c \subset G$ and the so-called *orbit space*

$$\mathcal{O} = G/G_c. \tag{1.3}$$

The relationship between the orbit space $\mathcal{O}$ and the reduced phase space F_c can be summarized as follows (see, e.g., [10, 11]):

- *the reduced phase space F_c is* **symplectic and diffeomorphic** *to the cotangent bundle $T^*\mathcal{O}$;*
- *the dynamics on the reduced degrees of freedom is Hamiltonian with a reduced Hamiltonian given by the projection of the original Hamiltonian function to F_c.*

These results are the modern generalizations of the classical theorems proving that the collection of holonomic constraints defines a configuration manifold M as a submanifold of $\mathbb{R}^n$ and that, in the absence of forces, the trajectories of mechanical system are geodesics of the induced Riemannian metric.

Note that the above results do not claim that the reduced phase space and the dynamics on the orbit space are isometric. Indeed, we know that on the reduced phase space we can define, at least locally, an induced metric that arises from the kinetic energy energy part of the reduced Hamiltonian

$$K_{\mathcal{O}} = \frac{1}{2}\,\mathbf{g}_{\mathcal{O}}(\xi_a, \xi_b) p_a\, p_b. \tag{1.4}$$

On the other hand, the map $\pi : G \to G/G_c$ induces the metric

$$\overline{\mathbf{g}}_{\mathcal{O}} = \pi_* \mathbf{g}_G. \tag{1.5}$$

We now pose a question about the relation between these two metrics.

> *When are the metrics $\mathbf{g}_{\mathcal{O}}$ and $\overline{\mathbf{g}}_{\mathcal{O}}$* **isometrically** *or, more weakly,* **geodesically** *equivalent?*

We do not know the general answer to this question, so in the present work we will focus our study on two examples: geodesic motion on the $SU(2)$ and $SU(3)$ group manifolds.

We start with a well-known example of Hamiltonian reduction $SU(2) \to SU(2)/U(1)$ and show that the reduced space *is indeed in isometrical correspondence with the cotangent bundle $T^*\mathbb{S}^2$* and the standard induced metric on the two-sphere $\mathbb{S}^2$. The case of the $SU(3) \to SU(3)/SU(2)$ reduction gives an example of the opposite result: the metric defined by the Hamiltonian flow on the orbit space $SU(3)/SU(2)$ *is not isometrically equivalent* to a standard round metric on the five-sphere $\mathbb{S}^5$. Furthermore, in this case, the stronger result is true: the reduced configuration space and the standard $\mathbb{S}^5$ are not even *geodesically equivalent*.

2. GEODESIC FLOW ON $SU(2)$

In this section we discuss the example of reduction of free motion on the $SU(2)$ group manifold. We start with a presentation of the key geometrical structures found on this group which are necessary for any further dynamical analysis.

2.1. The Euler Angle Parameterization. The special unitary group $SU(2)$, considered as a subgroup of the general matrix group $GL(2, \mathbb{C})$, is *topologically* the three-sphere $\mathbb{S}^3$ embedded into $\mathbb{C}^2$. This correspondence $SU(2) \approx \mathbb{S}^3$ follows immediately from the standard identification of an arbitrary element $g \in SU(2)$ as

$$g := \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \qquad |z_1|^2 + |z_2|^2 = 1\,. \tag{2.1}$$

The three-sphere $\mathbb{S}^3$ is a manifold which requires more than one chart to cover it and therefore there is no global parameterization of the $SU(2)$ group as a three-dimensional space. The local description usually adopted is given by the conventional symmetric *Euler representation* [14] for a group element

$$g \,=\, \exp\left(i\frac{\alpha}{2}\,\sigma_3\right)\,\exp\left(i\frac{\beta}{2}\,\sigma_2\right)\,\exp\left(i\frac{\gamma}{2}\,\sigma_3\right) \tag{2.2}$$

with the appropriately chosen range for the Euler angles $\alpha\,,\beta\,,\gamma\,.$

In this representation the generators of the one-parameter subgroups are the standard Pauli matrices $\sigma_1\,,\sigma_2$, and σ_3,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.3}$$

satisfying the $su(2)$ algebra

$$\sigma_a\sigma_b - \sigma_b\sigma_a = 2\,i\,\epsilon_{abc}\,\sigma_c\,. \tag{2.4}$$

Writing the complex numbers in (2.1) as $z_1 = x^1 + ix^2$ and $z_2 = x^3 + ix^4$ in polar form

$$z_1 := \mathrm{e}^{iu}\,\cos\theta\,, \qquad z_2 := \mathrm{e}^{iv}\,\sin\theta \tag{2.5}$$

and comparing (2.1) with the explicit form of the Euler matrix (2.2)

$$g = \begin{pmatrix} \mathrm{e}^{i\frac{\alpha+\gamma}{2}}\cos\left(\frac{\beta}{2}\right) & \mathrm{e}^{i\frac{\alpha-\gamma}{2}}\sin\left(\frac{\beta}{2}\right) \\ -\mathrm{e}^{-i\frac{\alpha-\gamma}{2}}\sin\left(\frac{\beta}{2}\right) & \mathrm{e}^{-i\frac{\alpha+\gamma}{2}}\cos\left(\frac{\beta}{2}\right) \end{pmatrix}, \tag{2.6}$$

we have

$$u=\frac{\alpha+\gamma}{2}\,,\qquad v=\frac{\alpha-\gamma}{2}\,,\qquad \theta=\frac{\beta}{2}\,. \tag{2.7}$$

The Euler decomposition (2.2) corresponds to the following parametric representation of the three-sphere embedded in $\mathbb{R}^4$:

$$\begin{aligned} x^1 &= \cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\,, \qquad x^2=\sin\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\,,\\ x^3 &= -\cos\left(\frac{\alpha-\gamma}{2}\right)\sin\left(\frac{\beta}{2}\right)\,, \qquad x^4=\sin\left(\frac{\alpha-\gamma}{2}\right)\sin\left(\frac{\beta}{2}\right)\,. \end{aligned} \tag{2.8}$$

To be more precise, though, this is not a valid parameterization for the entire three-sphere. In particular, the neighbourhood of the identity element of the group in this decomposition turns out to be degenerate. The identity element of $SU(2)$ corresponds to the whole set: $\beta=0$ and $\alpha+\gamma=0$. In order to properly cover the whole group manifold it is necessary to consider an atlas on the $SU(2)$ group and used different parameterizations on the different charts. Bearing this in mind, we proceed by assuming that we are working in a chart $(\mathcal{U},\phi)$ where $\alpha\,,\beta$ and γ serve as good local coordinates on $\mathbb{S}^3$ and calculate the Maurer–Cartan forms on $SU(2)$.

Using the following normalization:

$$g^{-1}\mathrm{d}g = \frac{i}{2}\sum_{a=1}^{3}\sigma_a\otimes\omega_L^a\,, \tag{2.9}$$

$$\mathrm{d}g\,g^{-1} = \frac{i}{2}\sum_{a=1}^{3}\sigma_a\otimes\omega_R^a \tag{2.10}$$

and performing the straightforward calculations with the Eulerian representation (2.2) we arrive at the well-known expressions for left-invariant 1-forms

$$\begin{aligned} \omega_L^1 &= \cos\gamma\sin\beta\,\mathrm{d}\alpha-\sin\gamma\,\mathrm{d}\beta\,,\\ \omega_L^2 &= \sin\beta\sin\gamma\,\mathrm{d}\alpha+\cos\gamma\,\mathrm{d}\beta\,,\\ \omega_L^3 &= \cos\beta\,\mathrm{d}\alpha+\mathrm{d}\gamma \end{aligned} \tag{2.11}$$

and the corresponding dual vectors, $\quad\omega_L^a(\,X_b^L\,)=\delta_b^a\,,\qquad a,b=1\,,2\,,3\,,$

$$\begin{aligned} X_1^L &= \frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha}-\sin\gamma\frac{\partial}{\partial\beta}-\cot\beta\cos\gamma\frac{\partial}{\partial\gamma}\,,\\ X_2^L &= \frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha}+\cos\gamma\frac{\partial}{\partial\beta}-\cot\beta\sin\gamma\frac{\partial}{\partial\gamma}\,,\\ X_3^L &= \frac{\partial}{\partial\gamma}\,. \end{aligned} \tag{2.12}$$

The right-invariant 1-forms and the corresponding dual vectors, $\omega^a_R(X^R_b) = \delta^a_b$, are:

$$\begin{aligned}
\omega^1_R &= \sin\alpha\, d\beta - \cos\alpha\sin\beta\, d\gamma\,,\\
\omega^2_R &= \cos\alpha\, d\beta + \sin\alpha\sin\beta\, d\gamma\,,\\
\omega^3_R &= d\alpha + \cos\beta\, d\gamma\,.
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
X^R_1 &= \cos\alpha\cot\beta\,\frac{\partial}{\partial\alpha} + \sin\alpha\,\frac{\partial}{\partial\beta} - \frac{\cos\alpha}{\sin\beta}\,\frac{\partial}{\partial\gamma}\,,\\
X^R_2 &= -\sin\alpha\cot\beta\,\frac{\partial}{\partial\alpha} + \cos\alpha\,\frac{\partial}{\partial\beta} + \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\,,\\
X^R_3 &= \frac{\partial}{\partial\alpha}\,.
\end{aligned} \tag{2.14}$$

The vector fields X^L_a and X^R_a obey the $su(2)\otimes su(2)$ algebra with respect to the Lie brackets operation

$$\left[X^L_a\,, X^L_b\right] = -\epsilon_{abc}\, X^L_c, \tag{2.15}$$

$$\left[X^R_a\,, X^R_b\right] = \epsilon_{abc}\, X^R_c, \tag{2.16}$$

$$\left[X^L_a\,, X^R_b\right] = 0\,. \tag{2.17}$$

Any compact Lie group can be endowed with the bi-invariant Riemannian metric build uniquely (up to a normalization factor) from the Cartan–Killing form over the algebra. It is convenient to choose the following normalization for the bi-invariant metric on the $SU(2)$ group:

$$\mathbf{g}_{SU(2)} = -\frac{1}{2}\,\mathrm{Tr}\left(g^{-1}dg \otimes g^{-1}dg\right)\,. \tag{2.18}$$

In terms of this left/right-invariant nonholonomic frame, (2.18) reads

$$\mathbf{g}_{SU(2)} = \frac{1}{4}\,\left(\omega^1_L\otimes\omega^1_L + \omega^2_L\otimes\omega^2_L + \omega^3_L\otimes\omega^3_L\right) \tag{2.19}$$

$$= \frac{1}{4}\,\left(\omega^1_R\otimes\omega^1_R + \omega^2_R\otimes\omega^2_R + \omega^3_R\otimes\omega^3_R\right)\,. \tag{2.20}$$

Substitution of the expressions (2.11) and (2.13) for the Maurer–Cartan forms ω_L and ω_R yields the metric in the coordinate frame $d\alpha\,, d\beta\,, d\gamma$ basis

$$\mathbf{g}_{SU(2)} = \frac{1}{4}\,\left(d\alpha\otimes d\alpha + d\beta\otimes d\beta + d\gamma\otimes d\gamma + 2\cos\beta\, d\alpha\otimes d\gamma\right)\,. \tag{2.21}$$

In order to understand the *metrical* characteristics of a group manifold viewed as an embedded space, it is instructive to compare this invariant metric with the metric induced from the ambient four-dimensional Euclidean space on the *unit* three-sphere (2.8)

$$\begin{aligned}\mathbf{g}_{\mathbb{S}^3} &= \mathrm{d}\bar{z}_1 \otimes \mathrm{d}z_1 + \mathrm{d}\bar{z}_2 \otimes \mathrm{d}z_2 = \\ &= \frac{1}{4}\left(\mathrm{d}\alpha \otimes \mathrm{d}\alpha + \mathrm{d}\beta \otimes \mathrm{d}\beta + \mathrm{d}\gamma \otimes \mathrm{d}\gamma + 2\cos\beta \mathrm{d}\alpha \otimes \mathrm{d}\gamma\right) .\end{aligned} \tag{2.22}$$

Comparing the metrics, (2.21) and (2.22), we conclude that the bi-invariant metric on $SU(2)$ is the same as the standard metric on the unit three-sphere $\mathbb{S}^3$ and its bi-invariant volume is

$$\begin{aligned}\mathrm{Vol}(SU(2)) &= \int \sqrt{\det \mathbf{g}_{SU(2)}}\mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\gamma = \\ &= \left(\frac{1}{2}\right)^3 \int_0^{2\pi} \mathrm{d}\alpha \int_0^{4\pi} \mathrm{d}\gamma \int_0^{\pi} \mathrm{d}\beta \, \sin(\beta) = 2\pi^2 = \mathrm{Vol}(\mathbb{S}^3).\end{aligned} \tag{2.23}$$

As a Riemannian manifold the $SU(2)$ group endowed with the metric (2.21) is a three-dimensional space of constant curvature with the Riemann scalar $\mathcal{R}_{SU(2)} = 6$ and the Ricci tensor $\mathcal{R}_{ab}$ given by

$$\mathcal{R}_{ab} = \frac{\mathcal{R}_{SU(2)}}{3} g_{ab} = 2\, g_{ab} \, . \tag{2.24}$$

The Gaussian curvature K of an n-dimensional manifold and the Riemann scalar are related via

$$K = \frac{\mathcal{R}}{n(n-1)} \, , \tag{2.25}$$

therefore $K_{SU(2)} = 1$, in agreement with the volume calculation (2.23).

2.2. Quotient $SU(2)/U(1)$**.** Here we recall the key ingredients of the construction of a quotient space G/H by considering the transitive action of the group G on a certain base space M. We have the result that*:

If the group G acts transitively on a set M with $H \subset G$ being an isotropy subgroup leaving a point $x_0 \in M$ fixed

$$H = \{g \in G \mid g \cdot x_0 = x_0\} \, ,$$

then the set M is in one-to-one correspondence with the left cosets gH of G.

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*For a rigorous statement we refer to *Theorem 3.2* in [13].

The explicit form of this map for the $SU(2)$ group is as follows. We identify the $su(2)$ algebra with $\mathbb{R}^3$ by the map, $x^a \in \mathbb{R}^3 \to \mathbf{X} \in su(2)$

$$\mathbf{X} = \sum_{a=1}^{3} x^a \sigma_a \, . \tag{2.26}$$

Consider now the *adjoint action* of $SU(2)$ on an element of its algebra $\mathbf{X} \in su(2)$

$$\mathrm{Ad}(g)(\mathbf{X}) = g\,\mathbf{X}\,g^{-1} \, .$$

The base point $x_0 = (0\,,0\,,1)$ (corresponding to the element σ_3) has a one-parameter isotropy subgroup

$$H = \exp\left(i\,\frac{\alpha}{2}\,\sigma_3\right) \, .$$

The orbit space of σ_3

$$\mathrm{Ad}(g)(\sigma_3) = g\,\sigma_3\,g^{-1}$$

is the coset $SU(2)/U(1)$. The proper atlas covering the $SU(2)$ group manifold provides the coset space parameterization. When $SU(2) \simeq \mathbb{S}^3$ is parameterized in terms of two complex coordinates z_1 and z_2 and the two-sphere is described by a unit vector $\mathbf{n} = (n^1\,, n^2\,, n^3)$, then the projection $\mathbb{S}^3 \to \mathbb{S}^2$ reads explicitly

$$(z_1, z_2) \to (n^1, n^2, n^3) = \big(2\,\Re[\bar{z}_1 z_2],\; 2\,\Im[\bar{z}_1 z_2],\; |z_1|^2 - |z_2|^2\big) \, . \tag{2.27}$$

This is the famous Hopf projection map $\pi : SU(2) \to \mathbb{S}^2$ showing that $SU(2)$ is a fibre bundle over $\mathbb{S}^2$ with nonintersecting circles $U(1) \equiv \mathbb{S}^1$ as fibres

$$\mathbb{S}^1 \hookrightarrow SU(2) \xrightarrow{\pi} \mathbb{S}^2 \, .$$

Using the Euler decomposition (2.6) the coset parameterization reads

$$g\,\sigma_3\,g^{-1} = n^a\,\sigma_a \, , \tag{2.28}$$

with the unit 3-vector

$$\mathbf{n} = (-\sin\beta\cos\alpha\,,\, \sin\beta\sin\alpha\,,\, \cos\beta\,) \, . \tag{2.29}$$

2.3. Lagrangian in Euler Coordinates. The bi-invariant Lagrangian

$$L_{SU(2)} = -\frac{1}{2}\,\mathrm{Tr}\left(g^{-1}(t)\frac{d}{dt}g(t)\,g^{-1}(t)\frac{d}{dt}g(t)\right) \tag{2.30}$$

in terms of left/right-invariant Maurer–Cartan forms (2.9) reads

$$L_{SU(2)} = \frac{1}{4}\sum_{a=1}^{3} i_{\dot U}\omega^a_L\, i_{\dot U}\omega^a_L = \frac{1}{4}\sum_{a=1}^{3} i_{\dot U}\omega^a_R\, i_{\dot U}\omega^a_R\,, \tag{2.31}$$

where $i_{\dot U}$ is the interior contraction of the vector field $\dot U = \dot\alpha\,\frac{\partial}{\partial\alpha} + \dot\beta\,\frac{\partial}{\partial\beta} + \dot\gamma\,\frac{\partial}{\partial\gamma}$.

Covering the group manifold with an atlas and considering the chart where the parameters α, β, γ in the Euler decomposition (2.2) serve as good coordinates, we arrive at

$$L_{SU(2)} = \frac{1}{4}\,\left(\dot\alpha^2 + \dot\beta^2 + \dot\gamma^2 + 2\cos(\beta)\dot\alpha\dot\gamma\right)\,. \tag{2.32}$$

Comparing (2.32) with expression (2.21) for the bi-invariant metric on $SU(2)$ we conclude that

$$L_{SU(2)} = \mathbf{g}_{SU(2)}(\dot U, \dot U)\,. \tag{2.33}$$

2.4. Hamiltonian Dynamics on $T^*SU(2)$. The Hamiltonian dynamics on the $SU(2)$ group is defined on the cotangent bundle $T^*SU(2)$ which can be identified with the trivialization $T^*SU(2) \approx SU(2)\times su(2)_L$ or with $T^*SU(2) \approx SU(2) \times su(2)_R$.

The canonical Hamiltonian describing geodesic motion on $SU(2)$ can be obtained by a Legendre transformation of the Lagrangian function (2.31). Introducing the Poincare–Cartan symplectic 1-form

$$\Theta = p_\alpha\,\mathrm{d}\alpha + p_\beta\,\mathrm{d}\beta + p_\gamma\,\mathrm{d}\gamma$$

with the canonically conjugated pairs

$$\{\alpha\,, p_\alpha\} = 1\,, \qquad \{\beta\,, p_\beta\} = 1\,, \qquad \{\gamma\,, p_\gamma\} = 1\,,$$

the Hamiltonian on $T^*SU(2)$ is defined as

$$H_{SU(2)} = \sum_{a=1}^{3} \xi^L_a\,\xi^L_a\,, = \sum_{a=1}^{3} \xi^R_a\,\xi^R_a\,, \tag{2.34}$$

where ξ^L_a and ξ^R_a are the values of the 1-form Θ on the left/right-invariant vector fields $X^L_a\,, X^R_a$ spanning the algebra $su(2)_{L,R}$

$$\xi^L_a := \Theta\left(X^L_a\right), \qquad \xi^R_a := \Theta\left(X^R_a\right)\,.$$

The set of functions ξ_a^L and ξ_a^R obey the $su(2)_L \times su(2)_R$ relations with respect to the Poisson brackets

$$\{\xi_a^L\,,\xi_b^L\} = -\epsilon_{abc}\,\xi_c^L\,, \tag{2.35}$$
$$\{\xi_a^R\,,\xi_b^R\} = \epsilon_{abc}\,\xi_c^R\,, \tag{2.36}$$
$$\{\xi_a^L\,,\xi_b^R\} = 0\,. \tag{2.37}$$

In the coordinate frame (2.32) the Hamiltonian (2.34) becomes

$$H_{SU(2)} = \frac{p_\alpha^2}{\sin^2(\beta)} + p_\beta^2 + \frac{p_\gamma^2}{\sin^2(\beta)} - \frac{2\,\cos(\beta)}{\sin^2(\beta)}\,p_\alpha\,p_\gamma\,. \tag{2.38}$$

Now noting that the components of the inverse of the bi-invariant metric (2.21) are

$$g^{-1}_{SU(2)} = \frac{4}{\sin^2(\beta)}\begin{pmatrix} 1 & 0 & -\cos(\beta) \\ 0 & \sin^2(\beta) & 0 \\ -\cos(\beta) & 0 & 1 \end{pmatrix}\,, \tag{2.39}$$

the Hamiltonian can be rewritten as

$$H_{SU(2)} = \frac{1}{4}\,\mathbf{g}^{-1}_{SU(2)}(\Theta,\Theta). \tag{2.40}$$

2.5. Hamiltonian Reduction to the Coset $SU(2)/U(1)$. The system with Hamiltonian function (2.38) has an obvious first integral

$$p_\alpha = k, \qquad \{p_\alpha\,,H_{SU(2)}\} = 0\,, \tag{2.41}$$

where k can be an arbitrary constant. The Hamiltonian on the level set $M_k := p_\alpha^{-1}(k)$ is, by definition, the projection of (2.38) onto this subspace:

$$H^{(k)} := H_{SU(2)}\,\Bigg|_{p_\alpha=k} = p_\beta^2 + \frac{p_\gamma^2}{\sin^2(\beta)} - k\,\frac{2\,\cos(\beta)}{\sin^2(\beta)}\,p_\gamma + \frac{k^2}{\sin^2(\beta)}\,. \tag{2.42}$$

The inverse Legendre transformation gives

$$L_{SU(2)/SU(1)} = \frac{1}{4}\left(\dot\beta^2 + \sin^2(\beta)\,\dot\gamma^2\right) + k\,\cos(\beta)\dot\gamma\,. \tag{2.43}$$

The interpretation of the system so obtained is the following [3]: the first two terms correspond to a particle moving on the two-sphere $\mathbb{S}^2$ endowed with the standard embedding metric, while the last term describes the particle interaction with a Dirac monopole whose potential is

$$A_\phi := k\,(1-\cos(\beta))\,.$$

3. GEODESIC FLOW ON $SU(3)$ USING GENERALIZED EULER COORDINATES

3.1. Generalized Euler Decomposition of $SU(3)$. Now we pass on to the description of the Euler decomposition of the $SU(3)$ group element. The Euler angle parameterization of the three-dimensional rotation group has been generalized for the higher orthogonal $SO(n)$ and special unitary $SU(n)$ groups [5–7, 15, 16] and [17]. Special attention has been paid to the study of the $SU(3)$ [18–21] and $SU(4)$ [4] groups.

The starting point for the derivation* of the Euler angle representation of the $SU(3)$ group is the so-called Cartan decomposition which holds for a real semi-simple Lie algebra $\mathcal{G}$. A decomposition of the algebra $\mathcal{G}$ into the direct sum of vector spaces $\mathcal{K}$ and $\mathcal{P}$

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{P} \tag{3.1}$$

is a *Cartan decomposition of the algebra* $\mathcal{G}$ if

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}\,, \tag{3.2}$$

$$[\mathcal{K}, \mathcal{P}] \subset \mathcal{P}\,, \tag{3.3}$$

$$[\mathcal{P}, \mathcal{P}] \subset \mathcal{K}\,. \tag{3.4}$$

The Cartan decomposition for a Lie algebra induces a corresponding *Cartan decomposition of the group* G

$$G = KP\,, \tag{3.5}$$

where K is a Lie subgroup of G with Lie algebra $\mathcal{K}$, and P is given by the exponential map $P = \exp(\mathcal{P})$.

An explicit realization of the Cartan decomposition for $SU(3)$ can be achieved using the standard traceless 3×3 Hermitian Gell-Mann matrices $\lambda_a\,, (a = 1\,, \dots\,, 8)$ (the explicit form of the λ matrices is given in Appendix (A.1)). Indeed, from the expressions for the commutation relations

$$[\lambda_a,\, \lambda_b] = 2i \sum_{c=1}^{8} \mathrm{f}_{abc}\, \lambda_c\,, \tag{3.6}$$

where the structure constants f_{abc} are antisymmetric in all indices and have the nonzero values

$$\begin{aligned} &\mathrm{f}_{123} = 1, \\ &\mathrm{f}_{147} = \mathrm{f}_{246} = \mathrm{f}_{257} = \mathrm{f}_{345} = \mathrm{f}_{516} = \mathrm{f}_{637} = 1/2, \\ &\mathrm{f}_{458} = \mathrm{f}_{678} = \sqrt{3}/2, \end{aligned} \tag{3.7}$$

*We follow the method of Robert Hermann [22], who attributed this construction to C. C. Moore.

it follows that the set of matrices $(\lambda_1\,,\lambda_2\,,\lambda_3\,,\lambda_8\,)$ can be used as the basis for the vector space $\mathcal{K}$ while the matrices $(\lambda_4\,,\lambda_5\,,\lambda_6\,,\lambda_7\,)$ span the Cartan subspace $\mathcal{P}$. Noting that the set of matrices $(\lambda_1,\lambda_2\,,\lambda_3\,,\lambda_8\,)$ comprise the generators $(\lambda_1\,,\lambda_2\,,\lambda_3\,)$ of the $SU(2)$ group, one can locally represent K as the product of the $SU(2)$ subgroup and a one-parameter subgroup

$$K = SU(2)\;\mathrm{e}^{i\phi\lambda_8}\,. \tag{3.8}$$

The second factor, $P = \exp(\mathcal{P})$ in the Cartan decomposition (3.5) can be represented as a product of one-parameter subgroups. Moreover, based on the algebra (3.6), it can be represented as a product of a one-parameter subgroup generated by an element* from $\lambda_4\,,\dots\,,\lambda_7$ «sandwiched» between two different copies of K. Fixing this generator to be, say, λ_4, we have

$$P = K'\,\mathrm{e}^{i\theta'\lambda_4}\;K''\,. \tag{3.9}$$

Now observing that $[\lambda_8\,,\lambda_4] = i\sqrt{3}\lambda_5$, the product KP can be reduced to

$$G = SU(2)\,\mathrm{e}^{i\theta\lambda_5}\,SU(2)'\,\mathrm{e}^{i\phi\lambda_8}\,. \tag{3.10}$$

Therefore, finally choosing the Euler representation for the elements of two subgroups $U \in SU(2)$ and $V \in SU(2)'$ in terms of two sets of angles (α,β,γ) and (a,b,c)

$$U(\alpha,\beta,\gamma) \;=\; \exp\left(i\,\frac{\alpha}{2}\,\lambda_3\right)\,\exp\left(i\,\frac{\beta}{2}\,\lambda_2\right)\,\exp\left(i\,\frac{\gamma}{2}\,\lambda_3\right)\,, \tag{3.11}$$

$$V(a,b,c) \;=\; \exp\left(i\,\frac{a}{2}\,\lambda_3\right)\,\exp\left(i\,\frac{b}{2}\,\lambda_2\right)\,\exp\left(i\,\frac{c}{2}\,\lambda_3\right)\,, \tag{3.12}$$

we arrive at the generalized Euler decomposition of an element of $g \in SU(3)$

$$g = U(\alpha,\beta,\gamma)\,Z(\theta,\phi)\,V(a,b,c), \tag{3.13}$$

with

$$Z(\theta,\phi) := \mathrm{e}^{i\,\theta\,\lambda_5}\,\mathrm{e}^{i\,\phi\,\lambda_8}\,. \tag{3.14}$$

Now it is necessary to fix the range of angles in (3.13). Just as in the case of the $SU(2)$ group where the Euler parameterization was not a global one, the $SU(3)$ group manifold cannot be covered by one chart. However there is a range of parameters such that the parameterization covers almost the whole manifold

*The freedom of choice in the one-parameter subgroups is analogous to the «x» or «y» Euler angle representation of $SU(2)$ with freedom to choose either σ_1 or σ_2.

except the set whose measure in the integral quantities, e.g. such as the invariant volume, is zero. The following ranges for the angles in (3.13)

$$0 \leqslant \alpha, a \leqslant 2\pi\,, \qquad 0 \leqslant \beta\,, b \leqslant \pi\,, \quad 0 \leqslant \gamma\,, c \leqslant 4\pi\,, \tag{3.15}$$

$$0 \leqslant \theta \leqslant \frac{\pi}{2}\,, \quad 0 \leqslant \phi \leqslant \sqrt{3}\pi \tag{3.16}$$

lead to the invariant volume for $SU(3)$

$$\mathrm{Vol}(SU(3)) = \int_{SU(3)} \cdot 1 = \sqrt{3}\,\pi^5\,. \tag{3.17}$$

Below this result will be checked by an explicit calculation of the volume of the $SU(3)$ manifold considered as the Riemannian space endowed with the bi-invariant metric

$$\mathbf{g}_{SU(3)} = -\frac{1}{2}\,\mathrm{Tr}\,\left(g^{-1}\,\mathrm{d}g \otimes g^{-1}\,\mathrm{d}g\right)\,. \tag{3.18}$$

In terms of the nonholonomic frame built up from the left/right-invariant forms

$$g^{-1}\mathrm{d}g \quad = \quad \frac{i}{2}\sum_{A=1}^{8} \lambda_A \otimes \omega_L^A\,, \tag{3.19}$$

$$\mathrm{d}g\,g^{-1} \quad = \quad \frac{i}{2}\sum_{A=1}^{8} \lambda_A \otimes \omega_R^A\,, \tag{3.20}$$

the Cartan–Killing metric (3.18) has the diagonal form

$$\mathbf{g}_{SU(3)} \quad = \quad \frac{1}{4}\,\left(\omega_L^1 \otimes \omega_L^1 + \omega_L^2 \otimes \omega_L^2 + \,\ldots\, + \omega_L^8 \otimes \omega_L^8\right) \tag{3.21}$$

$$\quad = \quad \frac{1}{4}\,\left(\omega_R^1 \otimes \omega_R^1 + \omega_R^2 \otimes \omega_R^2 + \,\ldots\, + \omega_R^8 \otimes \omega_R^8\right)\,, \tag{3.22}$$

while in the corresponding coordinate frame, with the Eulerian coordinates $(\alpha\,,\beta\,,\gamma\,,a\,,b\,,c\,,\theta\,,\phi)$, presented in Appendix (A.2), it becomes

$$\begin{aligned}\mathbf{g}_{SU(3)} = \frac{1}{4}\,&\big(\mathrm{d}\alpha \otimes \mathrm{d}\alpha + \mathrm{d}\beta \otimes \mathrm{d}\beta + \mathrm{d}\gamma \otimes \mathrm{d}\gamma + 2\cos\beta\,\mathrm{d}\alpha \otimes \mathrm{d}\gamma\big) + \\ &+ \frac{1}{4}\,\big(\mathrm{d}a \otimes \mathrm{d}a + \mathrm{d}b \otimes \mathrm{d}b + \mathrm{d}c \otimes \mathrm{d}c + 2\cos b\,\mathrm{d}a \otimes \mathrm{d}c\big) + \\ &+ \frac{1}{2}\,\cos\theta\Big[\sin(a+\gamma)\big(\sin\beta\,\mathrm{d}\alpha \otimes \mathrm{d}b + \sin b\mathrm{d}\beta \otimes \mathrm{d}c\big) + \\ + \cos(a+\gamma)\big(\mathrm{d}\beta \otimes \mathrm{d}b - \sin\beta\sin b\mathrm{d}\alpha \otimes \mathrm{d}c\big)\Big] &- \frac{\sqrt{3}}{2}\sin^2\theta\big(\cos\beta\mathrm{d}\alpha + \mathrm{d}\gamma\big) \otimes \mathrm{d}\phi + \\ + \frac{1}{4}(1+\cos^2\theta)\big(\cos\beta\mathrm{d}\alpha + \mathrm{d}\gamma\big) &\otimes \big(\mathrm{d}a + \cos b\mathrm{d}c\big) + \mathrm{d}\theta \otimes \mathrm{d}\theta + \mathrm{d}\phi \otimes \mathrm{d}\phi.\end{aligned} \tag{3.23}$$

Fixing the range of the Euler angles according to (3.15) and noting that the determinant of the Cartan–Killing metric (3.23) is

$$\det \mathbf{g}_{SU(3)} = \left(\frac{1}{2}\right)^{12} \sin^6(\theta)\cos^2(\theta)\sin^2(\beta)\sin^2(b),$$

one can check that the group-invariant volume on $SU(3)$ agrees with (3.17)

$$\begin{aligned}\mathrm{Vol}(SU(3)) = \int_{SU(3)} \sqrt{\det \mathbf{g}_{SU(3)}}\mathrm{d}\alpha\wedge\mathrm{d}\beta\wedge\mathrm{d}\gamma\wedge\mathrm{d}\theta\wedge\mathrm{d}a\wedge\mathrm{d}b\wedge\mathrm{d}c\wedge\mathrm{d}\phi = \\ = \left(\frac{1}{2}\right)^{6} \int_0^{2\pi}\mathrm{d}\alpha \int_0^{4\pi}\mathrm{d}\gamma \int_0^{2\pi}\mathrm{d}a \int_0^{4\pi}\mathrm{d}c \int_0^{\sqrt{3}\pi}\mathrm{d}\phi\times \\ \times \int_0^{\pi}\mathrm{d}\beta\,\sin(\beta) \int_0^{\pi/2}\mathrm{d}\theta\,\cos(\theta)\sin^3(\theta) \int_0^{\pi}\mathrm{d}b\,\sin(b) = \sqrt{3}\,\pi^5. \quad (3.24)\end{aligned}$$

This volume is in accordance with the general formula established by I. G. Macdonald in [23] and expresses the volume element of a compact Lie group in terms of the product of volume elements of odd-dimensional unit spheres

$$\mathrm{Vol}(SU(3)) \quad = \quad \frac{\sqrt{3}}{2}\times\mathrm{Vol}(\mathbb{S}^5)\times\mathrm{Vol}(\mathbb{S}^3) = \frac{\sqrt{3}}{2}\times\pi^3\times 2\pi^2\,. \quad (3.25)$$

In (3.25) the multiplier $\sqrt{3}/2$, comes from the volume of the maximal torus in $SU(3)$, interpreted sometimes as the *«stretching»* factor [24, 25]. This fact explicitly shows that the $SU(3)$ group is not a trivial product of the two spheres, $\mathbb{S}^3$ and $\mathbb{S}^5$.

The $SU(3)$ group endowed with the bi-invariant metric (3.23) has a constant positive Riemann scalar curvature

$$\mathcal{R}_{SU(3)} = 24\,,$$

and the Ricci tensor obeys the relations*

$$\mathcal{R}_{\mu\nu} = \frac{\mathcal{R}_{SU(3)}}{8}\,\mathrm{g}_{\mu\nu} = 3\,\mathrm{g}_{\mu\nu}. \quad (3.26)$$

*However, in contrast to the $SU(2)$ group the basic relation defining a space of constant curvature

$$\mathcal{R}_{\mu\nu\sigma\lambda} = \frac{\mathcal{R}}{n(n-1)}\left(\mathrm{g}_{\mu\sigma}\mathrm{g}_{\nu\lambda} - \mathrm{g}_{\mu\lambda}\mathrm{g}_{\nu\sigma}\right)$$

is not valid for the $SU(3)$ group.

3.2. Geometry of the Left Coset $SU(3)/SU(2)$. The group $SU(3)$ can be viewed as a *principal bundle over the base* $\mathbb{S}^5$ *with the structure group* $SU(2)$

$$SU(2) \hookrightarrow SU(3) \xrightarrow{\pi} \mathbb{S}^5,$$

with the canonical projection π from the $SU(3)$ onto the left coset $SU(3)/SU(2) \simeq \mathbb{S}^5$. This map can be realized in the following manner. Consider the general linear group $GL(3,\mathbb{C})$. An arbitrary element $\mathrm{M}_{3\times 3}$ can be written in the block form

$$\mathrm{M}_{3\times 3} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathrm{M}_{2\times 2}} & \begin{matrix} z_3 \\ z_2 \end{matrix} \\ \hline y_1 & y_2 & z_1 \end{array}\right) = \left(\begin{array}{c|c} \mathrm{M}_{2\times 2} & \mathbf{a} \\ \hline \mathbf{b} & z_1 \end{array}\right) \tag{3.27}$$

for complex 2×2 matrix $\mathrm{M}_{2\times 2}$ and $z_1\,, z_2\,, z_3\,, y_1\,, y_2 \in \mathbb{C}\,.$ The $U(3)$ subgroup of the $GL(3,\mathbb{C})$ group is defined by the two matrix equations

$$\mathrm{M}_{3\times 3}\mathrm{M}^{\dagger}_{3\times 3} = \mathbf{I}_{3\times 3}\,, \qquad \mathrm{M}^{\dagger}_{3\times 3}\mathrm{M}_{3\times 3} = \mathbf{I}_{3\times 3}\,. \tag{3.28}$$

When $\mathrm{M}_{3\times 3}$ is represented in block form, (3.27), the conditions (3.28) reduce to the quadratic equations

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1, \tag{3.29}$$

$$|z_1|^2 + |y_1|^2 + |y_2|^2 = 1 \tag{3.30}$$

and to the set of 2×2 matrix equations

$$\mathrm{M}_{2\times 2}\mathrm{M}^{\dagger}_{2\times 2} + \mathbf{a}\mathbf{a}^{\dagger} = \mathrm{I}_{2\times 2}, \tag{3.31}$$

$$\mathrm{M}^{\dagger}_{2\times 2}\mathrm{M}_{2\times 2} + \mathbf{b}^{\dagger}\mathbf{b} = \mathrm{I}_{2\times 2}, \tag{3.32}$$

$$z_1\,\mathbf{a} + \mathrm{M}_{2\times 2}\mathbf{a} = \mathbf{0}, \tag{3.33}$$

$$\bar{z}_1\,\mathbf{b} + \mathrm{M}^{\dagger}_{2\times 2}\mathbf{b} = \mathbf{0}. \tag{3.34}$$

Now let $\mathbb{S}^5$ be the five-sphere characterized by a unit complex vector $\mathbf{Z} := (z_1\,, z_2\,, z_3)^T$

$$\mathbf{Z}^{\dagger}\mathbf{Z} = 1\,.$$

The $SU(3)$ group element g then acts on this through left translations:

$$\mathbf{Z} \to \mathbf{Z}' = g\mathbf{Z}. \tag{3.35}$$

Let $\mathbf{Z}_0$ be the base point on this five-sphere with coordinates $\mathbf{Z}_0 = (0,0,1)^T$ whose isotropy group is

$$H_{3\times 3} = \left(\begin{array}{c|c} SU(2) & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right)\,. \tag{3.36}$$

Then the coset $SU(3)/SU(2)$ can be identified with the orbit

$$\mathbf{Z} = g \cdot (0\,,0\,,1)^T\,. \tag{3.37}$$

Using the explicit form of the representation (3.13), the subgroup $SU(2)$ is embedded into $SU(3)$ as follows:

$$SU(2) \to SU(3),\ V = \begin{pmatrix} \mathrm{e}^{-i\dfrac{a+c}{2}} \cos\left(\dfrac{b}{2}\right) & -\mathrm{e}^{-i\dfrac{a-c}{2}} \sin\left(\dfrac{b}{2}\right) & 0 \\ \mathrm{e}^{i\dfrac{a-c}{2}} \sin\left(\dfrac{b}{2}\right) & \mathrm{e}^{i\dfrac{a+c}{2}} \cos\left(\dfrac{b}{2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.38}$$

So the parameterization of a group element is

$$g = U\,Z\,V = W\,V\,,$$

where the factor W reads

$$W = \begin{pmatrix} \cos\theta\cos\dfrac{\beta}{2}\mathrm{e}^{i(u+\frac{1}{\sqrt{3}}\phi)} & \sin\dfrac{\beta}{2}\mathrm{e}^{i(v+\frac{1}{\sqrt{3}}\phi)} & \sin\theta\cos\dfrac{\beta}{2}\mathrm{e}^{i(u-\frac{2}{\sqrt{3}}\phi)} \\ -\cos\theta\sin\dfrac{\beta}{2}\mathrm{e}^{-i(v-\frac{1}{\sqrt{3}}\phi)} & \cos\dfrac{\beta}{2}\,\mathrm{e}^{-i(u-\frac{1}{\sqrt{3}}\phi)} & -\sin\theta\sin\dfrac{\beta}{2}\mathrm{e}^{-i(v+\frac{2}{\sqrt{3}}\phi)} \\ -\sin\theta\,\mathrm{e}^{\frac{i}{\sqrt{3}}\phi} & 0 & \cos\theta\,\mathrm{e}^{-i\frac{2}{\sqrt{3}}\phi} \end{pmatrix}.$$

$$u = \frac{\alpha+\gamma}{2}\,, \qquad v = \frac{\alpha-\gamma}{2}\,.$$

Using these representations in (3.37) we easily identify the projection onto the left coset as a five-sphere:

$$\pi : g \in SU(3) \to (z_1\,, z_2\,, z_3) \in \mathbb{S}^5\,,$$

which explicitly reads

$$z_1 = \cos\theta\ \mathrm{e}^{-i\frac{2}{\sqrt{3}}\,\phi}\,, \tag{3.39}$$

$$z_2 = -\sin\theta\sin\frac{\beta}{2}\,\mathrm{e}^{-\frac{i}{2}(\alpha-\gamma+\frac{4}{\sqrt{3}}\,\phi)}\,, \tag{3.40}$$

$$z_3 = \sin\theta\cos\frac{\beta}{2}\,\mathrm{e}^{\frac{i}{2}(\alpha+\gamma-\frac{4}{\sqrt{3}}\,\phi)}\,. \tag{3.41}$$

Under this projection the Euclidean metric $\mathrm{Tr}(\mathrm{d}M\mathrm{d}M^\dagger)$ on $GL(3,\mathbb{C})$ induces the following metric on a unit $\mathbb{S}^5$:

$$\mathbf{g}_{\mathbb{S}^5} = \mathrm{d}\bar{z}_1\otimes\mathrm{d}z_1 + \mathrm{d}\bar{z}_2\otimes\mathrm{d}z_2 + \mathrm{d}\bar{z}_3\otimes\mathrm{d}z_3 =$$

$$= \sin^2\theta\biggl(\frac{1}{4}\left(\mathrm{d}\alpha\otimes\mathrm{d}\alpha + \mathrm{d}\beta\otimes\mathrm{d}\beta + \mathrm{d}\gamma\otimes\mathrm{d}\gamma + 2\cos\beta\,\mathrm{d}\alpha\otimes\mathrm{d}\gamma\right) -$$

$$- \frac{2}{\sqrt{3}}\left(\cos\beta\mathrm{d}\alpha + \mathrm{d}\gamma\right)\otimes\mathrm{d}\phi\biggr) + \mathrm{d}\theta\otimes\mathrm{d}\theta + \frac{4}{3}\,\mathrm{d}\phi\otimes\mathrm{d}\phi, \quad (3.42)$$

whose determinant is

$$\det\mathbf{g}_{\mathbb{S}^5} = \frac{1}{48}\,\sin^6(\theta)\cos^2(\theta)\sin^2(\beta)\,. \quad (3.43)$$

The metric (3.42) defines a unit five-sphere $\mathbb{S}^5$ as a constant curvature Riemann manifold

$$\mathcal{R}_{\mathbb{S}^5} = 20\,, \quad (3.44)$$

which is in accordance with its Gaussian curvature

$$K_{\mathbb{S}^5} = \frac{\mathcal{R}_{\mathbb{S}^5}}{5(5-1)} = 1\,,$$

as well as with its volume

$$\mathrm{Vol}(\mathbb{S}^5) = \int_{\mathbb{S}^5}\sqrt{\det\mathbf{g}_{\mathbb{S}^5}}\,\mathrm{d}\alpha\wedge\mathrm{d}\beta\wedge\mathrm{d}\gamma\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi =$$

$$= \frac{1}{4\sqrt{3}}\int_0^{2\pi}\mathrm{d}\alpha\int_0^{4\pi}\mathrm{d}\gamma\int_0^{\sqrt{3}\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\beta\,\sin(\beta)\int_0^{\pi/2}\mathrm{d}\theta\,\cos(\theta)\sin^3(\theta) = \pi^3. \quad (3.45)$$

3.3. Lagrangian on $SU(3)$ in Terms of Generalized Euler Angles. Consider the Lagrangian describing the geodesic motion on the $SU(3)$ group manifold with respect to the bi-invariant metric (3.18)

$$L_{SU(3)} = -\frac{1}{2}\,\mathrm{Tr}\left(g^{-1}(t)\frac{d}{dt}g(t)\,g^{-1}(t)\frac{d}{dt}g(t)\right). \quad (3.46)$$

Using the generalized Euler angles on $SU(3)$ as the configuration space coordinates and (3.23) for the bi-invariant metric, one can write the Lagrangian (3.46) as

$$L_{SU(3)} = \frac{1}{4}\left(\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2 + 2\cos\beta\,\dot{\alpha}\dot{\gamma} + \dot{a}^2 + \dot{b}^2 + \dot{c}^2 + 2\cos b\,\dot{a}\dot{c}\right) +$$

$$+ \frac{1}{2}\cos\theta\left(\sin(a+\gamma)\bigl(\sin\beta\,\dot{\alpha}\dot{b} + \sin b\,\dot{\beta}\dot{c}\bigr) + \cos(a+\gamma)\bigl(\dot{\beta}\dot{b} - \sin\beta\sin b\,\dot{\alpha}\dot{c}\bigr)\right) -$$

$$- \frac{\sqrt{3}}{2}\sin^2\theta\bigl(\cos\beta\,\dot{\alpha}+\dot{\gamma}\bigr)\dot{\phi} + \frac{1}{4}\left(1+\cos^2\theta\right)\bigl(\cos\beta\,\dot{\alpha}+\dot{\gamma}\bigr)\left(\dot{a}+\cos b\,\dot{c}\right) + \dot{\theta}^2 + \dot{\phi}^2. \quad (3.47)$$

From this expression and (3.23) for it follows that:

$$L_{SU(3)} = \mathbf{g}_{SU(3)}(\dot{Z}, \dot{Z}), \tag{3.48}$$

where $\dot{Z}$ is the vector field on the tangent bundle $TSU(3)$

$$\dot{Z} = \dot{\alpha}\,\frac{\partial}{\partial\alpha} + \dot{\beta}\,\frac{\partial}{\partial\beta} + \dot{\gamma}\,\frac{\partial}{\partial\gamma} + \dot{\theta}\,\frac{\partial}{\partial\theta} + \dot{\phi}\,\frac{\partial}{\partial\phi} + \dot{a}\,\frac{\partial}{\partial a} + \dot{b}\,\frac{\partial}{\partial b} + \dot{c}\,\frac{\partial}{\partial c}\,. \tag{3.49}$$

It is worth to note that the Euler decomposition (3.13) for elements of $SU(3)$ in terms of the $SU(2)$ subgroups,

$$SU(3) = U(\alpha,\beta,\gamma)\ \exp(i\,\theta\,\lambda_5)\,V(a,b,c)\ \exp(i\,\phi\,\lambda_8)\,,$$

allows for the expression of the $SU(3)$ Lagrangian (3.47) in terms of the corresponding left- and right-invariant elements of the $SU(2)$ Maurer–Cartan 1-forms:

$$L_{SU(3)} = \frac{1}{4}\sum_{a=1}^{3} i_{\dot{U}}\omega_L^a\, i_{\dot{U}}\omega_L^a + \frac{1}{4}\sum_{a=1}^{3} i_{\dot{V}}\omega_L^a\, i_{\dot{V}}\omega_L^a + \frac{1}{2}\cos\theta \sum_{a=1}^{2} i_{\dot{U}}\omega_L^a\, i_{\dot{V}}\omega_R^i -$$
$$- \frac{1}{4}(1+\cos^2\theta)\, i_{\dot{U}}\omega_L^3\, i_{\dot{V}}\omega_R^3 - \frac{\sqrt{3}}{2}\,\sin^2\theta\, i_{\dot{U}}\omega_L^3\,\dot{\phi} + \dot{\theta}^2 + \dot{\phi}^2. \tag{3.50}$$

Here $i_{\dot{U}}$ and $i_{\dot{V}}$ denote the interior contraction of the vector field on each copy of the $SU(2)$ group, U and V, respectively,

$$\dot{U} = \dot{\alpha}\,\frac{\partial}{\partial\alpha} + \dot{\beta}\,\frac{\partial}{\partial\beta} + \dot{\gamma}\,\frac{\partial}{\partial\gamma}\,, \qquad \dot{V} = \dot{a}\,\frac{\partial}{\partial a} + \dot{b}\,\frac{\partial}{\partial b} + \dot{c}\,\frac{\partial}{\partial c}\,. \tag{3.51}$$

3.4. Hamiltonian Dynamics on $SU(3)$. Performing the Legendre transformation, we derive the canonical Hamiltonian generating the dynamics on the $SU(3)$ group manifold:

$$H_{SU(3)} = \frac{1}{\sin^2\theta}\left[\frac{p_\alpha^2}{\sin^2\beta} + p_\beta^2 + \left(\tan^2\,\theta + \frac{1}{\sin^2\beta}\right)p_\gamma^2 - 2\,\frac{\cos\beta}{\sin^2\beta}\;p_\alpha p_\gamma + \right.$$
$$\left. + \sin^2\theta\left(1 + \frac{1}{4}\cot^2\theta + \frac{1}{\sin^2 b}\right)p_a^2 + p_b^2 + \frac{1}{\sin^2 b}p_c^2 - 2\,\frac{\cos b}{\sin^2 b}p_a p_c\right] +$$
$$+ 2\,\frac{\cos\theta}{\sin^2\theta\sin\beta\sin b}\Big[\cos(a{+}\gamma)\Big((p_\alpha - \cos\beta\, p_\gamma)(p_c - \cos b\, p_a) - \sin b\, p_\beta\, p_b\Big) -$$
$$- \sin(a+\gamma)\Big(\sin b(p_\alpha - \cos\beta\, p_\gamma)p_b + \sin\beta(p_c - \cos b\, p_a)p_\beta\Big)\Big] +$$
$$+ \frac{1}{4}\,p_\theta^2 + \frac{1}{16}\left(1 + \frac{3}{\cos^2\,\theta}\right)\;p_\phi^2 + \frac{\sqrt{3}}{2}\,\frac{p_\gamma p_\phi}{\cos^2\theta} - \frac{\sqrt{3}}{4}\left(1 + \frac{1}{\cos^2\theta}\right)p_a p_\phi. \tag{3.52}$$

The Hamiltonian (3.52) can be rewritten in a compact form using the left- and right-invariant vector fields on the two $SU(2)$ group copies, U and V used in the Euler decomposition (3.13):

$$H_{SU(3)} = \sum_{a=1}^{3} \zeta^R_a\, \zeta^R_a + \frac{1}{\sin^2\theta} \sum_{a=1}^{2} (\xi^L_a - \cos\theta\, \zeta^R_a)^2 + \\ + \frac{1}{\sin^2 2\,\theta}(2\,\xi^L_3 - (1+\cos^2\theta)\,\zeta^R_3 - \frac{\sqrt{3}}{2}\sin^2\theta\, p_\phi)^2 + \frac{1}{4}\,p^2_\theta + \frac{1}{4}\,p^2_\phi. \quad (3.53)$$

Here ξ^L_a and ζ^R_a are functions defined through the relations

$$\xi^L_a := \Theta\left(X^L_a\right)\,, \qquad \zeta^R_a := \Theta\left(Y^R_a\right)\,,$$

with the $SU(2)$ left-invariant vector fields X^L_a on the tangent space to the U subgroup, TU, and the right-invariant fields Y^R_a on TV, correspondingly.

3.5. Hamiltonian Reduction to $SU(3)/SU(2)$. The representation (3.53) is very convenient for performing the reduction in degrees of freedom associated with the $SU(2)$ symmetry transformation. Due to the algebra of Poisson brackets (2.35) the functions ζ^L_1, ζ^L_2, and ζ^L_3 are the first integrals

$$\{\zeta^L_a\,, H_{SU(3)}\} = 0.$$

Let us consider the zero level of these integrals

$$\zeta^L_1 = 0\,, \qquad \zeta^L_2 = 0, \qquad \zeta^L_3 = 0. \quad (3.54)$$

Noting the relation between the left- and right-invariant vector fields on a group one can express the functions ζ^R_a entering in the Hamiltonian as

$$\zeta^R_c = \mathrm{Ad}(V)_{cb}\,\zeta^L_b\,,$$

where $\mathrm{Ad}(V)_{cb}$ is an adjoint matrix of an element $V \in SU(2)$. From this one can immediately find the reduced Hamiltonian on the integral level (3.54). Indeed, projecting the expression (3.53) on $\zeta^R_a = 0$ we find

$$H_{SU(3)/SU(2)} = \frac{1}{\sin^2\theta} \sum_{a=1}^{3} \xi^L_a\, \xi^L_a + \\ + \frac{1}{\sin^2 2\,\theta}(2\,\xi^L_3 - \frac{\sqrt{3}}{2}\sin^2\theta\, p_\phi)^2 + \frac{1}{4}\,p^2_\theta + \frac{1}{4}\,p^2_\phi\,, \quad (3.55)$$

or more explicitly in terms of the canonical coordinates

$$H_{SU(3)/SU(2)} = \frac{1}{\sin^2\theta}\left(\frac{p_\alpha^2}{\sin^2\beta} + p_\beta^2 + \left(\tan^2\,\theta + \frac{1}{\sin^2\beta}\right) p_\gamma^2 - \right.$$
$$\left. - \, 2\,\frac{\cos\beta}{\sin^2\beta}\; p_\alpha p_\gamma + \frac{\sqrt{3}}{2}\tan^2\theta\, p_\gamma p_\phi\right) + \frac{1}{4}\,p_\theta^2 + \frac{1}{16}\left(1 + \frac{3}{\cos^2\,\theta}\right)\; p_\phi^2. \quad (3.56)$$

Performing the inverse Legendre transformation we find the Lagrangian

$$L_{SU(3)/SU(2)} = \frac{1}{4}\,\sin^2\,\theta\left(\left(1 - \frac{1}{4}\cos^2\beta\,\sin^2\,\theta\right)\dot{\alpha}^2 + \dot{\beta}^2 + \frac{1}{4}\left(3 + \cos^2\,\theta\right)\dot{\gamma}^2 + \right.$$
$$\left. + \; \frac{1}{2}\,\cos\beta\,\left(3 + \cos^2\,\theta\right)\;\dot{\alpha}\dot{\gamma} - 2\sqrt{3}\,(\cos\beta\,\dot{\alpha} + \dot{\gamma})\dot{\phi}\right) + \dot{\theta}^2 + \dot{\phi}^2. \quad (3.57)$$

Now one can consider the bilinear form (3.57) as the metric $\mathbf{g}_{\mathcal{O}}$ on the orbit space $\mathcal{O} = SU(3)/SU(2)$

$$\mathbf{g}_{\mathcal{O}} = \frac{1}{4}\,\sin^2\,\theta\left(\left(1 - \frac{1}{4}\cos^2\beta\,\sin^2\,\theta\right)\mathrm{d}\alpha\otimes\mathrm{d}\alpha + \mathrm{d}\beta\otimes\mathrm{d}\beta + \frac{1}{4}\left(3 + \cos^2\,\theta\right)\times\right.$$
$$\left. \times\; \mathrm{d}\gamma\otimes\mathrm{d}\gamma + \frac{1}{2}\,\cos\beta\,\left(3 + \cos^2\,\theta\right)\;\mathrm{d}\alpha\otimes\mathrm{d}\gamma - 2\sqrt{3}\,(\cos\beta\,\mathrm{d}\alpha + \mathrm{d}\gamma)\otimes\mathrm{d}\phi\right) +$$
$$+ \,\mathrm{d}\theta\otimes\mathrm{d}\theta + \mathrm{d}\phi\otimes\mathrm{d}\phi. \quad (3.58)$$

Using our previous calculations (3.45) of $\mathrm{Vol}(\mathbb{S}^5)$ with respect to the metric (3.42) induced by the canonical projection to the left coset $\pi : SU(3) \to SU(3)/SU(2)$ and noting that the determinant of the new orbit metric (3.58) induced by the Hamiltonian reduction is

$$\det\mathbf{g}_{\mathcal{O}} \;=\; \frac{1}{64}\sin^6(\theta)\cos^2(\theta)\sin^2(\beta)\,, \quad (3.59)$$

we find

$$\mathrm{Vol}(SU(3)/SU(2)) = \frac{\sqrt{3}}{2}\,\mathrm{Vol}(\mathbb{S}^5), \quad (3.60)$$

with the same *stretching* factor $\sqrt{3}/2$ as found in (3.25) for the bi-invariant volume of the $SU(3)$ group.

4. RIEMANNIAN STRUCTURES ON THE QUOTIENT SPACE

Now we are ready to answer the questions about the relation between metric (3.42) induced on the left coset $SU(3)/SU(2)$ by canonical projection from the ambient Euclidean space and the metric (3.58) obtained as a result of performing the Hamiltonian reduction of the geodesic motion from $SU(3)$ to $SU(3)/SU(2)$.

Performing a straightforward calculation of the Riemannian curvature with respect to the metric (3.58) yields

$$\mathcal{R}\big(\mathbf{g}_{\frac{SU(3)}{SU(2)}}\big) = 21\,, \tag{4.1}$$

while, from the embedding argumentation we used before, the Riemann scalar of the unit five-sphere $\mathbb{S}^5$ with standard metric induced from the Euclidean space is

$$\mathcal{R}(\mathbf{g}_{\mathbb{S}^5}) = 20. \tag{4.2}$$

Furthermore, even though the Riemann scalar is a constant, calculations show that the metric (3.57) *is not the metric of a space of constant curvature*.

So, we have found that the Lagrangian of the reduced system defines local flows on the configuration space which are not isometric to those on $\mathbb{S}^5$ with its standard round metric.

We have shown above that the orbit space $SU(3)/SU(2)$ considered as a Riemannian space with metric $\mathbf{g}$ induced from the Cartan–Killing metric on $SU(3)$ is not isometric to the $\mathbb{S}^5$ with the standard round metric $\mathbf{g}_{\mathbb{S}^5}$. The next natural question is whether the metrics $\mathbf{g}$ and $\mathbf{g}_{\mathbb{S}^5}$ are *geodesically /projectively* equivalent.

There are several criteria on metrics to be geodesically equivalent. According to L. P. Eisenhart [26], two metrics $\mathbf{g}$ and $\overline{\mathbf{g}}$ on n-dimensional Riemann manifold *are geodesically equivalent if and only if*

$$2\,(n+1)\nabla_i(\mathbf{g})\,\overline{\mathbf{g}}_{jk} = 2\overline{\mathbf{g}}_{jk}\,\partial_i\Lambda + \overline{\mathbf{g}}_{ik}\,\partial_j\Lambda + \overline{\mathbf{g}}_{ji}\,\partial_k\Lambda\,, \tag{4.3}$$

where $\nabla_i(\mathbf{g})$ is covariant with respect the metric $\mathbf{g}$ and the scalar function Λ is

$$\Lambda = \ln\left(\frac{\det(\overline{\mathbf{g}})}{\det(\mathbf{g})}\right)\,. \tag{4.4}$$

According to our calculations

$$\det(\mathbf{g}_{\mathcal{O}}) = \frac{3}{4}\det(\mathbf{g}_{\mathbb{S}^5})$$

and

$$\nabla_i\big(\mathbf{g}_{\mathbb{S}^5}\big)\ \mathbf{g}_{\mathcal{O}\,jk} \neq 0\,,$$

and therefore $\mathbf{g}_{\mathbb{S}^5}$ and $\mathbf{g}_{\mathcal{O}}$ are not *geodesically /projectively* equivalent.

CONCLUSION

In this paper we have presented, for the first time, the explicit Hamiltonian reduction from free motion on $SU(3)$ to motion on the coset space $SU(3)/SU(2) \approx \mathbb{S}^5$. This has been made possible through a consistent parameterization of $SU(3)$ that generalizes the Euler angle parameterization of $SU(2)$. The full details for this parameterization of $SU(3)$ are, for completeness, collected together in Appendix. The results presented there have been checked independently using the computer algebra packages *Mathematica 5.0* and *Maple 9.5*.

Through this analysis we have seen that the resulting dynamics is not equivalent to the geodesic motion on $\mathbb{S}^5$ induced from its standard round metric. This result prompts the following questions.

• Is it possible to identify, a priori, the induced metric on the coset space in terms of the properties of $SU(3)$?

• Is it possible to formulate the dynamics on $SU(3)$ so that the reduced dynamics is the expected geodesic motion on $\mathbb{S}^5$?

• What happens if we reduce to a nonzero level set of the integrals (3.54)?

Progress in answering these questions will, we feel, throws much light on the dynamical aspects of the Hamiltonian reduction procedure and hence leads to a deeper understanding of the quantization of gauge theories.

APPENDIX

1. The $SU(3)$ Algebra Structure. The eight traceless 3×3 Gell-Mann matrices providing a basis for the $SU(3)$ algebra are listed below

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{A1}$$

Sometimes it is convenient to use instead of the Gell-Mann matrices the anti-Hermitian basis $\mathbf{t}_a := \frac{1}{2i}\lambda_a$, obeying the relations

$$\mathbf{t}_a\,\mathbf{t}_b \;=\; -\frac{1}{6}\,\delta_{ab}\,\mathbf{I} + \frac{1}{2}\sum_{c\,=\,1}^{8}\,(\,\mathrm{f}_{abc} - \imath\,\mathrm{d}_{abc})\;\mathbf{t}_c\,, \tag{A2}$$

where the structure constants d_{abc} are symmetric in their indices and the non-vanishing values are given in Table 1, the coefficients f_{abc} are skew symmetric in all indices. The constants f_{abc} determine the commutators between the basis elements

$$[\mathbf{t}_a\,,\mathbf{t}_b] \;=\; \sum_{c\,=\,1}^{8}\,\mathrm{f}_{abc}\,\mathbf{t}_c\,. \tag{A3}$$

Table 1. **The symmetric coefficients** d_{abc}

(abc)	(118)(228)(338)	(146),(157)(256)(344)(355)	(247)(366)(377)	(448)(558)(668)(778)(888)
d_{abc}	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$

Table 2. **Structure of the** $SU(3)$ **algebra**

	$\mathbf{t}_1$	$\mathbf{t}_2$	$\mathbf{t}_3$	$\mathbf{t}_4$	$\mathbf{t}_5$	$\mathbf{t}_6$	$\mathbf{t}_7$	$\mathbf{t}_8$
$\mathbf{t}_1$	0	$\mathbf{t}_3$	$-\mathbf{t}_2$	$\frac{1}{2}\,\mathbf{t}_7$	$-\frac{1}{2}\,\mathbf{t}_6$	$\frac{1}{2}\,\mathbf{t}_5$	$-\frac{1}{2}\,\mathbf{t}_4$	0
$\mathbf{t}_2$	$-\mathbf{t}_3$	0	$\mathbf{t}_1$	$\frac{1}{2}\,\mathbf{t}_6$	$\frac{1}{2}\,\mathbf{t}_7$	$-\frac{1}{2}\,\mathbf{t}_4$	$-\frac{1}{2}\,\mathbf{t}_5$	0
$\mathbf{t}_3$	$\mathbf{t}_2$	$-\mathbf{t}_1$	0	$\frac{1}{2}\,\mathbf{t}_5$	$-\frac{1}{2}\,\mathbf{t}_4$	$-\frac{1}{2}\,\mathbf{t}_7$	$\frac{1}{2}\,\mathbf{t}_6$	0
$\mathbf{t}_4$	$-\frac{1}{2}\,\mathbf{t}_7$	$-\frac{1}{2}\,\mathbf{t}_6$	$-\frac{1}{2}\,\mathbf{t}_5$	0	$\frac{1}{2}\,\mathbf{t}_3 + \frac{\sqrt{3}}{2}\,\mathbf{t}_8$	$\frac{1}{2}\,\mathbf{t}_2$	$\frac{1}{2}\,\mathbf{t}_1$	$-\frac{\sqrt{3}}{2}\,\mathbf{t}_5$
$\mathbf{t}_5$	$\frac{1}{2}\,\mathbf{t}_6$	$-\frac{1}{2}\,\mathbf{t}_7$	$\frac{1}{2}\,\mathbf{t}_4$	$-\frac{1}{2}\,\mathbf{t}_3 - \frac{\sqrt{3}}{2}\,\mathbf{t}_8$	0	$-\frac{1}{2}\,\mathbf{t}_1$	$\frac{1}{2}\,\mathbf{t}_2$	$\frac{\sqrt{3}}{2}\,\mathbf{t}_4$
$\mathbf{t}_6$	$-\frac{1}{2}\,\mathbf{t}_5$	$\frac{1}{2}\,\mathbf{t}_4$	$\frac{1}{2}\,\mathbf{t}_7$	$-\frac{1}{2}\,\mathbf{t}_2$	$\frac{1}{2}\,\mathbf{t}_1$	0	$-\frac{1}{2}\,\mathbf{t}_3+\frac{\sqrt{3}}{2}\,\mathbf{t}_8$	$-\frac{\sqrt{3}}{2}\,\mathbf{t}_7$
$\mathbf{t}_7$	$\frac{1}{2}\,\mathbf{t}_4$	$\frac{1}{2}\,\mathbf{t}_5$	$-\frac{1}{2}\,\mathbf{t}_6$	$-\frac{1}{2}\,\mathbf{t}_1$	$-\frac{1}{2}\,\mathbf{t}_2$	$\frac{1}{2}\,\mathbf{t}_3-\frac{\sqrt{3}}{2}\,\mathbf{t}_8$	0	$\frac{\sqrt{3}}{2}\,\mathbf{t}_6$
$\mathbf{t}_8$	0	0	0	$\frac{\sqrt{3}}{2}\,\mathbf{t}_5$	$-\frac{\sqrt{3}}{2}\,\mathbf{t}_4$	$\frac{\sqrt{3}}{2}\,\mathbf{t}_7$	$-\frac{\sqrt{3}}{2}\,\mathbf{t}_6$	0

2. The Basis of Invariant 1-Forms on the $SU(3)$ Group

a) The left-invariant 1-forms. Using the generalized Euler decomposition (3.13) for the $SU(3)$ group element, it is straightforward to calculate the left- and right-invariant 1-forms. The results are given below

$$\omega_L^1 = \Big(\cos[\beta]\sin[b]\cos[c](1-\frac{1}{2}\sin^2[\theta])+ \\ +\cos[\theta]\sin[\beta]\big(\cos[b]\cos[c]\cos[a+\gamma]-\sin[c]\sin[a+\gamma]\big)\Big)\mathrm{d}\alpha- \\ -\cos[\theta]\Big(\cos[a+\gamma]\sin[c]+\cos[b]\cos[c]\sin[a+\gamma]\Big)\mathrm{d}\beta+ \\ +\cos[c]\sin[b]\big(1-\frac{1}{2}\sin^2[\theta]\big)\mathrm{d}\gamma+\cos[c]\sin[b]\mathrm{d}a-\sin[c]\mathrm{d}b\,,$$

$$\omega_L^2 = \Big(\cos[\beta]\sin[b]\sin[c](1-\frac{1}{2}\sin^2[\theta])+ \\ +\cos[\theta]\sin[\beta]\big(\cos[b]\cos[a+\gamma]\sin[c]+\cos[c]\sin[a+\gamma]\big)\Big)\mathrm{d}\alpha+ \\ +\cos[\theta]\Big(\cos[c]\cos[a+\gamma]-\cos[b]\sin[c]\sin[a+\gamma]\Big)\mathrm{d}\beta+ \\ +\sin[b]\sin[c](1-\frac{1}{2}\sin^2[\theta])\mathrm{d}\gamma+\sin[b]\sin[c]\mathrm{d}a+\cos[c]\mathrm{d}b\,,$$

$$\omega_L^3 = \Big(\cos[b]\cos[\beta](1-\frac{1}{2}\sin^2[\theta])-\cos[a+\gamma]\cos[\theta]\sin[b]\sin[\beta]\Big)\mathrm{d}\alpha+ \\ +\cos[\theta]\sin[b]\sin[a+\gamma]\mathrm{d}\beta+\cos[b](1-\frac{1}{2}\sin^2[\theta])\mathrm{d}\gamma+\cos[b]\mathrm{d}a+\mathrm{d}c\,,$$

$$\omega_L^4 = \sin[\theta]\Big(\cos[\beta]\cos[\theta]\cos[\frac{b}{2}]\cos\big[\frac{a+c}{2}+\sqrt{3}\phi\big]- \\ -\cos\big[\frac{a-c}{2}+\gamma-\sqrt{3}\phi\big]\sin[\frac{b}{2}]\sin[\beta]\Big)\mathrm{d}\alpha+ \\ +\sin[\frac{b}{2}]\sin[\theta]\sin\big[\frac{a-c}{2}+\gamma-\sqrt{3}\phi\big]\mathrm{d}\beta+ \\ +\frac{1}{2}\cos[\frac{b}{2}]\cos\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\sin[2\theta]\mathrm{d}\gamma-2\,\cos[\frac{b}{2}]\sin\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\mathrm{d}\theta,$$

$$\omega_L^5 = \sin[\theta]\bigg(\sin[\frac{b}{2}]\sin[\beta]\sin\big[\frac{a-c}{2}+\gamma-\sqrt{3}\phi\big]+$$
$$+\cos[\frac{b}{2}]\cos[\beta]\cos[\theta]\sin\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\bigg)\mathrm{d}\alpha+$$
$$+\cos\big[\frac{a-c}{2}+\gamma-\sqrt{3}\phi\big]\sin[\frac{b}{2}]\sin[\theta]\mathrm{d}\beta+$$
$$+\frac{1}{2}\cos[\frac{b}{2}]\sin[2\theta]\sin\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\mathrm{d}\gamma+2\,\cos[\frac{b}{2}]\cos\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\mathrm{d}\theta,$$

$$\omega_L^6 = \sin[\theta]\bigg(\cos[\beta]\cos[\theta]\cos\big[\frac{a-c}{2}+\sqrt{3}\phi\big]\sin[\frac{b}{2}]+$$
$$+\sin[\beta]\cos\big[\frac{a+c}{2}+\gamma-\sqrt{3}\phi\big]\cos[\frac{b}{2}]\bigg)\mathrm{d}\alpha-$$
$$-\cos[\frac{b}{2}]\sin[\theta]\sin\big[\frac{a+c}{2}+\gamma-\sqrt{3}\phi\big]\mathrm{d}\beta+$$
$$+\frac{1}{2}\cos\big[\frac{a-c}{2}+\sqrt{3}\phi\big]\sin[\frac{b}{2}]\sin[2\theta]\mathrm{d}\gamma-2\,\sin[\frac{b}{2}]\sin\big[\frac{a-c}{2}+\sqrt{3}\phi\big]\mathrm{d}\theta,$$

$$\omega_L^7 = \sin[\theta]\bigg(\cos[\beta]\cos[\theta]\sin[\frac{b}{2}]\sin\big[\frac{a-c}{2}+\sqrt{3}\phi\big]-$$
$$-\cos[\frac{b}{2}]\sin[\beta]\sin\big[\frac{a+c}{2}+\gamma-\sqrt{3}\phi\big]\bigg)\mathrm{d}\alpha-$$
$$-\cos[\frac{b}{2}]\cos\big[\frac{a+c}{2}+\gamma-\sqrt{3}\phi\big]\sin[\theta]\mathrm{d}\beta+$$
$$+\frac{1}{2}\sin[\frac{b}{2}]\sin[2\theta]\sin\big[\frac{a-c}{2}+\sqrt{3}\phi\big]\mathrm{d}\gamma+2\,\cos\big[\frac{a-c}{2}+\sqrt{3}\phi\big]\sin[\frac{b}{2}]\mathrm{d}\theta,$$

$$\omega_L^8 = -\frac{\sqrt{3}}{2}\cos[\beta]\sin^2[\theta]\mathrm{d}\alpha-\frac{\sqrt{3}}{2}\sin^2[\theta]\mathrm{d}\gamma+2\,\mathrm{d}\phi\,.$$

b) The right-invariant 1-forms.

$$\omega_R^1 = \sin[\alpha]\mathrm{d}\beta-\cos[\alpha]\sin[\beta]\mathrm{d}\gamma-\cos[\alpha]\sin[\beta](1-\frac{1}{2}\sin^2[\theta])\,\mathrm{d}a+$$
$$+\cos[\theta]\bigg(\cos[a+\gamma]\sin[\alpha]+\cos[\alpha]\cos[\beta]\sin[a+\gamma]\bigg)\mathrm{d}b+$$
$$+\bigg(\cos[\theta]\sin[b]\big(-\cos[\alpha]\cos[\beta]\cos[a+\gamma]+\sin[\alpha]\sin[a+\gamma]\big)-$$
$$-\cos[\alpha]\cos[b]\sin[\beta](1-\frac{1}{2}\sin^2[\theta])\bigg)\mathrm{d}c+\sqrt{3}\cos[\alpha]\sin[\beta]\sin^2[\theta]\mathrm{d}\phi,$$

$$
\begin{aligned}
\omega_R^2 = {} & \cos[\alpha]\mathrm{d}\beta + \sin[\alpha]\sin[\beta]\mathrm{d}\gamma + \sin[\alpha]\sin[\beta]\big(1 - \frac{1}{2}\sin^2[\theta]\big)\mathrm{d}a + \\
& + \cos[\theta]\Bigg(\cos[\alpha]\cos[a+\gamma] - \cos[\beta]\sin[\alpha]\sin[a+\gamma]\Bigg)\mathrm{d}b + \\
& + \Bigg(\cos[\theta]\sin[b]\big(\cos[\beta]\cos[a+\gamma]\sin[\alpha] + \cos[\alpha]\sin[a+\gamma]\big) + \\
& + \cos[b]\sin[\alpha]\sin[\beta](1 - \frac{1}{2}\sin^2[\theta])\Bigg)\mathrm{d}c - \sqrt{3}\sin[\alpha]\sin[\beta]\sin^2[\theta]\mathrm{d}\phi
\end{aligned}
$$

$$
\begin{aligned}
\omega_R^3 = {} & \mathrm{d}\alpha + \cos[\beta]\mathrm{d}\gamma + \cos[\beta](1 - \frac{1}{2}\sin^2[\theta])\mathrm{d}a + \cos[\theta]\sin[\beta]\sin[a+\gamma]\mathrm{d}b + \\
& + \Bigg(\cos[b]\cos[\beta](1 - \frac{1}{2}\sin^2[\theta]) - \cos[a+\gamma]\cos[\theta]\sin[b]\sin[\beta]\Bigg)\mathrm{d}c - \\
& - \sqrt{3}\cos[\beta]\sin^2[\theta]\mathrm{d}\phi,
\end{aligned}
$$

$$
\begin{aligned}
\omega_R^4 = {} & 2\,\cos[\frac{\beta}{2}]\sin[\frac{\alpha+\gamma}{2}]\mathrm{d}\theta - \frac{1}{2}\cos[\frac{\beta}{2}]\cos[\frac{\alpha+\gamma}{2}]\sin[2\theta]\mathrm{d}a - \\
& - \sin[\frac{\beta}{2}]\sin[a - \frac{\alpha-\gamma}{2}]\sin[\theta]\mathrm{d}b + \sin[\theta]\Bigg(\cos[a - \frac{\alpha-\gamma}{2}]\sin[b]\sin[\frac{\beta}{2}] - \\
& - \cos[b]\cos[\frac{\beta}{2}]\cos[\theta]\cos[\frac{\alpha+\gamma}{2}]\Bigg)\mathrm{d}c - \\
& - \sqrt{3}\cos[\frac{\beta}{2}]\cos[\frac{\alpha+\gamma}{2}]\sin[2\theta]\mathrm{d}\phi,
\end{aligned}
$$

$$
\begin{aligned}
\omega_R^5 = {} & \cos[\frac{\beta}{2}]\cos[\frac{\alpha+\gamma}{2}]\mathrm{d}\theta + \frac{1}{2}\cos[\frac{\beta}{2}]\sin[\frac{\alpha+\gamma}{2}]\sin[2\theta]\mathrm{d}a + \\
& + \cos[a - \frac{\alpha-\gamma}{2}]\sin[\frac{\beta}{2}]\sin[\theta]\mathrm{d}b + \sin[\theta]\Bigg(\sin[b]\sin[\frac{\beta}{2}]\sin[a - \frac{\alpha-\gamma}{2}] + \\
& + \cos[b]\cos[\frac{\beta}{2}]\cos[\theta]\sin[\frac{\alpha+\gamma}{2}]\Bigg)\mathrm{d}c + \sqrt{3}\cos[\frac{\beta}{2}]\sin[\frac{\alpha+\gamma}{2}]\sin[2\theta]\mathrm{d}\phi,
\end{aligned}
$$

$$
\begin{aligned}
\omega_R^6 = {} & 2\,\sin[\frac{\beta}{2}]\sin[\frac{\alpha-\gamma}{2}]\mathrm{d}\theta + \frac{1}{2}\cos[\frac{\alpha-\gamma}{2}]\sin[\frac{\beta}{2}]\sin[2\theta]\mathrm{d}a - \\
& - \cos[\frac{\beta}{2}]\sin[a + \frac{\alpha+\gamma}{2}]\sin[\theta]\mathrm{d}b + \sin[\theta]\Bigg(\cos[\frac{\beta}{2}]\cos[a + \frac{\alpha+\gamma}{2}]\sin[b] + \\
& + \cos[b]\cos[\theta]\cos[\frac{\alpha-\gamma}{2}]\sin[\frac{\beta}{2}]\Bigg)\mathrm{d}c + \sqrt{3}\cos[\frac{\alpha-\gamma}{2}]\sin[\frac{\beta}{2}]\sin[2\theta]\mathrm{d}\phi\,,
\end{aligned}
$$

$$\omega_R^7 = -2\,\cos[\frac{\alpha-\gamma}{2}]\sin[\frac{\beta}{2}]\mathrm{d}\theta + \frac{1}{2}\sin[\frac{\beta}{2}]\sin[\frac{\alpha-\gamma}{2}]\sin[2\theta]\mathrm{d}a +$$
$$+\cos[\frac{\beta}{2}]\cos[a+\frac{\alpha+\gamma}{2}]\sin[\theta]\mathrm{d}b + \sin[\theta]\Big(\cos[\frac{\beta}{2}]\sin[b]\sin[a+\frac{\alpha+\gamma}{2}]+$$
$$+\cos[b]\cos[\theta]\sin[\frac{\beta}{2}]\sin[\frac{\alpha-\gamma}{2}]\Big)\mathrm{d}c + \sqrt{3}\sin[\frac{\beta}{2}]\sin[\frac{\alpha-\gamma}{2}]\sin[2\theta]\mathrm{d}\phi\,,$$

$$\omega_R^8 = -\frac{\sqrt{3}}{2}\sin^2[\theta]\mathrm{d}a - \frac{\sqrt{3}}{2}\cos[b]\sin^2[\theta]\mathrm{d}c + (2-3\sin^2[\theta])\mathrm{d}\phi.$$

3. The Basis of the Invariant Vector Fields on the $SU(3)$ Group. The expressions for the left-invariant vector fields basis in the Euler angles coordinate frame are given below

a) The left-invariant vector fields

$$X_1^L = \frac{\cos[c]}{\sin[b]}\frac{\partial}{\partial a} = \sin[c]\frac{\partial}{\partial b} - \cot[b]\cos[c]\frac{\partial}{\partial c}\,,$$

$$X_2^L = \frac{\sin[c]}{\sin[b]}\frac{\partial}{\partial a} + \cos[c]\frac{\partial}{\partial b} - \cot[b]\sin[c]\frac{\partial}{\partial c}\,,$$

$$X_3^L = \frac{\partial}{\partial c}\,,$$

$$X_4^L = -\frac{\sin[\frac{b}{2}]}{\sin[\beta]\sin[\theta]}\cos\big[\frac{a-c}{2}+\gamma-\sqrt{3}\phi\big]\frac{\partial}{\partial\alpha} + \frac{\sin[\frac{b}{2}]}{\sin[\theta]}\sin\big[\frac{a-c}{2}+\gamma-\sqrt{3}\phi\big]\frac{\partial}{\partial\beta} +$$
$$+\left(\frac{\sin[\frac{b}{2}]}{\sin[\theta]}\cot[\beta]\cos\big[\frac{a-c}{2}+\gamma-\sqrt{3}\phi\big] + \frac{2\cos[\frac{b}{2}]}{\sin[2\theta]}\cos\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\right)\frac{\partial}{\partial\gamma} -$$
$$-\frac{1}{2}\cos[\frac{b}{2}]\sin\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\frac{\partial}{\partial\theta} - \frac{1}{2}\left(\frac{\cot[\theta]}{\cos[\frac{b}{2}]} + \cos[\frac{b}{2}]\tan[\theta]\right)\times$$
$$\times\cos\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\frac{\partial}{\partial a} + \cot[\theta]\sin[\frac{b}{2}]\sin\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\frac{\partial}{\partial b} - \frac{\cot[\theta]}{2\cos[\frac{b}{2}]}\times$$
$$\times\cos\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\tfrac{\partial}{\partial c} + +\frac{\sqrt{3}}{4}\cos[\frac{b}{2}]\cos\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\tan[\theta]\frac{\partial}{\partial\phi},$$

$$X_5^L=\frac{\sin[\frac{b}{2}]}{\sin[\beta]\sin[\theta]}\sin\big[\frac{a-c}{2}+\gamma-\sqrt{3}\phi\big]\frac{\partial}{\partial\alpha}+\frac{\sin[\frac{b}{2}]}{\sin[\theta]}\cos\big[\frac{a-c}{2}+\gamma-\sqrt{3}\phi\big]\frac{\partial}{\partial\beta}=$$

$$=\left(\frac{\sin[\frac{b}{2}]}{\sin[\theta]}\cot[\beta]\sin\big[\frac{a-c}{2}+\gamma-\sqrt{3}\phi\big]-\frac{2\cos[\frac{b}{2}]}{\sin[2\theta]}\sin\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\right)\frac{\partial}{\partial\gamma}+$$

$$+\frac{1}{2}\cos[\frac{b}{2}]\cos\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\frac{\partial}{\partial\theta}-\frac{1}{2}\left(\frac{\cot[\theta]}{\cos[\frac{b}{2}]}+\cos[\frac{b}{2}]\tan[\theta]\right)\times$$

$$\times\sin\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\frac{\partial}{\partial a}-\cos\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\cot[\theta]\sin[\frac{b}{2}]\frac{\partial}{\partial b}-\frac{\cot[\theta]}{2\cos[\frac{b}{2}]}\times$$

$$\times\sin\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\tfrac{\partial}{\partial c}+\frac{\sqrt{3}}{4}\cos[\frac{b}{2}]\sin\big[\frac{a+c}{2}+\sqrt{3}\phi\big]\tan[\theta]\frac{\partial}{\partial\phi},$$

$$X_6^L=\frac{\cos[\frac{b}{2}]}{\sin[\beta]\sin[\theta]}\cos\big[\frac{a+c}{2}+\gamma-\sqrt{3}\phi\big]\frac{\partial}{\partial\alpha}-\frac{\cos[\frac{b}{2}]}{\sin[\theta]}\sin\big[\frac{a+c}{2}+\gamma-\sqrt{3}\phi\big]\frac{\partial}{\partial\beta}-$$

$$-\left(\frac{\cos[\frac{b}{2}]}{\sin[\theta]}\cot[\beta]\cos\big[\frac{a+c}{2}+\gamma-\sqrt{3}\phi\big]-\frac{2\sin[\frac{b}{2}]}{\sin[2\theta]}\cos\big[\frac{a-c}{2}+\sqrt{3}\phi\big]\right)\frac{\partial}{\partial\gamma}-$$

$$-\frac{1}{2}\sin[\frac{b}{2}]\sin\big[\frac{a-c}{2}+\sqrt{3}\phi\big]\frac{\partial}{\partial\theta}-\frac{1}{2}\left(\frac{\cot[\theta]}{\sin[\frac{b}{2}]}+\sin[\frac{b}{2}]\tan[\theta]\right)\times$$

$$\times\cos\big[\frac{a-c}{2}+\sqrt{3}\phi\big]\frac{\partial}{\partial a}-\cos[\frac{b}{2}]\cot[\theta]\sin\big[\frac{a-c}{2}+\sqrt{3}\phi\big]\frac{\partial}{\partial b}+\frac{\cot[\theta]}{2\sin[\frac{b}{2}]}\times$$

$$\times\cos\big[\frac{a-c}{2}+\sqrt{3}\phi\big]\tfrac{\partial}{\partial c}+\frac{\sqrt{3}}{4}\cos\big[\frac{a-c}{2}+\sqrt{3}\phi\big]\sin[\frac{b}{2}]\tan[\theta]\frac{\partial}{\partial\phi},$$

$$X_7^L = -\frac{\cos[\frac{b}{2}]}{\sin[\beta]\sin[\theta]}\sin\left[\frac{a+c}{2}+\gamma-\sqrt{3}\phi\right]\frac{\partial}{\partial\alpha} - \frac{\cos[\frac{b}{2}]}{\sin[\theta]}\cos\left[\frac{a+c}{2}+\gamma-\sqrt{3}\phi\right]\frac{\partial}{\partial\beta} +$$

$$+\left(\frac{\cos[\frac{b}{2}]}{\sin[\theta]}\cot[\beta]\sin\left[\frac{a+c}{2}+\gamma-\sqrt{3}\phi\right] + \frac{2\sin[\frac{b}{2}]}{\sin[2\theta]}\sin\left[\frac{a-c}{2}+\sqrt{3}\phi\right]\right)\frac{\partial}{\partial\gamma} +$$

$$+\frac{1}{2}\cos\left[\frac{a-c}{2}+\sqrt{3}\phi\right]\sin[\frac{b}{2}]\frac{\partial}{\partial\theta} - \frac{1}{2}\left(\frac{\cot[\theta]}{\sin[\frac{b}{2}]} + \sin[\frac{b}{2}]\tan[\theta]\right)\times$$

$$\times\sin\left[\frac{a-c}{2}+\sqrt{3}\phi\right]\frac{\partial}{\partial a} + \cos[\frac{b}{2}]\cos\left[\frac{a-c}{2}+\sqrt{3}\phi\right]\cot[\theta]\frac{\partial}{\partial b} + \frac{\cot[\theta]}{2\sin[\frac{b}{2}]}\times$$

$$\times\sin\left[\frac{a-c}{2}+\sqrt{3}\phi\right]\frac{\partial}{\partial c} + \frac{\sqrt{3}}{4}\sin[b]\sin\left[\frac{a-c}{2}+\sqrt{3}\phi\right]\tan[\theta]\frac{\partial}{\partial\phi},$$

$$X_8^L = \frac{1}{2}\frac{\partial}{\partial\phi}\,.$$

b) The right-invariant vector fields

$$X_1^R = \cos[\alpha]\cot[\beta]\frac{\partial}{\partial\alpha} + \sin[\alpha]\frac{\partial}{\partial\beta} - \frac{\cos[\alpha]}{\sin[\beta]}\frac{\partial}{\partial\gamma}\,,$$

$$X_2^R = -\sin[\alpha]\cot[\beta]\frac{\partial}{\partial\alpha} + \cos[\alpha]\frac{\partial}{\partial\beta} + \frac{\sin[\alpha]}{\sin[\beta]}\frac{\partial}{\partial\gamma}\,,$$

$$X_3^R = \frac{\partial}{\partial\alpha}\,,$$

$$X_4^R = \frac{\cot[\theta]}{2\cos[\frac{\beta}{2}]}\cos[\frac{\alpha+\gamma}{2}]\frac{\partial}{\partial\alpha} - \cot[\theta]\sin[\frac{\beta}{2}]\sin[\frac{\alpha+\gamma}{2}]\frac{\partial}{\partial\beta} +$$

$$+\cos[\frac{\alpha+\gamma}{2}]\left(\frac{\cot[\theta]}{2\cos[\frac{\beta}{2}]} - \cos[\frac{\beta}{2}]\tan[\theta]\right)\frac{\partial}{\partial\gamma} + \frac{1}{2}\cos[\frac{\beta}{2}]\sin[\frac{\alpha+\gamma}{2}]\frac{\partial}{\partial\theta} -$$

$$-\left(\frac{\cot[b]}{\sin[\theta]}\cos[a - \frac{\alpha-\gamma}{2}]\sin[\frac{\beta}{2}] + \frac{\cos[\frac{\beta}{2}]}{\sin[2\theta]}\cos[\frac{\alpha+\gamma}{2}](2 - 3\sin^2[\theta])\right)\frac{\partial}{\partial a} -$$

$$-\frac{\sin[\frac{\beta}{2}]}{\sin[\theta]}\sin[a - \frac{\alpha-\gamma}{2}]\frac{\partial}{\partial b} + \frac{\sin[\frac{\beta}{2}]}{\sin[b]\sin[\theta]}\cos[a - \frac{\alpha-\gamma}{2}]\frac{\partial}{\partial c} -$$

$$-\frac{\sqrt{3}}{4}\cos[\frac{\beta}{2}]\cos[\frac{\alpha+\gamma}{2}]\tan[\theta]\frac{\partial}{\partial\phi},$$

$$X_5^R = -\frac{\cot[\theta]}{2\cos[\frac{\beta}{2}]}\sin[\frac{\alpha+\gamma}{2}]\frac{\partial}{\partial\alpha} - \cos[\frac{\alpha+\gamma}{2}]\cot[\theta]\sin[\frac{\beta}{2}]\frac{\partial}{\partial\beta} -$$

$$-\sin[\frac{\alpha+\gamma}{2}]\left(\frac{\cot[\theta]}{2\cos[\frac{\beta}{2}]} - \cos[\frac{\beta}{2}]\tan[\theta]\right)\frac{\partial}{\partial\gamma} + \frac{1}{2}\cos[\frac{\beta}{2}]\cos[\frac{\alpha+\gamma}{2}]\frac{\partial}{\partial\theta} -$$

$$-\left(\frac{\cot[b]}{\sin[\theta]}\sin[a - \frac{\alpha-\gamma}{2}]\sin[\frac{\beta}{2}] - \frac{\cos[\frac{\beta}{2}]}{\sin[2\theta]}\sin[\frac{\alpha+\gamma}{2}](2 - 3\sin^2[\theta])\right)\frac{\partial}{\partial a} +$$

$$+\frac{\sin[\frac{\beta}{2}]}{\sin[\theta]}\cos[a - \frac{\alpha+\gamma}{2}]\frac{\partial}{\partial b} + \frac{\sin[\frac{\beta}{2}]}{\sin[b]\sin[\theta]}\sin[a - \frac{\alpha-\gamma}{2}]\frac{\partial}{\partial c} +$$

$$+\frac{\sqrt{3}}{4}\cos[\frac{\beta}{2}]\sin[\frac{\alpha+\gamma}{2}]\tan[\theta]\frac{\partial}{\partial\phi},$$

$$X_6^R = \frac{\cot[\theta]}{2\sin[\frac{\beta}{2}]}\cos[\frac{\alpha-\gamma}{2}]\frac{\partial}{\partial\alpha} + \cos[\frac{\beta}{2}]\cot[\theta]\sin[\frac{\alpha-\gamma}{2}]\frac{\partial}{\partial\beta} -$$

$$-\cos[\frac{\alpha-\gamma}{2}]\left(\frac{\cot[\theta]}{2\sin[\frac{\beta}{2}]} - \sin[\frac{\beta}{2}]\tan[\theta]\right)\frac{\partial}{\partial\gamma} + \frac{1}{2}\sin[\frac{\beta}{2}]\sin[\frac{\alpha-\gamma}{2}]\frac{\partial}{\partial\theta} -$$

$$-\left(\frac{\cot[b]}{\sin[\theta]}\cos[a+\frac{\alpha+\gamma}{2}]\cos[\frac{\beta}{2}] - \frac{\sin[\frac{\beta}{2}]}{\sin[2\theta]}\cos[\frac{\alpha-\gamma}{2}](2-3\sin^2[\theta])\right)\frac{\partial}{\partial a} -$$

$$-\frac{\cos[\frac{\beta}{2}]}{\sin[\theta]}\sin[a+\frac{\alpha+\gamma}{2}]\frac{\partial}{\partial b} + \frac{\cos[\frac{\beta}{2}]}{\sin[b]\sin[\theta]}\cos[a+\frac{\alpha+\gamma}{2}]\frac{\partial}{\partial c} +$$

$$+\frac{\sqrt{3}}{4}\cos[\frac{\alpha-\gamma}{2}]\sin[\frac{\beta}{2}]\tan[\theta]\frac{\partial}{\partial\phi},$$

$$X_7^R = \frac{\cot[\theta]}{2\sin[\frac{\beta}{2}]}\sin[\frac{\alpha-\gamma}{2}]\frac{\partial}{\partial\alpha} - \cos[\frac{\beta}{2}]\cos[\frac{\alpha-\gamma}{2}]\cot[\theta]\frac{\partial}{\partial\beta} -$$

$$-\sin[\frac{\alpha-\gamma}{2}]\left(\frac{\cot[\theta]}{2\sin[\frac{\beta}{2}]} - \sin[\frac{\beta}{2}]\tan[\theta]\right)\frac{\partial}{\partial\gamma} - \frac{1}{2}\cos[\frac{\alpha-\gamma}{2}]\sin[\frac{\beta}{2}]\frac{\partial}{\partial\theta} -$$

$$-\left(\frac{\cot[b]}{\sin[\theta]}\cos[\frac{\beta}{2}]\sin[a+\frac{\alpha+\gamma}{2}] - \frac{\sin[\frac{\beta}{2}]}{\sin[2\theta]}\sin[\frac{\alpha-\gamma}{2}](2-3\sin^2[\theta])\right)\frac{\partial}{\partial a} +$$

$$+\frac{\cos[\frac{\beta}{2}]}{\sin[\theta]}\cos[a+\frac{\alpha+\gamma}{2}]\frac{\partial}{\partial b} + \frac{\cos[\frac{\beta}{2}]}{\sin[b]\sin[\theta]}\sin[a+\frac{\alpha+\gamma}{2}]\frac{\partial}{\partial c} +$$

$$+\frac{\sqrt{3}}{4}\sin[\frac{\beta}{2}]\sin[\frac{\alpha-\gamma}{2}]\tan[\theta]\frac{\partial}{\partial\phi},$$

$$X_8^R = \sqrt{3}\frac{\partial}{\partial\gamma} - \sqrt{3}\frac{\partial}{\partial a} + \frac{1}{2}\frac{\partial}{\partial\phi}.$$

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