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ON THE EXISTENCE OF POSITIVE RADIAL SOLUTIONS FOR SYSTEMS OF SUPERLINEAR SECOND-ORDER ELLIPTIC EQUATIONS

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О существовании положительных радиальных решений для систем суперлинейных эллиптических уравнений второго порядка

Рассматривается система

$$-\Delta u + cu = g(u,v) + u^p,\ u = u(x),\ x \in B \subset \mathbb{R}^N,\ u\big|_{\partial B} = 0,$$

$$-\Delta v + dv = h(u,v) + v^q, \quad v = v(x),\ v\big|_{\partial B} = 0,$$

где $c, d \geqslant 0$, B-шар и $1 < p, q < p^\star$ с $p^\star = (N+2)/(N-2)$, если $N \geqslant 3$, и с $p^\star = +\infty$, если $N = 1, 2$. Помимо прочего, предполагается, что $g(0,v) = h(u,0) = g'_u(0,v) = h'_v(u,0) = 0$ и что g и h — неубывающие функции каждого из своих аргументов, подчиненные определенным условиям роста на бесконечности. Доказано существование радиального решения (u,v), удовлетворяющего условию $u, v > 0$ в B.

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On the Existence of Positive Radial Solutions for Systems of Superlinear Second-Order Elliptic Equations

The system under consideration is

$$-\Delta u + cu = g(u,v) + u^p, \quad u = u(x),\ x \in B \subset \mathbb{R}^N,\ u\big|_{\partial B} = 0,$$

$$-\Delta v + dv = h(u,v) + v^q, \quad v = v(x),\ v\big|_{\partial B} = 0,$$

where $c, d \geqslant 0$, B is a ball and $1 < p, q < p^\star$ with $p^\star = (N+2)/(N-2)$ if $N \geqslant 3$ and $p^\star = +\infty$ if $N = 1, 2$. Among others, it is assumed that $g(0,v) = h(u,0) = g'_u(0,v) = h'_v(u,0) = 0$ and that g and h are nondecreasing functions in each of their arguments obeying certain growth conditions at infinity. We prove the existence of a radial solution (u,v) satisfying $u, v > 0$ in B.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

We consider the problem

$$-\Delta u + cu = u^{a_1} v^{b_1} + u^p, \quad u = u(x),$$

$$-\Delta v + dv = u^{a_2} v^{b_2} + v^q, \quad v = v(x),\ x \in B \subset \mathbb{R}^N,$$

$$u\big|_{\partial B} = v\big|_{\partial B} = 0$$

and its generalizations. Here N is a positive integer, $B = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : \ |x| < 1\}$, $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \ldots + \dfrac{\partial^2}{\partial x_N^2}$ is the Laplace operator, $c, d \geqslant 0$ are constants in B, $a_i + b_i \leqslant \overline{p} = \min\{p; q\}$, $a_1, b_2 > 1$, $a_2, b_1 \geqslant 0$ and $1 < p, q < \dfrac{N+2}{N-2}$ for $N \geqslant 3$ (to be interpreted as $1 < p, q < \infty$ if $N = 1, 2$). For this problem, we prove the existence of a solution (u, v) radial in B (that is, depending only on $r = |x|$) and satisfying $u(r),\ v(r) > 0,\ u'(r) \leqslant 0$ and $v'(r) \leqslant 0$ for $r \in (0, 1)$.

The problem above naturally arises as one of the simplest systems of superlinear second-order elliptic equations. Studies of the scalar equations of this type have a long history and the basic results in this direction are well known; we mention the result by P. H. Rabinowitz [4] according to which a scalar equation similar to the system above has a positive solution and, if the nonlinearity is odd, an infinite sequence of pairwise different solutions. We also indicate monograph [5] where some results on the existence of solutions for scalar equations in $\mathbb{R}^N$ are reviewed. As for systems of similar equations, to our knowledge, in the literature there are only certain results for variational problems, i. e., for systems for which there exist corresponding functionals whose critical points are solutions of these systems. On the other hand, the system we consider and similar ones seem to be not variational and at the same time, they have a lot of applications, for example, in diffusion theory, physical and chemical kinetics, etc. Because the problem seems to be difficult, we analyze one of the simplest cases.

2. PRECISE STATEMENT OF THE PROBLEM. RESULT

In fact, we consider the problem

$$-\Delta u + cu = g(u, v) + u^p,\ u = u(x), \tag{1}$$

$$-\Delta v + dv = h(u,v) + v^q,\ v = v(x),\ x \in B \subset \mathbb{R}^N, \tag{2}$$

$$u\big|_{\partial B} = v\big|_{\partial B} = 0. \tag{3}$$

Here $N \geqslant 1$ is integer and $B = \{x \in \mathbb{R}^N :\ |x| < 1\}$. Hereafter, all quantities we consider are real. We consider classical $C^2(B) \cap C(\overline{B})$-solutions of (1)–(3). Our assumptions are the following:
(h1) c, d *are nonnegative constants in* B*;*
(h2) $1 < p, q < \dfrac{N+2}{N-2}$ *for* $N \geqslant 3$ *and* $1 < p, q < \infty$ *if* $N = 1, 2$*;*
(h3) the functions g, h *are locally Hölger continuous in* $[0,\infty) \times [0,\infty)$*;*
(h4) there exists $C > 0$ *such that* $|g(u,v)| \leqslant C(|u|^{\overline{p}} + |v|^{\overline{p}})$ *and* $|h(u,v)| \leqslant C(|u|^{\overline{p}} + |v|^{\overline{p}})$ *for all* $u, v \geqslant 1$*;*
(h5) the functions g *and* h *are nondecreasing in* $[0,\infty) \times [0,\infty)$ *in each of two arguments;*
(h6) for any $A > 0$ *there exists a function* $\gamma(s) \to +0$ *as* $s \to +0$ *such that* $g(u,v) \leqslant \gamma(u)u$ *for all* $v \in [0, A]$ *and all sufficiently small* $u > 0$ *and* $h(u,v) \leqslant \gamma(v)v$ *for all* $u \in [0, A]$ *and for all sufficiently small* $v > 0$.
Denote $r = |x|$. Our main result is the following.

Theorem. *Under assumptions (h1)–(h6) there exists a radial solution* $(u(r), v(r))$ *of problem (1)–(3) that satisfies* $u(r) > 0,\ v(r) > 0,\ u'(r) \leqslant 0$ *and* $v'(r) \leqslant 0$ *for all* $r \in [0, 1)$.

This result admits a natural generalization for systems of n equations, $n \geqslant 3$. Our method of its proving is mainly based on two points. First, we obtain a priori estimates in $C(B)$ for positive radial nonincreasing in r solutions of (1)–(3). For this aim, we apply (and partly modernize) the approach by D. G. de Figueiredo, P. L. Lions and R. D. Nussbaum [2]. In particular, we establish a simple derivation of the Pohozaev identity for our system for which we proceed in the way well known in physics. Then, to prove the existence of a solution, we apply an abstract result presented in [1] and based on the concept of the index of a compact operator in a Banach space.

Everywhere in the following it is accepted that assumptions (h1)–(h6) are valid. We also continue the functions g and h by 0 for negative values of their arguments.

3. A PRIORI ESTIMATES OF SOLUTIONS

In this and the next sections, we assume in addition to hypotheses (h1)–(h6) that the functions g and h are continuously differentiable. In the class of radial solutions, problem (1)–(3) reduces to the following:

$$-u'' - \frac{N-1}{r}u' + cu = g(u,v) + u^p, \quad u = u(r), \tag{4}$$

$$-v'' - \frac{N-1}{r}v' + dv = h(u,v) + v^q, \quad v = v(r),\ r \in (0,1), \tag{5}$$

$$u'(0) = v'(0) = u(1) = v(1) = 0, \tag{6}$$

where the prime denotes the derivative in r. Denote by $L_s(B)$, $s \geqslant 1$, the standard Lebesgue space with the norm $|w|_s = \left\{\int\limits_B |w(x)|^s dx\right\}^{1/s}$ and by $H_0^1(B)$ the usual Sobolev space of functions in B equal to zero a. e. on the boundary ∂B, equipped with the norm $||w|| = \left\{\int\limits_B |\nabla w(x)|^2 dx\right\}^{1/2}$; for radial $u \in L_s(B)$ and $v \in H_0^1(B)$ one has respectively: $|u|_s^s = \int\limits_0^1 r^{N-1}|u(r)|^s dr$ and $||v||^2 = \int\limits_0^1 r^{N-1}|v'(r)|^2 dr$, where we omit a positive coefficient C_N depending only on N. Denote by H_r^1 the subspace of $H_0^1(B)$ consisting of radial functions. We identify this space with the space of functions $u(r)$ of $r \in (0,1)$ equal to 0 at $r = 1$ with the same norm. Introduce two quantities

$$I(u,v) = \int\limits_0^1 r^{N-1}\left\{\frac{1}{2}u'^2(r) + \frac{c}{2}u^2(r) - G(u,v) - \frac{1}{p+1}u^{p+1}\right\} dr$$

and

$$J(u,v) = \int\limits_0^1 r^{N-1}\left\{\frac{1}{2}v'^2(r) + \frac{d}{2}v^2(r) - H(u,v) - \frac{1}{q+1}v^{q+1}\right\} dr,$$

where $G(u,v) = \int\limits_0^u g(s,v)ds$ and $H(u,v) = \int\limits_0^v h(u,s)ds$. As one can easily verify (and as it is well known, see, for example, [5]), for any fixed radial solution $(u_0, v_0) \in C^2(B) \cap C(\overline{B})$ of problem (1)–(3) the functional $I_1(u) = I(u, v_0(r))$ taken with the fixed $v_0(r)$ is continuously differentiable in H_r^1 and $u_0(r)$ is its critical point in this space; by analogy, the functional $J_1(v) = J(u_0(r), v)$ taken with the fixed $u_0(r)$ is continuously differentiable in H_r^1 and v_0 is its critical point in this space.

Now, take two functions $\alpha, \beta \in C^3([0,1])$ satisfying $\alpha'(0) = \beta'(0) = \alpha(1) = \beta(1) = 0$ and consider the following problem:

$$-(r^{N-1}(u' - \alpha'))' + cr^{N-1}(u - \alpha) = r^{N-1}[g(u,v) + u^p], \tag{7}$$

$$-(r^{N-1}(v'-\beta'))' + dr^{N-1}(v-\beta) = r^{N-1}[h(u,v)+v^q], \tag{8}$$

$$u'(0) = v'(0) = u(1) = v(1) = 0. \tag{9}$$

As above, the corresponding functionals such that solutions of (7)–(9) are their critical points are the following:

$$I_\alpha(u,v) = \int_0^1 r^{N-1}\left\{\frac{1}{2}(u'-\alpha')^2 + \frac{c}{2}(u-\alpha)^2 - G(u,v) - \frac{1}{p+1}u^{p+1}\right\} dr$$

and

$$I_\beta(u,v) = \int_0^1 r^{N-1}\left\{\frac{1}{2}(v'-\beta')^2 + \frac{c}{2}(v-\beta)^2 - H(u,v) - \frac{1}{q+1}v^{q+1}\right\} dr.$$

The statement below is a variant and an extension of several results. The first one was obtained by S. I. Pohozaev [3]. Here, we apply another method to derive it; in our opinion, our method is simpler.

Lemma 1. *Let $v \in C^2([0,1)) \cap C([0,1])$ and α (resp., $u \in C^2([0,1)) \cap C([0,1])$ and β) be fixed, let $u \in C^2([0,1)) \cap C([0,1])$ (resp. $v \in C^2([0,1)) \cap C([0,1])$) be a critical point of $I_2(u) = I_\alpha(u,v)$ (resp., of $J_2(v) = J_\beta(u,v)$) in H_r^1 and $u'(r) \leqslant 0$ in $[0,1]$ (resp., $v'(r) \leqslant 0$ in $[0,1]$). Then, the following relations hold:*

$$\int_0^1 \left\{r^{N-1}u'^2 + cr^{N-1}u^2 - cr^{N-1}\alpha u + u(r^{N-1}\alpha')'\right\} dr =$$
$$= \int_0^1 r^{N-1}\left\{ug(u,v) + u^{p+1}\right\} dr \tag{10}$$

(respectively

$$\int_0^1 \left\{r^{N-1}v'^2 + dr^{N-1}v^2 - dr^{N-1}\beta v + v(r^{N-1}\beta')'\right\} dr =$$
$$= \int_0^1 r^{N-1}\left\{vh(u,v) + v^{q+1}\right\} dr)$$

and

$$\int_0^1 \left\{ \frac{2-N}{2} r^{N-1} u'^2 + (1-N) u (r^{N-1}\alpha')' - \right.$$

$$- \frac{N}{2} r^{N-1} \alpha'^2 - u (r^N \alpha'')' + r^N \alpha' \alpha'' - \frac{cN}{2} r^{N-1} u^2 -$$

$$\left. - \frac{cN}{2} r^{N-1} \alpha^2 + cN r^{N-1} \alpha u + c r^N u \alpha' - c r^N \alpha \alpha' \right\} dr =$$

$$= \int_0^1 r^{N-1} \left\{ -\frac{N}{p+1} u^{p+1} + r g(u,v) u' \right\} dr \quad (11)$$

(respectively

$$\int_0^1 \left\{ \frac{2-N}{2} r^{N-1} v'^2 + (1-N) v (r^{N-1}\beta')' - \right.$$

$$- \frac{N}{2} r^{N-1} \beta'^2 - v (r^N \beta'')' + r^N \beta' \beta'' - \frac{dN}{2} r^{N-1} v^2 -$$

$$\left. - \frac{dN}{2} r^{N-1} \beta^2 + dN r^{N-1} \beta v + d r^N v \beta' - d r^N \beta \beta' \right\} dr =$$

$$= \int_0^1 r^{N-1} \left\{ -\frac{N}{q+1} v^{q+1} + r h(u,v) v' \right\} dr).$$

Proof. We derive only the first equalities (10) and (11) because the second ones can be obtained by analogy. To obtain (10), multiply Eq. (7) by u and integrate the result from 0 to 1. To derive (11), we consider the parameter $a \in (0,1]$ and the function $w(a,r)$ equal to $u(ar)$ for $r \in [0,1]$ and to 0 for $r \in (1, a^{-1}]$. Then, the mapping $w(a,\cdot) : \ (0,1] \to H^1_r$ is continuously differentiable. Hence, since u is a critical point of $I_\alpha(u,v)$ with a fixed v, it must be $\left.\frac{\partial I_\alpha(w,v)}{\partial a}\right|_{a=1-0} = 0$. So, calculating the quantity in the left-hand side of this equality, after simple transformations with an application of the integration by parts for several times, we obtain (11).$\square$

Now fix functions $\alpha_0, \beta_0 \in C^3([0,1])$ positive in $[0,1)$ and satisfying $\alpha_0'(0) = \beta_0'(0) = \alpha_0(1) = \beta_0(1) = 0$. Denote by K the set of such positive nonincreasing solutions of (7)–(9) taken with $\alpha = \lambda\alpha_0$ and $\beta = \mu\beta_0$ that $\lambda, \mu \geqslant 0$ satisfy the inequalities $\lambda\alpha_0(r) \leqslant u(r)$ and $\mu\beta_0(r) \leqslant v(r)$ for all $r \in [0,1]$.

Lemma 2. *There exists $D_0 > 0$ such that for any functions g and h continuously differentiable and satisfying hypotheses (h1)–(h6) and for any $(u, v) \in K$ one has $|u|_{p+1} \leqslant D_0$ and $|v|_{q+1} \leqslant D_0$, where we mean that the norms of the functions $u = u(|x|)$ and $v = v(|x|)$ in the Lebesgue spaces $L_{p+1}(B)$ and $L_{q+1}(B)$ are taken.*

We establish the **Proof** only for $N \geqslant 3$ and estimate only $|u|_{p+1}$ because in all other cases the proof can be made by analogy. In view of the definition of λ, one has

$$0 \leqslant \lambda \int_0^1 r^{N-1}\alpha_0(r)dr \leqslant \int_0^1 r^{N-1}u(r)dr,$$

hence,

$$0 \leqslant \lambda \leqslant C|u|_2 \tag{12}$$

for a positive constant C because the function α_0 is fixed. Now, multiply (10) by $\dfrac{2-N}{2}$ and subtract the result from (11). Then, in view of (12) and since α_0 and β_0 are fixed, we obtain after simple transformations: $C_1|u|_2^2 \geqslant |u|_{p+1}^{p+1}$ where $C_1 > 0$ does not depend on $(u, v) \in K$. But by the Hölder's inequality $|u|_2^2 \leqslant C_2|u|_{p+1}^2$ and thus, $|u|_{p+1} \leqslant$ Constant.□

Remark. Note that the constant D_0 does not depend on the choice of the functions g and h obeying hypotheses (h1)–(h6).

Now, we establish the main result of this section.

Proposition 1. *Let $\{g_\lambda\}$ and $\{h_\lambda\}$, where $\lambda \in \Lambda$, be arbitrary families of continuously differentiable functions (Λ is an arbitrary set) that satisfy hypotheses (h1)–(h6) and let in addition these functions satisfy hypothesis (h4) with the same constant $C > 0$ for all λ. Then, there exists $D > 0$ such that for any $(u_\lambda, v_\lambda) \in K_\lambda$, where $K = K_\lambda$ is the above-defined set of solutions of problem (7)–(9) corresponding to $(g, h) = (g_\lambda, h_\lambda)$, one has $u_\lambda(0) = \max\limits_{r\in[0,1]} u_\lambda(r) \leqslant D$ and $v_\lambda(0) = \max\limits_{r\in[0,1]} v_\lambda(r) \leqslant D$.*

Proof. By Lemma 2 one has $|u_\lambda|_{p+1} \leqslant D_0$ and $|v_\lambda|_{q+1} \leqslant D_0$ for any $(u_\lambda, v_\lambda) \in K_\lambda$ and for a constant $D_0 > 0$ independent of λ and of $(u_\lambda, v_\lambda) \in K_\lambda$. By these estimates and hypothesis (h4)

$$\int_B u_\lambda(|x|)g_\lambda(u_\lambda(|x|), v_\lambda(|x|))dx \leqslant C_1$$

and

$$\int_B v_\lambda(|x|)h_\lambda(u_\lambda(|x|),v_\lambda(|x|))dx \leqslant C_1$$

for a constant $C_1 > 0$ independent of λ and of $(u_\lambda, v_\lambda) \in K_\lambda$. Hence, according to Lemma 2 and (10)

$$\|u_\lambda\| \leqslant C_2 \quad \text{and} \quad \|v_\lambda\| \leqslant C_2$$

for a constant $C_2 > 0$ independent of $\lambda \in \Lambda$ and of $(u_\lambda, v_\lambda) \in K_\lambda$. Now, it can be proved completely as in [5], section 2.2, by using well-known arguments that $|u_\lambda(0)| \leqslant D$ and $|v_\lambda(0)| \leqslant D$ for a constant $D > 0$ independent of λ and of $(u_\lambda, v_\lambda) \in K_\lambda$.$\square$

4. PROOF OF THE THEOREM. THE CASE OF SMOOTH g AND h

In this section, we assume in addition to hypotheses (h1)–(h6) that g and h are C^1 functions. Then, Proposition 1 holds. In the following, we apply results presented by H. Amann in [1], sections 11 and 12. Let $C_0([0,1])$ be the space of functions $s(r)$ continuous in $[0,1]$ and satisfying $s(1) = 0$, equipped with the uniform norm. Denote by X the set of functions $s(|x|)$, where $s \in C_0([0,1])$, and let R be the subset of X consisting of functions $s(|x|)$ such that the corresponding functions $s(r)$ are nonnegative and nonincreasing in $(0,1)$. Then, since R is a closed convex set in X, according to [1] R is a retract in X which means by definition that there exists a continuous function (retraction) $\theta : X \to R$ satisfying $\theta\big|_R = Id$ where Id denotes the identity. In addition, it is easily seen that $R \times R$ is a retract in $X \times X$ with one of the retractions $\theta \times \theta$ where θ is one of the retractions in X.

For $s, t \in X$, consider the operators $S_u(s) := (-\Delta + c)^{-1}s$, $S_v(t) := (-\Delta + d)^{-1}t$, $S = S_u \times S_v$, $(u,v) = T(s,t) := S((g(s,t) + s^p), (h(s,t) + t^q))$ and $(u_\lambda, v_\lambda) = T_\lambda(s,t) := S((\lambda g(s,t) + s^p), (\lambda h(s,t) + t^q))$, where $\lambda \in [0,1]$ is a parameter. Clearly, $T_0 = T_u \times T_v$ with $T_u(s) = (-\Delta + c)^{-1}(s^p)$ and $T_v(t) = (-\Delta + d)^{-1}(t^q)$. Denote also $B_\rho = \{u \in R : \|u\|_{C(B)} < \rho\}$ where $\rho > 0$. By Proposition 1, $T_\lambda(u,v) \neq (u,v)$, $T_u(u) \neq u$ and $T_v(v) \neq v$ for any $u, v \in \partial B_{2D}$ and $\lambda \in [0,1]$ where the boundary ∂B_{2D} is taken in the topological space R with the induced topology.

Now, consider three sets $A, B \subset R$ and $F \subset R \times R$ bounded respectively in X and in $X \times X$ and open in the induced topologies of the spaces R and $R \times R$, respectively. According to [1], sections 11 and 12, if $T_u(u) \neq u$ for any $u \in \partial A$, $v \neq T_v(v)$ for any $v \in \partial B$ and/or $T_\lambda(u,v) \neq (u,v)$ for any $(u,v) \in \partial F$, then the indexes of the operators T_u, T_v and T_λ are determined: $i(T_u, A, R)$, $i(T_v, B, R)$ and/or $i(T_\lambda, F, R \times R)$, respectively. In fact, indexes take integer values and by definition, for example, $i(T_u, A, R) = \deg(Id - T_u \circ \theta, \theta^{-1}(A), X)$, where θ is an arbitrary retraction in X corresponding to the retract R (the index does not depend on the retraction θ). In addition, $i(T_0, A \times B, R \times R) = i(T_u, A, R) \cdot i(T_v, B, R)$ provided $T_u(u) \neq u$ and $T_v(v) \neq v$ for any $u \in \partial A$ and $v \in \partial B$ and if $T_\lambda(u, v) \neq (u, v)$ for all $\lambda \in [0, 1]$ and for all $(u, v) \in \partial F$, then $i(T, F, R \times R) = i(T_0, F, R \times R)$.

Proposition 2. *For any $\lambda \in [0, 1]$, the operator T_λ transforms $R \times R$ into $R \times R$.*

Proof. We prove this statement for $\lambda = 1$ to make the notation simpler. Let $s, t \in R$ and $(u, v) = T(s, t)$. Since S_u and S_v are bounded as linear operators from $L_s(B)$ into the Sobolev space $W_s^2(B)$ for any $s \geqslant 1$ and due to the compactness of the embedding of $W_s^2(B)$ into $C^1(\overline{B})$ for sufficiently large s, one has $u, v \in C^1(\overline{B})$ and $(u(r), v(r))$ is a solution (maybe, a weak solution) of the problem

$$\begin{aligned} -(r^{N-1}u')' + cr^{N-1}u &= r^{N-1}[g(s(r), t(r)) + s^p(r)], \\ -(r^{N-1}v')' + dr^{N-1}v &= r^{N-1}[h(s(r), t(r)) + t^q(r)], \\ u'(0) = v'(0) &= u(1) = v(1) = 0. \end{aligned} \tag{13}$$

It is well known that $u, v \in C^2((0, 1])$. Therefore, by the maximum principle, $u(r) \geqslant 0$ and $v(r) \geqslant 0$ for all $r \in [0, 1]$. We have also to prove that $u'(r) \leqslant 0$ in $[0, 1)$. On the contrary, suppose that $u'(r_1) > 0$ at some $r_1 \in (0, 1)$. Denote by $[r_0, r_1] \subset [0, r_1]$ the maximal left half-interval such that $u'(r) \geqslant 0$ for all $r \in [r_0, r_1]$. Then, $u''(r_0) \geqslant 0$ and hence, in view of the differential equation $cu(r_0) \geqslant g(s(r_0), t(r_0)) + s^p(r_0)$. Therefore, it is clear that $cu(r_1) > g(s(r_1), t(r_1)) + s^p(r_1)$ and so, it is easy to see from the equation that $u'(r) \geqslant 0$ everywhere in $[r_1, 1]$ which contradicts the boundary condition $u(1) = 0$. So, our proposition is proved.□

Proposition 3. *For any $\lambda \in [0, 1]$ one has $i(T_\lambda, B_\rho \times B_\rho, R \times R) = i(T_u, B_\rho, R) = i(T_v, B_\rho, R) = 1$ if $\rho > 0$ is sufficiently small and $i(T_\lambda, B_{2D} \times B_{2D}, R \times R) = i(T_u, B_{2D}, R) = i(T_v, B_{2D}, R) = 0$.*

Proof. This result is, in fact, a variant of Lemma 12.1 in [1]. In view of Proposition 1, the second claim immediately follows from this result. Further, since according to hypothesis (h6) $[\lambda g(s, t) + s^p] = o(s)$ as $s \to +0$ uniformly in $t \in [0, 2D]$ and $[\lambda h(s, t) + t^q] = o(t)$ as $t \to +0$ uniformly in $s \in [0, 2D]$ and in $\lambda \in [0, 1]$, for any $\lambda \in [0, 1]$, sufficiently small $\rho > 0$ and $s, t \in B_\rho$ one has $\|u_\lambda\|_{C(B)} < \rho/3$ and $\|v_\lambda\|_{C(B)} < \rho/3$ where $(u_\lambda, v_\lambda) = T_\lambda(s, t)$. Hence, by Lemma 12.1 in [1], $i(T, B_\rho \times B_\rho, R \times R) = i(T_\lambda, B_\rho \times B_\rho, R \times R) = i(T_u, B_\rho, R) \cdot i(T_v, B_\rho, R) = 1$ for the same ρ and λ.□

In $R \times R$, consider the set

$$A = ((((B_{2D} \times B_{2D}) \setminus \overline{[(B_{2D} \setminus B_\rho) \times B_\rho]}) \setminus \overline{[B_\rho \times (B_{2D} \setminus B_\rho)]}) \setminus \overline{(B_\rho \times B_\rho)})$$

where $\rho > 0$ is sufficiently small. By Proposition 3, the arguments above and the results in [1], section 11, one has: $i(T, (B_{2D} \setminus B_\rho) \times B_\rho, R \times R) = i(T_0, (B_{2D} \setminus B_\rho) \times B_\rho, R \times R) = i(T_u, B_{2D} \setminus B_\rho, R) \cdot i(T_v, B_\rho, R) = (0-1) \times (+1) = -1$. By analogy, $i(T, B_\rho \times (B_{2D} \setminus B_\rho), R \times R) = -1$. So, we have: $i(T, A, R \times R) = 0 - (-1) - (-1) - (+1) = 1$. Therefore, the operator T has a fixed point $(u, v) \in A$. By construction $u \not\equiv 0$ and $v \not\equiv 0$, $u(r)$ and $v(r)$ are nonnegative and nonincreasing functions in $(0,1)$ and $(u(|x|), v(|x|))$ is a $C^2(B) \cap C(\overline{B})$ solution of problem (1)–(3). By standard arguments $u(r) > 0$ and $v(r) > 0$ in $[0,1)$ (because otherwise $u(r_0) = u'(r_0) = 0$ or $v(r_0) = v'(r_0) = 0$ at some point $r \in (0,1)$, but this relations imply $u(r) \equiv 0$ (resp., $v(r) \equiv 0$) in $(0,1)$ by the uniqueness theorem). Our theorem is proved in the case when g and h are continuously differentiable functions.□

5. PROOF OF THE THEOREM. THE CASE OF NON-SMOOTH g AND h

Take two sequences $\{g_n\}$ and $\{h_n\}$ of smooth functions g_n and h_n converging to g and h, respectively, uniformly in $[0,\infty) \times [0,\infty)$ and satisfying hypotheses (h1)–(h6) uniformly in n (that is, with the same function γ in (h6) and the constant $C > 0$ in (h4) for all n). Note that the constant $D > 0$ introduced in Proposition 1 can be chosen the same for all g_n and h_n. For each n, by (u_n, v_n) denote an arbitrary radial solution of (1)–(3) taken with $g = g_n$ and $h = h_n$ such that $u_n(r), v_n(r) \geqslant 0$ and $u_n'(r), v_n'(r) \leqslant 0$ in $(0,1)$. Then, we have $u_n(0) \leqslant D$ and $v_n(0) \leqslant D$, therefore, $\|u_n\|_{W_s^2(B)} \leqslant \overline{C}_s$ and $\|v_n\|_{W_s^2(B)} \leqslant \overline{C}_s$ for all $s > 1$ and for constants $\overline{C}_s > 0$ independent of n. Hence, the sequence (u_n, v_n) contains a subsequence still denoted (u_n, v_n) that converges in $C^1(\overline{B}) \times C^1(\overline{B})$. Denote by (u, v) its limit. Let us prove that $u \not\equiv 0$ and $v \not\equiv 0$. On the contrary, suppose that, for example, $u \equiv 0$ in $[0,1]$. Then, $u_n \to 0$ in $C^1(\overline{B})$. But we have from hypothesis (h6) and (10) by the Sobolev embedding:

$$C_1 |u_n|_{p+1}^2 \leqslant C_2 \gamma(u_n(0)) |u_n|_{p+1}^2 + |u_n|_{p+1}^{p+1}$$

with constants $C_1, C_2 > 0$ independent of n. This relation easily implies $|u_n|_{p+1} \geqslant C_3 > 0$ with a constant C_3 independent of sufficiently large n, and we get a contradiction. So, it is proved that $u \not\equiv 0$ and $v \not\equiv 0$. In addition, obviously u and v are radial functions, $u(r), v(r) \geqslant 0$ and $u'(r), v'(r) \leqslant 0$ in $(0,1)$ and $(u(|x|), v(|x|))$ is a fixed point of T in $R \times R$, hence, a $C^2(B) \cap C(\overline{B})$ solution of system (1)–(3). The fact that $u(|x|), v(|x|) > 0$ in B is well known and can be proved as in the previous section. So, our theorem is completely proved.□

REFERENCES

1. *Amann H.* Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces // SIAM Rev. 1976. V. 18. P. 620–709.

2. *de Fegueiredo D. G., Lions P. L., Nussbaum R. D.* A priori estimates and existence of positive solutions of semilinear elliptic equations // J. Math. Pures Appl. 1982. V. 61. P. 41–63.

3. *Pohozaev S. I.* Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$ // Sov. Math. Dokl. 1965. V. 6. P. 1408–1411.

4. *Rabinowitz P. H.* Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS 65, Providence: AMS Publ., 1986.

5. *Zhidkov P. E.* Korteweg–de Vries and Nonlinear Schrödinger Equations: Qualitative Theory. Lecture Notes in Mathematics. V. 1756. Heidelberg: Springer-Verlag, 2001.

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