E2-2006-185

D. J. Cirilo-Lombardo

ON THE MATHEMATICAL STRUCTURE AND HIDDEN SYMMETRIES OF THE BORN–INFELD FIELD EQUATIONS

Submitted to «Journal of Mathematical Physics»

Сирило-Ломбардо Д. Х. E2-2006-185
О математической структуре и скрытых симметриях полевых уравнений Борна–Инфельда

В данной работе математическая структура полевых уравнений Борна–Инфельда проанализирована с точки зрения симметрий. С этой целью полевые уравнения были записаны в сжатой форме при помощи кватернионных операторов, построенных согласно всем симметриям теории, включая расширение до некоммутативной структуры. Кватернионная структура получена в явном виде и описана с точки зрения гамильтониана, а также явно показано сходство теории Борна–Инфельда с (линейной) электродинамикой Максвелла. Наши результаты согласуются с наблюдениями Гиббонса и Рашида в том, что существует дискретная симметрия в структуре полевых уравнений, которая уникальна в случае нелинейной электродинамики Борна–Инфельда.

Работа выполнена в Лаборатории теоретической физики им. Н. Н. Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 2006

Cirilo-Lombardo D. J. E2-2006-185
On the Mathematical Structure and Hidden Symmetries of the Born–Infeld Field Equations

The mathematical structure of the Born–Infeld field equations was analyzed from the point of view of the symmetries. To this end, the field equations were written in the most compact form by means of quaternionic operators constructed according to all the symmetries of the theory, including the extension to a non-commutative structure. The quaternionic structure of the phase space was explicitly derived and described from the Hamiltonian point of view, and the analogy and similarities between the BI theory and the Maxwell (linear) electrodynamics in a curved space–time was explicitly shown. Our results agree with the observation of Gibbons and Rasheed that there exists a discrete symmetry in the structure of the field equations that is unique in the case of the Born–Infeld nonlinear electrodynamics.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 2006

INTRODUCTION

The most significative nonlinear theory of electrodynamics is, by excellence, the Born–Infeld (BI) theory. The Lagrangian density describing BI theory (in arbitrary space–time dimensions) is

$$\mathcal{L}_{BI} = \sqrt{-g} L_{BI} = \frac{b^2}{4\pi} \left\{ \sqrt{-g} - \sqrt{\left| \det(g_{\mu\nu} + b^{-1} F_{\mu\nu}) \right|} \right\}, \tag{1}$$

where b is a fundamental parameter of the theory with field dimensions*. In four space-time dimensions the determinant in (1) may be expanded out to give

$$L_{BI} = \frac{b^2}{4\pi} \left\{ 1 - \sqrt{1 + \frac{1}{2} b^{-2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16} b^{-4} \left(F_{\mu\nu} \widetilde{F}^{\mu\nu} \right)^2} \right\} \tag{2}$$

which coincides with the usual Maxwell Lagrangian in the weak field limit. Similarly, if we consider the second rank tensor $\mathbb{F}^{\mu\nu}$ defined by

$$\mathbb{F}^{\mu\nu} = -\frac{1}{2} \frac{\partial L_{BI}}{\partial F_{\mu\nu}} = \frac{F^{\mu\nu} - \frac{1}{4} b^{-2} \left(F_{\rho\sigma} \widetilde{F}^{\rho\sigma} \right) \widetilde{F}^{\mu\nu}}{\sqrt{1 + \frac{1}{2} b^{-2} F_{\rho\sigma} F^{\rho\sigma} - \frac{1}{16} b^{-4} \left(F_{\rho\sigma} \widetilde{F}^{\rho\sigma} \right)^2}} \tag{3}$$

(so that $P^{\mu\nu} \approx F^{\mu\nu}$ for weak fields) this second kind of antisymmetrical tensor satisfies the electromagnetic equations of motion

$$\nabla_\mu \mathbb{F}^{\mu\nu} = 0 \tag{4}$$

which are highly nonlinear in $F_{\mu\nu}$. Another interesting object to analyze is the energy-momentum tensor that can be written as

$$T_{\mu\nu} = \frac{1}{4\pi} \left\{ \frac{F_{\mu}{}^{\lambda} F_{\nu\lambda} + b^2 \left[\mathbb{R} - 1 - \frac{1}{2} b^{-2} F_{\rho\sigma} F^{\rho\sigma} \right] g_{\mu\nu}}{\mathbb{R}} \right\}, \tag{5}$$

$$\mathbb{R} \equiv \sqrt{1 + \frac{1}{2} b^{-2} F_{\rho\sigma} F^{\rho\sigma} - \frac{1}{16} b^{-4} \left(F_{\rho\sigma} \widetilde{F}^{\rho\sigma} \right)^2}.$$

---

*In open superstring theory (2 dimensions), for example, loop calculations lead to this Lagrangian with $b^{-1} = 2\pi\alpha'$ ($\alpha' \equiv$ inverse of the string tension).

But besides these more or less obvious statements which were observed by Born and Infeld in their original work [1] that the tensor $\mathbb{F}$ has to F a relation similar to that, which in Maxwell theory of macroscopical bodies is the dielectric displacement and magnetic induction, but in the BI case this relation is a discrete electric-magnetic duality invariance [5] that is associated to an underlying $SO(2)$ symmetry. In Ref. [1] the relations that put in evidence the symmetries of these transformations that are the characteristics of the BI field equations only, are

$$\mathbb{F}^{\mu\nu} = \frac{F^{\mu\nu} - G\widetilde{F}^{\mu\nu}}{\sqrt{1+S-G^2}}, \tag{6}$$

$$F^{\mu\nu} = \frac{\mathbb{F}^{\mu\nu} + Q\widetilde{\mathbb{F}}^{\mu\nu}}{\sqrt{1+P-Q^2}}, \quad Q \equiv G, \tag{7}$$

where $G,\ Q,\ S$ and P are the electromagnetic invariants constructed from the two types of fields F and $\mathbb{F}$, and which we will express explicitly in the next section. Although it is by no means obvious, it may be verified that equations (3), (6) and (7) are invariant under the electric-magnetic rotations of duality $F \longleftrightarrow \widetilde{\mathbb{F}}$, but notice that the BI Lagrangian (1) is not. This fact was pointed out first from the general publications on the electromagnetic duality rotations by Gaillard and Zumino [4] and more recently and specifically for the BI case, in the papers of Gibbons and Rasheed [5, 6].

The main task of this work is to complete in any sense the analysis given in Refs. [4, 5, 6] for the BI theory showing explicitly the quaternionic structure of the field equations. The starting point to complete such analysis is based on the previous paper of the author [7] where was explicitly shown that the transformations (6), (7) are produced by quaternionic operator acting over vectors in which the components are the corresponding electromagnetic fields:

$$\begin{pmatrix} \mathbb{F} \\ \widetilde{\mathbb{F}} \end{pmatrix}^{\mu\nu} = \overbrace{\frac{1}{\mathbb{R}}\left(\sigma_0 - i\sigma_2\mathbb{P}\right)}^{\equiv A} \begin{pmatrix} F \\ \widetilde{F} \end{pmatrix}^{\mu\nu}, \tag{8}$$

$$\begin{pmatrix} F \\ \widetilde{F} \end{pmatrix}^{\mu\nu} = \overbrace{\frac{\mathbb{R}}{1+\mathbb{P}^2}\left(\sigma_0 + i\sigma_2\mathbb{P}\right)}^{\equiv \overline{B}} \begin{pmatrix} \mathbb{F} \\ \widetilde{\mathbb{F}} \end{pmatrix}^{\mu\nu}, \tag{9}$$

where $\widetilde{\mathbb{F}}^{kl} \equiv \dfrac{\partial L_{\rm BI}}{\partial \widetilde{F}_{kl}}$; $\mathbb{R} \equiv \sqrt{1 + \frac{1}{2}b^{-2}F_{\rho\sigma}F^{\rho\sigma} - \frac{1}{16}b^{-4}\left(F_{\rho\sigma}\widetilde{F}^{\rho\sigma}\right)^2}$ and $\mathbb{P} = -\dfrac{1}{4b^2}F_{\mu\nu}\widetilde{F}^{\mu\nu}$ ($b \equiv$ absolute field of the BI theory) and the complex conjugation is indicated by the horizontal bar over the operators. The Pauli matrix is defined as (Landau–Lifshitz, 1968)

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \quad \sigma_0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

The norms of the operators A and B are

$$\overline{A}A = A\overline{A} = \frac{1+\mathbb{P}^2}{\mathbb{R}^2},$$

$$\overline{B}B = B\overline{B} = \frac{\mathbb{R}^2}{1+\mathbb{P}^2},$$

where from expressions (8), (9) we have

$$A\overline{B} = \overline{A}B = 1.$$

The plan of this paper is as follows In Sec. 1 the quaternionic structure of the BI field equations is manifestly presented and the mathematical structure is carefully analyzed and extended. In Sec. 2 we describe the phase space determined by the symmetries of the BI field equations from the Hamiltonian point of view. In Sec. 3 the constitutive-like relations of the BI theory are studied comparing them with the ordinary Maxwell electrodynamics in a Riemannian space with arbitrary metric and the Fresnel equation is explicitly given for the BI case. Finally, remarks and conclusions are given.

Our convention is as in Ref. [2] with signatures of the metric, Riemann and Einstein tensors (–++), the internal indexes (gauge group) are denoted by $a, b, c...$, space–time indexes by Greek letters $\mu, \nu, \rho...$ and the tetrad indexes by capital Latin letters $A, B, C...$

1. THE QUATERNIONIC STRUCTURE

Now we can see in an explicit and compact form how the transformations (6), (7) can be realized by means of a quaternionic structure. We will start with the following definitions for the invariants of the electromagnetic field $S \equiv \frac{1}{2b^2}F_{\rho\sigma}F^{\rho\sigma}$, $G = \frac{1}{2b^2}F_{\mu\nu}\widetilde{F}^{\mu\nu}$. $\mathbb{R} \equiv \sqrt{1+S-G^2}$ and the following signature for the metric tensor is adopted $g_{\mu\nu} = (---+)$. Starting from expressions (6), (7) with the new definitions for the invariants we have

$$\begin{pmatrix} \mathbb{F} \\ \\ \widetilde{\mathbb{F}} \end{pmatrix}^{\mu\nu} = \frac{1}{\mathbb{R}} \overbrace{(\sigma_0 - i\sigma_2 G)}^{\equiv \mathbb{Q}} \begin{pmatrix} F \\ \\ \widetilde{F} \end{pmatrix}^{\mu\nu}. \tag{10}$$

It is interesting to notice that, because the following identity holds $F_{\mu\nu}\widetilde{F}^{\mu\nu} = \mathbb{F}_{\mu\nu}\widetilde{\mathbb{F}}^{\mu\nu}$, the quaternion $\mathbb{Q}$ is *invariant* from the topological point of view. It is a very important property because the mapping between the different set of fields, F and $\mathbb{F}$, respectively, preserves the topological charge unaltered. This means that the topological charge is a fixed point of the $\mathbb{Q}$ transformation. Defining the «spinors»

$$\Psi = \begin{pmatrix} F \\ \\ \widetilde{F} \end{pmatrix} \quad \overline{\Psi} = (\sigma_3 \Psi)^{\dagger} \ ,$$

and, in such manner,

$$\Phi = \begin{pmatrix} \mathbb{F} \\ \\ \widetilde{\mathbb{F}} \end{pmatrix} \quad \overline{\Phi} = (\sigma_3 \Phi)^{\dagger} \ ,$$

the square root $\mathbb{R}$ in (10) is simplified to the following expression:

$$\sqrt{1+S-G^2} = \sqrt{1+\frac{1}{4}\left(\overline{\Psi}\mathbb{Q}\Psi\right)},$$

and relation (10) takes the compact form

$$\Phi = \frac{\mathbb{Q}\Psi}{\sqrt{1+\frac{1}{4}\left(\overline{\Psi}\mathbb{Q}\Psi\right)}}. \tag{11}$$

As we see in Introduction [1], in such manner, it is possible to invert the above equation and to put all as a function of the spinor Ψ. In order to do this, it is sufficient to consider: $P \equiv \frac{1}{2b^2}\widetilde{\mathbb{F}}_{\rho\sigma}\widetilde{\mathbb{F}}^{\rho\sigma}$, $Q = G = \frac{1}{2b^2}\mathbb{F}_{\mu\nu}\widetilde{\mathbb{F}}^{\mu\nu}$and the following property $F_{\rho\sigma}F^{\rho\sigma} = -\widetilde{F}_{\rho\sigma}\widetilde{F}^{\rho\sigma}$. The square root in this inverted transformation is ($\overline{\mathbb{Q}} \equiv (\sigma_0 + i\sigma_2 G)$)

$$\sqrt{1+P-Q^2} = \sqrt{1-\frac{1}{4}\left(\overline{\Phi}\mathbb{Q}\Phi\right)},$$

and the inverse transformation becomes

$$\Psi = \frac{\overline{\mathbb{Q}}\Phi}{\sqrt{1-\frac{1}{4}\left(\overline{\Phi}\overline{\mathbb{Q}}\Phi\right)}}. \tag{12}$$

We stop here to consider in more detail the mathematical structure of the operators $\mathbb{Q}$. From (10) we can see that the $\mathbb{Q}$ form a part of a commutative ring of complex operators $\mathbb{Q} \equiv \{\alpha + i\beta\mathbb{I} \ /\alpha, \beta \in \mathbb{C}\}$, equipped with addition and multiplication laws induced by those in $\mathbb{C}$, such as addition and multiplication on $\mathbb{Q}$ are given by the usual matrix addition and multiplication, with $\mathbb{I}$ having the following form*:

$$\mathbb{I} \ = \pm \begin{pmatrix} 0 & 1_{d/2} \\ -1_{d/2} & 0 \end{pmatrix}.$$

It is easily seen that $\mathbb{Q}$ is a commutative ring with zero divisors

$$\mathbb{Q}^0_{\pm} \equiv \left\{ \lambda \left(1_d \pm i \mathbb{I}_d \right), \lambda \in \mathbb{C} \right\},$$

$\mathbb{Q}^0_{-}, \mathbb{Q}^0_{+}$ are the only multiplicative ideals in $\mathbb{Q}$, for instance, they are maximal ideals. Thus, the only fields that we can construct from $\mathbb{Q}$ are

$$\frac{\mathbb{Q}}{\mathbb{Q}^0_{\pm}} \cong \mathbb{Q}^0_{\mp} \cong \mathbb{C}.$$

In the general case that $\alpha, \beta \in \mathbb{C}$, the map $|\cdot|^2 : \mathbb{Q} \to \mathbb{R} / \left|\mathbb{Q}\right|^2 \equiv \mathbb{Q}\overline{\mathbb{Q}} = \alpha^2 + \beta^2$ can be seen as a semi-modulus on the ring $\mathbb{Q}$

$$\mathbb{Q} = \mathbb{Q}^{+} \cup \mathbb{Q}^{0} \cup \mathbb{Q}^{-}$$

according to the sign of the modulus of $\mathbb{Q}$. It is important to note that, in contrast with the analysis of reference [9], for the BI case $\alpha, \beta \in \mathbb{R}$ (α, β — the identity and the pseudoscalar invariants of the electromagnetic field, respectively) and the commutative ring described by $\mathbb{Q}$ has no pseudo-complex structure.

Another interesting thing about this commutative ring of complex operators is that it permits us to define for $d = 2$ the following exponential mapping:

$$\mathrm{e}^{(\alpha\sigma_0 - i\beta\sigma_2)} = \mathrm{e}^{a} \left(\cos\beta - i\sigma_2 \sin\beta \right)$$

that puts in a concrete and more clear form the mathematical structure described in a more abstract way earlier.

The important thing is that the correct analysis of the algebraic and divisor ring structure of the BI-field equations is a crucial point which goes towards a truly non-commutative BI theory. The generalization of the transformations (10) will be realized with the operators over the non-commutative field of full-quaternions in the following manner:

$$\frac{1}{\mathbb{R}} \left[\sigma_0 \delta - iG \left(\sigma_1 \alpha + \sigma_2 \beta + \sigma_3 \gamma \right) \right].$$

*Here d is the dimension.

We assume the coefficients a, b, c :reals and d :complex, in principle, being the final form of the operator

$$\frac{1}{\mathbb{R}}\left[\begin{pmatrix} \delta & \\ & {}^{*}\delta \end{pmatrix} - iG\begin{pmatrix} \gamma & \alpha - i\beta \\ \alpha + i\beta & -\gamma \end{pmatrix}\right],$$

where the star means complex conjugate, and the quantity G will have another meaning as in the initial expression (10), obviously. The question that immediately arises is: Is it possible to impose conditions over the coefficients α, β, γ and δ in the above expression in order to obtain a full-quaternionic non-commutative operator from the equations of motion of a determinant-geometrical action? The answer is affirmative: if and only if $c = 0$ and $\delta = \alpha - i\beta$, in such case the square root of the determinant in the BI action, where the equations of motion that determine the mapping coming from, is

$$\sqrt{|\det(g_{\mu\nu} + b^{-1}\chi F_{\mu\nu})|},$$

where $\chi^4 = i(\alpha - i\beta)$ and $G = \dfrac{\chi^2}{2b^2} F_{\mu\nu}\widetilde{F}^{\mu\nu}$, following the same conventions from the beginning. This issue with a carefully study of the possible physical meaning will be analyzed in our future work.

2. THE HAMILTONIAN POINT OF VIEW

We can show that the $SO(2)$ structure of the BI theory is more easily seen in the following operator form [7]:

$$\frac{1}{R}\left(\sigma_0 - i\sigma_2\overline{\mathbb{P}}\right) L = \mathbb{L},$$

$$\frac{\mathbb{R}}{\left(1 + \overline{\mathbb{P}}^2\right)}\left(\sigma_0 + i\sigma_2\overline{\mathbb{P}}\right) \mathbb{L} = \mathbb{L},$$

$$\overline{\mathbb{P}} \equiv \frac{\mathbb{P}}{b},$$

where we defined the following quaternionic operators:

$$L = F - i\sigma_2\widetilde{F},$$

$$\mathbb{L} = \mathbb{F} - i\sigma_2\widetilde{\mathbb{F}},$$

the pseudo-scalar of the electromagnetic tensor $F^{\mu\nu}$

$$\mathbb{P} = -\frac{1}{4}F_{\mu\nu}\widetilde{F}^{\mu\nu},$$

and σ_0, σ_2 — the well-know Pauli matrices that we defined previously. Now, with the definitions given before, we pass to describe the phase space from the Hamiltonian point of view in the similar form as in Ref. [5].

The 6-dimensional space $V = \Lambda^2\left(\mathbb{R}^4\right) \rightarrow$2-forms $\in \mathbb{R}^4$, has coordinates $F_{\mu\nu}$ and carries a Lorentz-invariant metric with signature $(+++---)$ defined by

$$k_H\left(F, F\right) \equiv L\widetilde{L} = 2F\widetilde{F}.$$

The dual space V^* of V consists of the skew-symmetric second rank contravariant tensors $\mathbb{F}^{\mu\nu}$. The phase space $P = V \oplus V^*$ carries a natural quaternionic symplectic structure given by

$$d\mathbb{L} \wedge d\widetilde{L} = d\mathbb{F} \wedge dF - d\widetilde{F} \wedge d\widetilde{\mathbb{F}}.$$

Notice that now, from the mathematical description of the phase space, the $SO\left(2\right)$ symmetry is, in fact, embedded in a large quaternionic structure.

3. MAXWELL EQUATIONS IN A RIEMANNIAN SPACE AND THE BORN–INFELD THEORY

We want to give now some curious aspects about the relation between the BI field equations and the Maxwell equations in a Riemannian space. From Ref. [2] we know that when gravitational field exists (i. e. curved space–time), it is possible to write the Maxwell equations in vacuum as the same equations in a hypothetic medium as*

$$\mathbf{D} = \frac{\mathbf{E}}{\sqrt{\mathbf{h}}} + \left[\mathbf{B} \times \mathbf{g}\right], \quad \mathbf{H} = \frac{\mathbf{B}}{\sqrt{\mathbf{h}}} - \left[\mathbf{E} \times \mathbf{g}\right]$$

(i. e. for a *girotropic* medium [3]). Analogously to the Born–Infeld case, we can put these constitutive relations in the following form**:

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{H} \end{pmatrix}^{\alpha} = \overbrace{\left[\frac{\sigma_0}{\sqrt{h}} + i\sigma_2 \epsilon_{\beta\gamma} \mathbf{g}^{\beta}\right]^{\alpha}}^{\equiv \mathbb{Q}} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}^{\gamma}. \tag{13}$$

Notice the remarkable analogy with the similar expression (10) from the BI theory that makes it possible to be formulated in an effective metric theory as was shown

---

*Here $g_\alpha = -\dfrac{g_{0\alpha}}{g_{00}}$, $\gamma_{\alpha\beta} = -g_{\alpha\beta} + \dfrac{g_{0\alpha} g_{0\beta}}{g_{00}}$ and $h = g_{00}$ as in Ref. [2].

**Here ϵ is the full-antisymmetric tensor, as usual.

in [8]. For the BI case the constitutive-like relations give $\mathbf{D}$ and $\mathbf{H}$ in terms of $\mathbf{E}$ and $\mathbf{B}$ [5]:

$$\mathbf{D} = \frac{\mathbf{E} + b^{-2}\left(\mathbf{E}\cdot B\right)\mathbf{B}}{\sqrt{1 + b^{-2}\left(\mathbf{B^2} - \mathbf{E^2}\right) - b^{-4}\left(\mathbf{E}\cdot B\right)^2}},$$

$$\mathbf{H} = \frac{\mathbf{B} - b^{-2}\left(\mathbf{E}\cdot B\right)\mathbf{E}}{\sqrt{1 + b^{-2}\left(\mathbf{B}^2 - \mathbf{E}^2\right) - b^{-4}\left(\mathbf{E}\cdot B\right)^2}}. \tag{14}$$

From Introduction, we know that these equations can be solved to give E and H in terms of E and B:

$$\mathbf{E} = \frac{\left(1 + b^{-2}\mathbf{B}^2\right)\mathbf{D} + b^{-2}\left(\mathbf{D}\cdot B\right)\mathbf{B}}{\sqrt{\left(1 + b^{-2}\mathbf{B}^2\right)\left(1 + b^{-2}\mathbf{D}^2\right) - b^{-4}\left(\mathbf{D}\cdot B\right)^2}},$$

$$\mathbf{B} = \frac{\left(1 + b^{-2}\mathbf{D}^2\right)\mathbf{B} + b^{-2}\left(\mathbf{D}\cdot B\right)\mathbf{D}}{\sqrt{\left(1 + b^{-2}\mathbf{B}^2\right)\left(1 + b^{-2}\mathbf{D}^2\right) - b^{-4}\left(\mathbf{D}\cdot B\right)^2}}$$

that make easy the explicit comparison between (13) and (14) when the fields $\mathbf{D}$ and $\mathbf{H}$ are the same in both cases: BI fields in flat space–time and linear field in curved space–time:

$$\left.\frac{E_\alpha}{\mathbb{R}}\right|_{BI} = \left.\frac{E_\alpha}{\sqrt{h}}\right|_f,$$

$$\left.\left(\gamma^{\beta\gamma}B_\beta E_\gamma\right)\frac{B_\alpha}{\mathbb{R}}\right|_{BI} = \left.\sqrt{\gamma}\epsilon_{\alpha\beta\gamma}g^{0\alpha}B^\gamma\right|_f,$$

where the subindexes BI and f indicate the fields in BI theory (flat space–time) and the Maxwell fields in any frame (curved), respectively.

Following the same procedure as in [3] for the Maxwell case, without any background (gravitatory and/or electromagnetic) the Fresnel equation in the Born–Infeld case, the flat space–time takes the following form:

$$-n^2\left(C_{xx}n_x^2 + C_{yy}n_y^2 + C_{zz}n_z^2\right) + n_x^2 C_{xx}\left(C_{yy} + C_{zz}\right) +$$
$$+ n_y^2 C_{yy}\left(C_{xx} + C_{zz}\right) + n_z^2 C_{zz}\left(C_{xx} + C_{yy}\right) - C_{xx}C_{yy}C_{zz} = 0, \tag{15}$$

where n_i are the coordinates of the surface of propagation (wave number) and

$$C_{ij} \equiv \frac{\left(\delta_{ij} + \left(\mathbf{E}\cdot\mathbf{B}\right)E_iB_j\right)}{1 + b^{-2}\left(\mathbf{B^2} - \mathbf{E^2}\right) - b^{-4}\left(\mathbf{E}\cdot\mathbf{B}\right)^2}. \tag{16}$$

Notice that expression (15) has the same form as in [3] but with the ε_{ij} replaced by the C_{ij} given by (16). Notice also that, in the presence of any electromagnetic background the particular form of the Fresnel equation (15) can take a more general form depending on those components for C_{ij} with $c \neq j$ (i. e., Ref. [3]). And it is interesting in order to have a theoretical tool to test the nonlinearity of the BI field as a deviation of the Maxwell theory being of particular importance in astrophysical phenomena [10].

CONCLUDING REMARKS

In this work the Born–Infeld field equations were written in the most compact form by means of the quaternionic operators constructed according to the symmetries of the theory.

We also show that the $\mathbb{Q}$ operators defined here form a part of a commutative ring of complex operators and the $SO\,(2)$ symmetry of the BI field equations is in such manner embedded into a larger quaternionic structure. This extension can be realized transforming the commutative ring of complex operators to a non-commutative ring. Our results agree with the observation of Gibbons and Rasheed in [5, 6] that there exists a discrete symmetry in the structure of the field equations that is unique in the case of the nonlinear electrodynamics of Born and Infeld: this fact is easily seen in our work because these discrete symmetries that are generated by the $\mathbb{Q}$ operators are invertible.

The quaternionic structure of the phase space was explicitly derived and described from the Hamiltonian point of view, showing, at the same time, that the results on the structure of the phase space of Ref. [5] are naturally included in this large quaternionic symmetry.

Finally, the analogy and similarities between the BI theory and the Maxwell (linear) electrodynamics in a curved space–time were explicitly shown and the Fresnel equation in the nonlinear BI case without background was explicitly given and proposed as a theoretical tool to test this particularly interesting nonlinear electrodynamics of M. Born and L. Infeld.

Acknowledgements. I am very thankful to Profs. B. M. Barbashov and A. Dorokhov for their advisement and, in particular, to Prof. J. A. Helayel-Neto for his interest to put in this research. I am very grateful to the Directorate of JINR, in particular, to the Bogoliubov Laboratory of Theoretical Physics, for the hospitality and support.

REFERENCES

1. *Born M., Infeld L.* // Proc. Roy. Soc. (London) 1934. V. 144. P. 425.

2. *Landau L. D., Lifshitz E. M. Teoría clásica de los Campos*. Buenos Aires: Reverté, 1974.

3. *Landau L. D., Lifshitz E. M. Electrodinamica de los medios continuos*. Buenos Aires: Reverté, 1974.

4. *Gaillard M. K., Zumino B.* // Nucl. Phys. B. 1981. V. 193. P. 221.

5. *Gibbons G. W., Rasheed D. A.* // Nucl. Phys. B. 1995. V. 454. P. 185.

6. *Gibbons G. W., Rasheed D. A.* // Phys. Lett. B. 1996. V. 365. P. 46.

7. *Cirilo-Lombardo D. J.* // Class. Quntum. Grav. 2004. V. 21. P. 1407, and references therein.

8. *Novello M. et al.* // Phys. Rev. D. 2000. V. 61. P. 45001, and references therein.

9. *Schuller F. P.* // Ann. Phys. 2002. V. 299. P. 174.

10. *Zeldovich Y. B.* // JETP Lett. 1965. V. 1. P. 95.

Received on December 28, 2006.

Корректор *Т. Е. Попеко*

Подписано в печать 23.03.2007.
Формат 60 × 90/16. Бумага офсетная. Печать офсетная.
Усл. печ. л. 0,93. Уч.-изд. л. 1,32. Тираж 415 экз. Заказ № 55728.

Издательский отдел Объединенного института ядерных исследований
141980, г. Дубна, Московская обл., ул. Жолио-Кюри, 6.
E-mail: publish@jinr.ru
www.jinr.ru/publish/