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ON $(W_2^1(\mathbb{R}) \cap W_\infty^1(\mathbb{R}))$-SOLUTIONS OF THE EQUATION $u_{tt} = (a(u)u_x)_x + f(x,t)$

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Жидков П. Е. E5-2013-105
О $(W_2^1(\mathbb{R})\cap W_\infty^1(\mathbb{R}))$-решениях уравнения $u_{tt} = (a(u)u_x)_x + f(x,t)$

Рассматривается задача с начальными условиями для уравнения в заголовке с $u(x,0) = u_0(x) \in W_2^1(\mathbb{R})\cap W_\infty^1(\mathbb{R})$ и $u_t(x,0) = u_1(x) \in L_2(\mathbb{R})\cap L_\infty(\mathbb{R})$ в случае, когда это уравнение является равномерно гиперболическим. Доказаны существование и единственность локального слабого решения $u(x,t)$ этой задачи, такого, что, в частности, $(u(\cdot,t), u_t(\cdot,t)) \in (W_2^1(\mathbb{R})\cap W_\infty^1(\mathbb{R})) \times (L_2(\mathbb{R}) \cap L_\infty(\mathbb{R}))$ для любого фиксированного t из интервала существования. Для гладких начальных данных доказано, что время жизни гладкого решения совпадает с временем жизни нашего слабого решения.

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Zhidkov P. E. E5-2013-105
On $(W_2^1(\mathbb{R})\cap W_\infty^1(\mathbb{R}))$-Solutions of the Equation $u_{tt} = (a(u)u_x)_x + f(x,t)$

We consider the initial value problem for the equation in the title with $u(x,0) = u_0(x) \in W_2^1(\mathbb{R})\cap W_\infty^1(\mathbb{R})$ and $u_t(x,0) = u_1(x) \in L_2(\mathbb{R})\cap L_\infty(\mathbb{R})$ in the case when this equation is uniformly hyperbolic. We prove the existence and uniqueness of a local weak solution $u(x,t)$ of this problem such that in particular $(u(\cdot,t), u_t(\cdot,t)) \in (W_2^1(\mathbb{R})\cap W_\infty^1(\mathbb{R})) \times (L_2(\mathbb{R}) \cap L_\infty(\mathbb{R}))$ for any fixed t in the interval of existence. For smooth initial data, it is proved that the life time of the smooth solution coincides with the life time of our weak solution.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION. NOTATION. STATEMENTS OF THE MAIN RESULTS

We continue our studies of nonlinear second-order hyperbolic problems begun in [7]. Here we consider the following problem:

$$u_{tt} = (a(u)u_x)_x + f(x,t), \quad (x,t) \in \mathbb{R} \times I, \ u = u(x,t), \tag{1}$$

$$u(\cdot,0) = u_0(\cdot) \in W_2^1 \cap W_\infty^1, \quad u_t(\cdot,0) = u_1(\cdot) \in L_2 \cap L_\infty. \tag{2}$$

Hereafter all the quantities that we deal with are real, $I \subset \mathbb{R}$ is an interval that contains 0, $a(\cdot)$ is a sufficiently smooth function that satisfies in particular

$$0 < a_1 \leqslant a(s) \leqslant a_2 < +\infty \quad \text{and} \quad |a'(s)| \leqslant a_3 \tag{3}$$

for some positive constants a_1, a_2 and a_3 that do not depend on $s \in \mathbb{R}$, and $W_p^1 = W_p^1(\mathbb{R})$ and $L_p = L_p(\mathbb{R})$ are the standard Sobolev and Lebesgue spaces that will be introduced more carefully in the following (here $1 \leqslant p \leqslant +\infty$). For simplicity, the function f in equation (1) is assumed to be in particular sufficiently smooth and bounded with its gradient.

Studies of the initial value problems for quasi-linear hyperbolic equations have a long history and currently the basic results on the well-posedness for such problems are known. Generally speaking, such a problem has a unique local sufficiently smooth solution provided the initial data is sufficiently regular (on this subject, see, for example, [1–3]). A specific feature of this class of problems is that their solutions have a tendency to form singularities in finite intervals of time t and therefore, a standard result states only the local existence of a solution. At the same time, generally speaking, again, it is known only that higher derivatives of such smooth solutions of quasi-linear second-order hyperbolic equations may blow up (on this subject, see, for example, [4]). At least for this reason, it seems to be of an interest to study weak solutions of problems of type (1)–(2), supposing in particular that, maybe, such a weak solution (that has no higher derivatives) could be global, that is, defined for all $(x,t) \in \mathbb{R} \times \mathbb{R}$. In addition, to the author's knowledge, concretely for solutions of equation (1) the blow up phenomenon is unknown.

A known result on the existence of a local lower regularity solution for a problem of the type of (1)–(2) is due to Th. J. R. Hughes, T. Kato and J. E. Marsden [2]: these authors have proved that if the functions a and f obey certain regularity assumptions and the initial data are in $H^s \times H^{s-1}$, $s > \frac{3}{2}$, then there exist an interval $I \ni 0$ and a unique local solution of a problem similar to our one in the space $C(I; H^s)$ (in fact, in this article, a much more complicated problem is considered, in particular, a multidimensional one, when $x \in \mathbb{R}^d$ with integer $d \geqslant 1$). We also refer the reader to the recent book [6] for a theory of linear hyperbolic equations and systems.

So, in the present paper, our aim is to prove the existence and uniqueness of a local weak solution for problem (1)–(2). As for applications of problems of this type, there are a lot of them. Equations of this type arise in the elasticity theory, in the description of various wave and oscillation processes, etc. In this context, one may consider equations (1) and (2) as a model problem and the present article as a step in the way of studies of weak solutions of more realistic quasi-linear second-order hyperbolic problems, solutions of which possess only the first partial derivatives.

Now, we introduce some *notation.* Let $\Omega \subset \mathbb{R}^d$ be an open domain with a smooth boundary, where $d \geqslant 1$ is integer. By $L_p(\Omega)$ with $1 \leqslant p < +\infty$ we denote the standard Lebesgue space of functions $g, h, \ldots$ measurable in Ω for which the norm

$$\|g\|_{L_p} = \left\{ \int\limits_{\Omega} |g(x)|^p dx \right\}^{\frac{1}{p}}$$

is finite. We set $\|g\|_{L_\infty(\Omega)} = \operatorname{ess\,sup}_{x \in \Omega} |g(x)|$ and $L_p = L_p(\mathbb{R})$. As is well known, $L_2(\Omega)$ is a separable Hilbert space with the scalar product

$$(g, h)_{L_2(\Omega)} = \int\limits_{\Omega} g(x) h(x) dx.$$

Denote now by $x = (x_1, \ldots, x_d)$ the points of Ω and let $w \in L_1(\Omega)$. Everywhere we understand the weak derivative $\varphi(x) = \dfrac{\partial w}{\partial x_i}$ in the sense of distributions. By $W_p^l(\Omega)$, where $1 \leqslant p \leqslant +\infty$ and $l \geqslant 0$ is integer, we denote the standard Sobolev space that consists of all functions $g \in L_p(\Omega)$ such that $\dfrac{\partial^l g}{\partial x_{i_1} \ldots \partial x_{i_l}} \in L_p(\Omega)$ for any indexes $i_1, \ldots, i_l = 1, 2, \ldots, d$. The space $W_p^l(\Omega)$ is equipped with the norm $\|g\|_{W_p^l(\Omega)} = \|g\|_{L_p(\Omega)} + \sum\limits_{|k|=l} \|D_x^k g\|_{L_p(\Omega)}$, where $k = (k_1, \ldots, k_d)$, k_i are nonnegative integer, $|k| = k_1 + \ldots + k_d$ and $D_x^k = \dfrac{\partial^{|k|}}{\partial x_1^{k_1} \ldots \partial_d^{k_d}}$. Then, $W_2^l(\Omega)$

is a separable Hilbert space for any $l \geqslant 0$, with the standard scalar product. We set $W_p^l = W_p^l(\mathbb{R})$. In addition, $L_p(\Omega)$ and $W_p^l(\Omega)$ are Banach spaces.

For an interval I, integer $k \geqslant 0$ and a Banach space B with a norm $\|\cdot\|_B$ denote by $C^k(I;B)$ the Banach space of functions $g: I \to B$ continuous and bounded with all their derivatives of all orders up to k. The space $C^k(I;B)$ is equipped with the norm $\|g(\cdot)\|_{C^k(I;B)} = \sum\limits_{m=0}^{k} \sup\limits_{t\in I} \|\frac{d^m g(t)}{dt^m}\|_B$. By $L_b(I;B)$ we denote the space of functions $g: I \to B$ bounded and such that $\|g(t)\|_B$ is a measurable function of $t \in I$, taken with the norm $\|g\|_{L_b(I;B)} = \sup\limits_{t\in I} \|g(t)\|_B$. We denote by $C_w(I;B)$ the space of functions $g: I \to B$ bounded in the norm of B and continuous in the topology of weak convergence of B. We write $g \in C_{loc}^k(I;B)$ if for any compact subinterval $I_1 \subset I$ one has: $g \in C^k(I_1;B)$.

For noninteger $s > 0$, we denote by $[s]$ the maximal integer number smaller than s and set for any interval I and any function g which is $l = [s]$ times continuously differentiable in I:

$$\|g\|_{C^s(I;\mathbb{R})} = \sum_{k=0}^{l} \sup_{t\in I} \left| \frac{d^k g(x)}{dt^k} \right| + \sup_{\substack{x,y\in I \\ x\neq y}} \frac{|g^{(l)}(x) - g^{(l)}(y)|}{|x-y|^{s-l}}.$$

Then, the set of all functions g above, for which $\|g\|_{C^s(I;\mathbb{R})} < \infty$, equipped by the norm $\|\cdot\|_{C^s(I;\mathbb{R})}$ is a Banach space. We denote $C^s(\mathbb{R};\mathbb{R})$ by $C^s(\mathbb{R})$.

For $s \in \mathbb{R}$, by $H^s(\mathbb{R}^d)$ we denote the Hilbert space which is the completion of the set of infinitely differentiable finite functions g in $\mathbb{R}^d$ with respect to the norm $\|g\|_{H^s(\mathbb{R}^d)} = \|\mathcal{F}^{-1}(1+|r|^2)^{\frac{s}{2}}(\mathcal{F}g)(r)\|_{L_2}$, where $\mathcal{F}$ is the Fourier transform, taken with the corresponding scalar product. We set $H^s = H^s(\mathbb{R})$. Then, as is known, $H^{s'}$ is continuously embedded in H^s for any s, s' satisfying $s' > s$. In addition, H^s is isomorphic to W_2^s for any integer $s \geqslant 0$ and $L_p(\Omega) = W_p^0(\Omega)$ for any $1 \leqslant p \leqslant +\infty$.

Sometimes we denote by $c, c_1, C, C_1, C_2, C', C'', \ldots$ positive constants not indicating especially what they are independent of if it is clear from the context.

For some $s > 0$, we shall impose the following assumptions on the functions a and f:

(a) *Let* $a(\cdot) \in C^s(\mathbb{R})$ *and estimates (3) hold.*

(f) *Let* $f(\cdot,t) \in C_{loc}(\mathbb{R};H^s) \cap C_{loc}(\mathbb{R};C^s(\mathbb{R}))$.

In accordance with the book [5], we accept the following definition of a weak solution of problem (1)–(2).

Definition. *Let* $I = [0,T)$*, where* $T > 0$*, and let assumptions (a) and (f) be valid with* $s = 1$*. We say that a function* $u(x,t) \in W_2^1(\mathbb{R} \times I)$ *is a weak solution* *of problem (1)–(2), if for any $\eta(x,t)$ continuously differentiable and finite in $\mathbb{R}\times I$ one has:*

$$\int_0^T dt \int_{\mathbb{R}} \{-u_t(x,t)\eta_t(x,t) + a(u(x,t))u_x(x,t)\eta_x(x,t) - f(x,t)\eta(x,t)\}dx+$$

$$+\int_{\mathbb{R}} u_1(x)\eta(x,0)dx = 0. \quad (4)$$

One can define a weak solution of problem (1)–(2) in the same sense for $t < 0$ by complete analogy.

Sometimes, in what follows, we shall consider in place of equations (1) and (2) the following problem:

$$u_{tt} = a(u)u_{xx} + f_1(x,t,u,u_x), \quad (x,t) \in \mathbb{R}\times I,\ u = u(x,t), \quad (5)$$

$$u(\cdot,0) = u_0(\cdot), \quad u_t(\cdot,0) = u_1(\cdot), \quad (6)$$

where I is an interval containing 0, $f_1(x,t,u,u_x) = f(x,t) + a'(u)u_x^2$ and now $(u_0,u_1) \in H^s \times H^{s-1}$ with integer $s \geqslant 2$. Clearly, (5) is equation (1) written in another form. According to [2], the following result takes place.

Theorem 0. *Let for some $s > \frac{3}{2}$ assumptions (a) and (f) be valid and let in (6) $(u_0,u_1) \in H^s \times H^{s-1}$. Then, there exist $T > 0$, that depends only on $\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}$, and a unique solution $u^s(\cdot,t) \in C([-T,T];H^s)\cap C^1([-T,T];H^{s-1})$ of problem (5)–(6). We call such a solution of problem (5)–(6) the H^s-solution of this problem. This solution can be uniquely continued on a maximal interval $(-T_s^-, T_s^+)$, where $T_s^-, T_s^+ > 0$, such that either $T_s^+ = +\infty$ (resp. $T_s^- = +\infty$) or*

$$\limsup_{t\to T_s^+ - 0} [\|u^s(\cdot,t)\|_{H^s} + \|u_t^s(\cdot,t)\|_{H^{s-1}}] = +\infty$$

(resp. $\limsup_{t\to -T_s^- + 0} [\|u^s(\cdot,t)\|_{H^s} + \|u_t^s(\cdot,t)\|_{H^{s-1}}] = +\infty$). This H^s-solution depends on initial data continuously in the sense that for any compact interval $J \subset (-T_s^-, T_s^+)$ and initial data in (6) sufficiently close to (u_0,u_1) in $H^s\times H^{s-1}$ the corresponding H^s-solution can be continued on the whole interval J and the correspondence $(u_0,u_1) \to (u(\cdot,t),u_t(\cdot,t))$ as a mapping of $H^s \times H^{s-1}$ in $C(J;H^s)\times C^1(J;H^{s-1})$ is continuous. If $s' > s > \frac{3}{2}$ and $(u_0,u_1) \in H^{s'} \times H^{s'-1}$, problem (5)–(6) has a unique H^s-solution $u^s(x,t)$ and a unique $H^{s'}$-solution $u^{s'}(x,t)$ defined in the maximal intervals of time I and I', respectively. Then, in fact, $I = I'$ and $u^s(x,t) = u^{s'}(x,t)$ for all $(x,t)\in\mathbb{R}\times I$.

For the Proof of this result, see [2].□

Now, we can establish our main results. They are as follows.

Theorem 1. *Let assumptions (a) and (f) hold with $s = 3$. Then, for any $R > 0$ there exists $T > 0$ such that for any initial data $(u_0, u_1) \in (W_2^1 \cap W_\infty^1) \times (L_2 \cap L_\infty)$ which satisfy $\|u_0\|_{W_2^1 \cap W_\infty^1} + \|u_1\|_{L_2 \cap L_\infty} \leqslant R$ problem (1)–(2) has a weak solution $u(x,t)$ such that $(u(\cdot,t), u_t(\cdot,t)) \in (C(I; L_2) \cap C_w(I; W_2^1) \cap L_b(I; W_\infty^1)) \times (C_w(I; L_2) \cap L_b(I; L_\infty))$, where $I = [-T, T]$. Such a weak solution that belongs to the indicated function spaces is unique. There exists a maximal interval $(-T^-, T^+)$ with $T^-, T^+ > 0$ such that our weak solution can be uniquely continued on this interval and either $T^+ = +\infty$ (resp. $T^- = +\infty$) or*

$$\limsup_{t \to T^+ - 0} \{\|u(\cdot,t)\|_{W_2^1 \cap W_\infty^1} + \|u_t(\cdot,t)\|_{L_2 \cap L_\infty}\} = +\infty$$

(resp. $\limsup\limits_{t \to -T^- + 0} [\|u(\cdot,t)\|_{W_2^1 \cap W_\infty^1} + \|u_t(\cdot,t)\|_{L_2 \cap L_\infty}] = +\infty$).

Theorem 2. *Let for some $s' \geqslant 2$ assumptions (a) and (f) be valid with some $s \geqslant \max\{s'; 3\}$ and let in (6) $(u_0, u_1) \in H^{s'} \times H^{s'-1}$. Now, in the formulation of this claim, we rename s' by s. Then, according to theorem 0 above, problem (5)–(6) has a unique H^s-solution $u^s(x,t)$ whose maximal interval of existence is denoted $I^s = (-T_s^-, T_s^+)$ $(T_s^-, T_s^+ > 0)$. Clearly, according to Theorem 1 our problem still has a unique weak solution $u(x,t)$. We denote by I the maximal interval of existence of this weak solution. Clearly, $I^s \subset I$ and $u^s(\cdot,t) = u(\cdot,t)$ for any $(x,t) \in \mathbb{R} \times I^s$. We state that, in fact, $I^s = I$ and $u^s(x,t) \equiv u(x,t)$.*

Remark 1. The author of this article believes that the result stated with Theorem 1 is incomplete and that, in fact, for our weak solution $u(x,t)$ one has: $(u(\cdot,t), u_t(\cdot,t)) \in (C(I; W_2^1) \cap L_b(I; W_\infty^1)) \times (C(I; L_2) \cap L_b(I; L_\infty))$. However, it is not proved in this article.

Remark 2. Suppose that $(u_0, u_1) \in H^s \times H^{s-1}$ with some $s \geqslant 2$ and that the functions a and f are sufficiently regular. Then, according to Theorem 2 our weak solution $u(x,t)$ blows up in a finite interval of time if and only if it is so for the H^s-solution, and the time T of this blowing up for these two solutions is the same. It gives a negative answer to our question, whether it may be that the H^s-solution blows up at a time $T > 0$, but the corresponding weak solution is global, that is, the latter one can be continued on the entire real line $t \in \mathbb{R}$.

The next Sec. 2 contains preliminary considerations including a derivation of a priori estimates of H^s-solutions of equations (5) and (6) with $s = 2$ which are given by Theorem 0. Our proofs of Theorems 1 and 2 are established in Sec. 3.

2. PRELIMINARIES. A PRIORI ESTIMATES OF SMOOTH SOLUTIONS

Everywhere in this section $s = 2$, if otherwise is not stated, and we accept that assumptions (a) and (f) are valid with $s = 2$ except for Lemmas 3 and 4, where $s = 3$. Let us now consider problem (5)–(6) for $(u_0, u_1) = (u_0^\epsilon, u_1^\epsilon) \in H^2 \times H^1$, where $\epsilon \in (0, 1]$ is a parameter. According to Theorem 0, for any $\epsilon \in (0, 1]$ there exists $T^\epsilon > 0$ and a unique H^2-solution $u^\epsilon(x, t)$ of problem (5)–(6) that belongs to the space $C([-T^\epsilon, T^\epsilon]; H^2) \cap C^1([-T^\epsilon, T^\epsilon]; H^1)$, where T^ϵ depends only on $\|u_0^\epsilon\|_{H^2} + \|u_1^\epsilon\|_{H^1}$. We take such initial data $(u_0^\epsilon, u_1^\epsilon)$ that $(u_0^\epsilon, u_1^\epsilon) \to (u_0, u_1)$ as $\epsilon \to +0$ in $W_2^1 \times L_2$ and that there exists $D > 0$ such that

$$\|u_0^\epsilon\|_{W^1_\infty} + \|u_1^\epsilon\|_{L_\infty} \leqslant D$$

for any $\epsilon \in (0, 1]$. In the following in this section, we shall prove that there exists $T > 0$ such that T^ϵ can be chosen not smaller than T for any $\epsilon > 0$ sufficiently small. More precisely, we shall derive upper estimates for the pair $(u^\epsilon(x, t), u_t^\epsilon(x, t))$ in the space $C([-T, T]; H^2) \times C^1([-T, T]; H^1)$ for some $T > 0$ independent of ϵ which according to [2] implies

$$T^\epsilon \geqslant T \tag{7}$$

for any $\epsilon > 0$ sufficiently small. We shall derive all our estimates including (7) only for $t > 0$, because for $t < 0$ they can be obtained by complete analogy.

So, consider problem (5)–(6). Then, for $a_1(x, t) = a(u(x, t))$ and $f_1(x, t) = f(x, t) + a'(u(x, t))u_x^2(x, t)$, where $u(x, t)$ is the corresponding H^2-solution of equations (5) and (6), all the assumptions of Lemma 12 in [7] are valid. In our case, according to this result the following relation occurs (here $n \geqslant 1$ is arbitrary integer):

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}} \big\{ &(|u_t(x,t)| + a^{\frac{1}{2}}(u(x,t))|u_x(x,t)|)^{2n} + \\
&+ (|u_t(x,t)| - a^{\frac{1}{2}}(u(x,t))|u_x(x,t)|)^{2n} \big\} dx = \\
= \sum_{k=0}^{n-1} 2n \binom{2n-1}{2k} &\int\limits_{\mathbb{R}} a^k(u(x,t)) f_1(x,t) u_t^{2n-2k-1}(x,t) u_x^{2k}(x,t)dx - \\
- \sum_{k=1}^{n} k \binom{2n}{2k-1} &\int\limits_{\mathbb{R}} a^{k-1}(u(x,t))a'(u(x,t))u_t^{2n-2k+1}(x,t)u_x^{2k}(x,t)dx + \\
+ \sum_{k=1}^{n} k \binom{2n}{2k} &\int\limits_{\mathbb{R}} a^{k-1}(u(x,t))a'(u(x,t))u_t^{2n-2k+1}(x,t)u_x^{2k}(x,t)dx,
\end{aligned} \tag{8}$$

where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$, $0 \leqslant m \leqslant n$ are positive integer and $0! = 1$.

Lemma 1. *Let assumptions (a) and (f) be valid with* $s = 2$ *and let* $(-T_2^-, T_2^+)$ *denote the maximal interval on which the* H^2*-solution* $u(x,t)$ *of problem (5)–(6) taken with* $(u_0, u_1) \in H^2 \times H^1$ *can be continued. Then, for any* $R > 0$ *there exist* $T>0$ *and* $C>0$ *such that for any such* (u_0, u_1) *that satisfy* $\|u_0\|_{W^1_\infty} + \|u_1\|_{L_\infty} \leqslant R$ *for any* $t \in [-T, T] \cap (-T_2^-, T_2^+)$ *one has:* $\|u(\cdot,t)\|_{W^1_\infty} + \|u_t(\cdot,t)\|_{L_\infty} \leqslant C$.

Proof. To prove this claim, we apply identity (8). Let $t \geqslant 0$. Using the binomial formula and the Hölder inequality, from (8) we obtain

$$\begin{aligned} \frac{d}{dt}\Bigg\{ \int\limits_{\mathbb{R}} [(|u_t(x,t)| + a^{\frac12}(u(x,t))|u_x(x,t)|)^{2n} + \\ + (|u_t(x,t)| - a^{\frac12}(u(x,t))|u_x(x,t)|)^{2n}]dx \Bigg\}^{\frac{1}{2n}} \leqslant \\ \leqslant C_1 \Bigg\{ \int\limits_{\mathbb{R}} [(|u_t(x,t)| + a^{\frac12}(u(x,t))|u_x(x,t)|)^{2n} + \\ + (|u_t(x,t)| - a^{\frac12}(u(x,t))|u_x(x,t)|)^{2n}]dx \Bigg\}^{\frac{1}{2n}-1} \times \\ \times \Bigg\{ \Bigg[\Bigg(\int\limits_{\mathbb{R}} |f(x,t)|^{2n} dx \Bigg)^{\frac{1}{2n}} + \|u_x(\cdot,t)\|^2_{L_{4n}} \Bigg] \times \\ \times \Bigg[\int\limits_{\mathbb{R}} ((|u_t(x,t)| + a^{\frac12}(u(x,t))|u_x(x,t)|)^{2n} + \\ + (|u_t(x,t)| - a^{\frac12}(u(x,t))|u_x(x,t)|)^{2n}) dx \Bigg]^{1-\frac{1}{2n}} + \\ + (\|u_t(\cdot,t)\|_{L_\infty} + \|u_x(\cdot,t)\|_{L_\infty}) \times \\ \times \int\limits_{\mathbb{R}} [(|u_t(x,t)| + a^{\frac12}(u(x,t))|u_x(x,t)|)^{2n} + \\ + (|u_t(x,t)| - a^{\frac12}(u(x,t))|u_x(x,t)|)^{2n})]dx \Bigg\}, \end{aligned}$$

where the constant $C_1 > 0$ does not depend on $R > 0$, $t \in (-T_2^-, T_2^+)$ and on $n \geqslant 1$. From this relation, we derive:

$$\left\{ \int_{\mathbb{R}} [(|u_t(x,t)| + a^{\frac{1}{2}}(u(x,t))|u_x(x,t)|)^{2n} + \right.$$

$$\left. + (|u_t(x,t)| - a^{\frac{1}{2}}(u(x,t))|u_x(x,t)|)^{2n}]dx \right\}^{\frac{1}{2n}} \leqslant$$

$$\leqslant \left\{ \int_{\mathbb{R}} [(|u_1(x)| + a^{\frac{1}{2}}(u_0(x))|u_{0x}(x)|)^{2n} + \right.$$

$$\left. + (|u_1(x)| - a^{\frac{1}{2}}(u_0(x))|u_{0x}(x)|)^{2n}]dx \right\}^{\frac{1}{2n}} +$$

$$+C_2 \int_0^t (\|f(\cdot,s)\|_{L_{2n}} + \|u_x(\cdot,s)\|^2_{L_{4n}})ds + C_3 \int_0^t \left\{ (\|u_s(\cdot,s)\|_{L_\infty} + \|u_x(\cdot,s)\|_{L_\infty}) \times \right.$$

$$\times \int_{\mathbb{R}} [(|u_s(x,s)| + a^{\frac{1}{2}}(u(x,s))|u_x(x,s)|)^{2n} +$$

$$\left. + (|u_s(x,s)| - a^{\frac{1}{2}}(u(x,s))|u_x(x,s)|)^{2n}]dx \right\}^{\frac{1}{2n}} ds,$$

where the positive constants C_2 and C_3 do not depend on $R > 0$, $t \in [0, T_2^+)$ and on $n \geqslant 1$. For $t \in [0, T_2^+)$, take the limit $n \to +\infty$ in the latter relation. Then, we obtain

$$\||u_t(\cdot,t)| + a^{\frac{1}{2}}(u(\cdot,t))|u_x(\cdot,t)|\|_{L_\infty} \leqslant$$

$$\leqslant \||u_1(\cdot)| + a^{\frac{1}{2}}(u_0(\cdot))|u_{0x}(\cdot)|\|_{L_\infty} + C_2 \int_0^t \|f(\cdot,s)\|_{L_\infty} ds +$$

$$+ C_4 \int_0^t \{1 + \||u_s(\cdot,s)| + a^{\frac{1}{2}}(u(\cdot,s))|u_x(\cdot,s)|\|^2_{L_\infty}\} ds$$

for a constant $C_4 > 0$ independent of $R > 0$ and $t \in [0, T_2^+)$ (here we used the known relations $\|g\|_{L_{2n}} \to \|g\|_{L_\infty}$ as $n \to +\infty$ and $(|p|^{2n} + |q|^{2n})^{\frac{1}{2n}} \to \max\{|p|; |q|\}$ as $n \to +\infty$ for any $g \in L_2 \cap L_\infty$ and $p, q \in \mathbb{R}$, respectively). Now Lemma 1 immediately follows from the latter estimate.□

In the following, we fix this choice of the constant $T > 0$.

Lemma 2. *Let assumptions (a) and (f) be valid with* $s = 2$*. Then, for any* $R > 0$ *and* $R_1 > 0$ *there exists* $C > 0$ *such that for any initial data* $(u_0, u_1) \in$

$H^2 \times H^1$ *that satisfy* $\|u_0\|_{W^1_\infty} + \|u_1\|_{L_\infty} \leqslant R$ *and* $\|u_0\|_{W^1_2} + \|u_1\|_{L_2} \leqslant R_1$ *for any* $t \in [-T, T] \cap (-T_2^-, T_2^+)$ *one has*

$$\|u(\cdot, t)\|_{W^1_2} + \|u_t(\cdot, t)\|_{L_2} \leqslant C,$$

where $u(x,t)$ *is the corresponding* H^2*-solution of equations (5) and (6).*

Proof. Again, we use identity (8), taken now with $n = 1$. Applying the Hölder inequality and Lemma 1, we obtain from this relation:

$$\frac{d}{dt}\int_{\mathbb{R}} \{(|u_t(x,t)| + a^{\frac{1}{2}}(u(x,t))|u_x(x,t)|)^2 + \\ + (|u_t(x,t)| - a^{\frac{1}{2}}(u(x,t))|u_x(x,t)|)^2\}dx = \\ = 4\int_{\mathbb{R}} f(x,t)u_t(x,t)dx + 10\int_{\mathbb{R}} a'(u(x,t))u_t(x,t)u_x^2(x,t)dx \leqslant \\ \leqslant 2\|f(\cdot,t)\|^2_{L_2} + 10(1 + a_3\|u_t(\cdot,t)\|_{L_\infty})\int_{\mathbb{R}} [u_t^2(x,t) + u_x^2(x,t)]dx \leqslant \\ \leqslant C_1 + C_2\int_{\mathbb{R}} \{(|u_t(x,t)| + a^{\frac{1}{2}}(u(x,t))|u_x(x,t)|)^2 + \\ + (|u_t(x,t)| - a^{\frac{1}{2}}(u(x,t))|u_x(x,t)|)^2\}dx,$$

where the constants $C_1 > 0$ and $C_2 > 0$ do not depend on $t \in [-T,T] \cap (-T_2^-, T_2^+)$. Now, applying Gronwell's lemma, we obtain

$$\|u_t(\cdot,t)\|_{L_2} + \|u_x(\cdot,t)\|_{L_2} \leqslant C_3, \tag{9}$$

with a constant $C_3 > 0$ independent of $t \in [-T,T] \cap (-T_2^-, T_2^+)$. Finally, using (9),

$$\|u(\cdot,t)\|_{L_2} \leqslant \|u_0\|_{L_2} + \int_0^t \|u_s(\cdot,s)\|_{L_2} ds \leqslant \|u_0\|_{L_2} + TC_3.$$

Our proof of Lemma 2 is complete.□

Lemma 3. *Let assumptions (a) and (f) be valid with* $s = 3$*. Then, for any* $R > 0$ *and* $R_2 > 0$ *there exists* $C > 0$ *such that for any initial data* $(u_0, u_1) \in H^2 \times H^1$ *in (6) that satisfy* $\|u_0\|_{W^1_\infty} + \|u_1\|_{L_\infty} \leqslant R$ *and* $\|u_0\|_{H^2} + \|u_1\|_{H^1} \leqslant R_2$ *for any* $t \in [-T,T] \cap (-T_2^-, T_2^+)$ *one has:*

$$\|u(\cdot,t)\|_{H^2} + \|u_t(\cdot,t)\|_{H^1} \leqslant C,$$

where $u(x,t)$ *is the corresponding* H^2*-solution of problem (5)–(6).*

Proof. Let $(u_0, u_1) \in H^2 \times H^1$ satisfy the assumptions of Lemma 3 and $(\overline{u}_0, \overline{u}_1) \in H^3 \times H^2$. Denote

$$P(\overline{u}) = \frac{1}{2} \int\limits_{\mathbb{R}} \{\overline{u}_{xt}^2(x,t) + a(\overline{u}(x,t))\overline{u}_{xx}^2(x,t)\}dx.$$

Then, according to Theorem 0, for any sufficiently small $\delta > 0$ the corresponding H^3-solution $\overline{u}(x,t)$ can be continued on the whole interval $[-T_2^- + \delta, T_2^+ - \delta]$, provided $(\overline{u}_0, \overline{u}_1)$ are sufficiently close to (u_0, u_1) in $H^2 \times H^1$, and we have:

$$\frac{dP(\overline{u})}{dt} = \int\limits_{\mathbb{R}} \{\overline{u}_{xt}(x,t)\overline{u}_{xtt}(x,t) + a(\overline{u}(x,t))\overline{u}_{xx}(x,t)\overline{u}_{xxt}(x,t) + \\ + \frac{1}{2}a'(\overline{u}(x,t))\overline{u}_t(x,t)\overline{u}_{xx}^2(x,t)\}dx.$$

Apply Lemmas 1 and 2, integration by parts and the Hölder inequality to the expression in the right-hand side of this equality. Then,

$$\frac{dP(\overline{u})}{dt} = \int\limits_{\mathbb{R}} \{\overline{u}_{xt}(x,t)[\overline{u}_{xtt}(x,t) - (a(\overline{u}(x,t))\overline{u}_{xx}(x,t))'_x] + \\ + \frac{1}{2}a'(\overline{u}(x,t))\overline{u}_t(x,t)\overline{u}_{xx}^2(x,t)\}dx \leqslant \\ \leqslant \int\limits_{\mathbb{R}} |\overline{u}_{xt}(x,t)| \cdot [|f'_x(x,t)| + |(a'(\overline{u}(x,t))\overline{u}_x^2(x,t))'_x|]dx + \\ + C_1\|\overline{u}_t\|_{L_b([-T,T]\cap[-T_2^-+\delta,T_2^+-\delta];L_\infty)}P(\overline{u}) \leqslant C_2 + C_3P(\overline{u}) \quad (10)$$

for constants $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ independent of $t \in [-T,T] \cap (-T_2^-, T_2^+)$. Now, the result for our H^3-solution follows by Gronwell's lemma. To obtain this claim for H^2-solutions, it suffices now to rewrite (10) in the integral form and to apply the continuous dependence of our H^2-solution on initial data stated in Theorem 0. Lemma 3 is proved.□

Remark 3. Note that our proved Lemmas 1 and 3 immediately imply the following. Let $(u_0, u_1) \in H^2 \times H^1$ and $u(x,t)$ be the corresponding H^2-solution of problem (5)–(6). Suppose that $\|u(\cdot,t)\|_{W^1_\infty} + \|u_t(\cdot,t)\|_{L_\infty}$ is bounded in a bounded interval of time J. Then, the expression $\|u(\cdot,t)\|_{H^2} + \|u(\cdot,t)\|_{H^1}$ is also bounded in J.

Lemma 4. *Let assumptions (a) and (f) be valid with $s = 3$. Then, for any $R > 0$ there exist $C > 0$ and $T > 0$, where $T > 0$ was defined earlier, such that* *for any initial data* $(u_0^\epsilon, u_1^\epsilon) \in H^2 \times H^1$*, that satisfy* $\|u_0^\epsilon\|_{W_2^1 \cap W_\infty^1} + \|u_1^\epsilon\|_{L_2 \cap L_\infty} \leqslant R$*, the corresponding* H^2*-solution* $u^\epsilon(x,t)$ *can be continued on the whole interval* $[-T,T]$ *and*

$$\|u^\epsilon(\cdot,t)\|_{W_2^1 \cap W_\infty^1} + \|u_t^\epsilon(\cdot,t)\|_{L_2 \cap L_\infty} \leqslant C$$

for any $t \in [-T,T]$.

Proof. Immediately follows from Theorem 0 and proved Lemmas 1–3.□

3. PROOFS OF THEOREMS 1 AND 2

We begin with proving Theorem 1. Again, we shall establish our proof only for $t > 0$ because for $t < 0$ it can be made by analogy. Let us prove the existence of a solution of problem (1)–(2) stated with Theorem 1. We use ϵ-approximations of problem (5)–(6) described in Sec. 2. So, let $(u_0^\epsilon, u_1^\epsilon) \in H^2 \times H^1$ and let $u^\epsilon(x,t)$ be the corresponding H^2-solution of equations (5) and (6). Here $(u_0^\epsilon, u_1^\epsilon) \to (u_0, u_1) \in W_2^1 \times L_2$ as $\epsilon \to +0$ and

$$\|u_0^\epsilon\|_{W_\infty^1} + \|u_1^\epsilon\|_{L_\infty} \leqslant \|u_0\|_{W_\infty^1} + \|u_1\|_{L_\infty} + 1$$

for any $\epsilon \in (0,1]$. According to Lemma 4, there exists $T > 0$ depending only on $\|u_0\|_{W_\infty^1} + \|u_1\|_{L_\infty}$ such that for any $\epsilon \in (0,1]$ the corresponding H^2-solution of problem (5)–(6) can be continued on the whole interval $[-T,T]$.

Let us prove that in a sense the functions $u^\epsilon(x,t)$ converge to a function $u(x,t)$ as $\epsilon \to +0$. Denote $A(u) = \int\limits_0^u a(r)dr$, $w(x,t) = u^{\epsilon_1}(x,t) - u^{\epsilon_2}(x,t)$ and take $b \in [0,T]$ and $\eta(x,t) = \int\limits_b^t [A(u^{\epsilon_1}(x,s)) - A(u^{\epsilon_2}(x,s))]ds$ for $t \in [0,b]$ and $\eta \equiv 0$ if $t > b$. Introduce also $k(x,t) = \dfrac{A(u^{\epsilon_1}(x,t)) - A(u^{\epsilon_2}(x,t))}{u^{\epsilon_1}(x,t) - u^{\epsilon_2}(x,t)}$ if $u^{\epsilon_1}(x,t) \neq u^{\epsilon_2}(x,t)$ and $k(x,t) = a(u^{\epsilon_1}(x,t))$ otherwise.

It is easily seen that in particular $\eta(x,t) \in W_2^1(\mathbb{R} \times [0,T])$ and that $\eta = 0$ for $t \geqslant b$. In addition, using results in Sec. 2, one may verify that $k(x,t)$ is a continuously differentiable function in $\mathbb{R} \times [0,b]$ and there exists $C > 0$ such that

$$a_1 \leqslant k(x,t) \leqslant a_2 \quad \text{and} \quad |k_t'(x,t)| \leqslant C \tag{11}$$

for any $(x,t) \in \mathbb{R} \times [0,b]$ and for any sufficiently small $\epsilon_1, \epsilon_2 > 0$.

Observe that identity (4) with this function η still holds. Now, from (4), applying Lemma 3:

$$0=\int\limits_0^b dt\int\limits_{\mathbb{R}}\{w_t(x,t)[A(u^{\epsilon_1}(x,t))-A(u^{\epsilon_2}(x,t))]-[A(u^{\epsilon_1}(x,t))-A(u^{\epsilon_2}(x,t))]'_x\times$$

$$\times\int\limits_b^t ds[A(u^{\epsilon_1}(x,s))-A(u^{\epsilon_2}(x,s))]'_x\}dx-\int\limits_{\mathbb{R}}[u_1^{\epsilon_1}(x)-u_1^{\epsilon_2}(x)]\eta(x,0)dx=$$

$$=\beta(\epsilon_1,\epsilon_2)+\int\limits_0^b dt\int\limits_{\mathbb{R}}k(x,t)w(x,t)w_t(x,t)dx+$$

$$+\int\limits_{\mathbb{R}}dx\int\limits_0^b ds[A(u^{\epsilon_1}(x,s))-A(u^{\epsilon_2}(x,s))]'_x\times$$

$$\times\int\limits_0^s dt[A(u^{\epsilon_1}(x,t))-A(u^{\epsilon_2}(x,t))]'_x=\beta(\epsilon_1,\epsilon_2)+\frac{1}{2}\int\limits_{\mathbb{R}}k(x,b)w^2(x,b)dx-$$

$$-\frac{1}{2}\int\limits_0^b dt\int\limits_{\mathbb{R}}k'_t(x,t)w^2(x,t)dx+\frac{1}{2}\int\limits_{\mathbb{R}}\left\{\int\limits_0^b dt[A(u^{\epsilon_1}(x,t))-A(u^{\epsilon_2}(x,t))]'_x\right\}^2dx, \tag{12}$$

where $\beta(\epsilon_1,\epsilon_2)\to 0$ as $\epsilon_1,\epsilon_2\to+0$. From this estimate, in view of (11), it follows by applying Gronwell's lemma that:

$$\|w\|_{C([0,T];L_2)}\to 0 \quad \text{as} \quad \epsilon_1,\epsilon_2\to+0.$$

Therefore, the functions $u^\epsilon(x,t)$ converge to a function $u(x,t)$ in $C([0,T];L_2)$ as $\epsilon\to+0$. In addition, it follows from Lemma 4 that there exists $C>0$ such that

$$\|u_t^\epsilon\|_{C([0,T];L_2)\cap L_b([0,T];L_\infty)}+\|u^\epsilon\|_{C([0,T];W_2^1)\cap L_b([0,T];W_\infty^1)}\leqslant C \tag{13}$$

for all $\epsilon\in(0,1]$ sufficiently small.

Let us prove that $u(\cdot,t)\in W_p^1(\mathbb{R}\times[0,T])$ for any $p\in[2,\infty)$. According to (13), there exists a sequence $\epsilon_n\to+0$ such that u^{ϵ_n} converges to some $w(x,t)\in W_p^1(\mathbb{R}\times[0,T])$ weakly in this space as $\epsilon_n\to+0$. But this sequence u^{ϵ_n} converges to u in $C([0,T];L_2)$. Hence, $u\equiv w\in W_p^1(\mathbb{R}\times[0,T])$.

In view of (13) and the arguments above, we have in addition that $u^\epsilon\to u$ in $C([0,t];L_p)$ for any $p\in[2,\infty)$ and $\|u(\cdot,t)\|_{L_p}\leqslant C_1$ for a constant $C_1>0$

independent of p and $t \in [0,T]$. Hence, $\|u(\cdot,t)\|_{L_\infty} = \lim_{p\to+\infty} \|u(\cdot,t)\|_{L_p} \leqslant C_1$ which implies in addition that $\|u(\cdot,t\|_{L_\infty}$ is a measurable function of t. Therefore, $u(\cdot,t) \in L_b([0,T];L_\infty)$.

Let us prove that $u(\cdot,t) \in L_b([0,T];W^1_\infty)$. Take arbitrary $t \in [0,T]$ and $p \in [2,\infty)$. Then, in view of (13), there exists a sequence $\epsilon_n \to +0$ such that $u^{\epsilon_n}(\cdot,t)$ converges weakly in W^1_p to some $w \in W^1_p$. By the known property and (13) $\|w\|_{W^1_p} \leqslant \liminf_{n\to\infty} \|u^{\epsilon_n}(\cdot,t)\|_{W^1_p} \leqslant C_2$ for a constant $C_2 > 0$ independent of $p \in [2,\infty)$ and of t. But the sequence $\{u^{\epsilon_n}(\cdot,t)\}_{n=1,2,3,\dots}$ converges strongly in L_2 to $u(\cdot,t)$. Thus, $u(\cdot,t) \equiv w(\cdot) \in W^1_p$ and $\|u(\cdot,t)\|_{W^1_p} \leqslant C_2$ for any $p \in [2,\infty)$ and $t \in [0,T]$. The latter estimate implies that $\|u_x(\cdot,t)\|_{L_\infty} = \lim_{p\to\infty} \|u_x(\cdot,t)\|_{L_p} \leqslant C_2$. Hence, we need only to prove that $\|u_x(\cdot,t)\|_{L_\infty}$ is a measurable function of t. By the Fubini theorem and the proved fact that $u \in W^1_p(\mathbb{R}\times[0,T])$ for any $p \in [2,\infty)$ $\|u_x(\cdot,t)\|_{L_p}$ is a measurable function of t. But hence, indeed $\|u_x(\cdot,t)\|_{L_\infty} = \lim_{p\to\infty} \|u_x(\cdot,t)\|_{L_p}$ is measurable in t. So, indeed $u(\cdot,t) \in L_b([0,T];W^1_\infty)$. Note that we have proved in addition that $u(\cdot,t) \in L_b([0,T];W^1_p)$, for any $p \in [2,\infty)$.

Let us prove that $u(\cdot,t) \in C_w([0,T];W^1_2)$. Let $t \in [0,T]$ and $\{t_n\}_{n=1,2,3,\dots} \subset [0,T]$ be arbitrary, such that $t_n \to t$ as $n \to \infty$. Let us prove that $u(\cdot,t_n) \to u(\cdot,t)$ weakly in W^1_2. According to (13) and the proved above, the sequence $\{u(\cdot,t_n)\}_{n=1,2,3,\dots}$ is relatively compact in the topology of weak convergence of the space W^1_2. Let $w(\cdot) \in W^1_2$ be an arbitrary limit point of this sequence in this sense. We have, at the same time, $u(\cdot,t_n) \to u(\cdot,t)$ strongly in L_2. Therefore, $u(\cdot,t) \equiv w(\cdot)$ and $u(\cdot,t_n) \to u(\cdot,t)$ as $n \to \infty$ weakly in W^1_2.

Let us prove that $u_t(\cdot,t) \in C_w([0,T];L_2)$. For this aim, we recall that for any $s \geqslant 0$ a norm $[|\cdot|]_{H^{-s}}$ equivalent to $\|\cdot\|_{H^{-s}}$ can be introduced as follows: $[|g|]_{H^{-s}} = \sup_{h\in H^s:\ \|h\|_{H^s}=1} |(g,h)_{L_2}|$, where the expression $(g,h)_{L_2}$ can be defined by the standard limiting procedure. Using this fact, let us prove that the expression $u^\epsilon_t(\cdot,t) = \int_0^t \{(a(u^\epsilon(\cdot,s))u^\epsilon_x(\cdot,s))_x + f(\cdot,s)\}ds$ is converging in the space $C([0,T];H^{-2})$ as $\epsilon \to +0$. We have formally for any h in the unit sphere in H^2:

$$\left|(h,(a(u^{\epsilon_1})u^{\epsilon_1}_x)_x - (a(u^{\epsilon_2})u^{\epsilon_2}_x)_x)_{L_2}\right| \leqslant \left|([a(u)h_x]'_x, u^{\epsilon_1} - u^{\epsilon_2})_{L_2}\right| +$$

$$+\left|([h_x u^{\epsilon_1}_x, a(u^{\epsilon_1}) - a(u))_{L_2}\right| + \left|([h_x u^{\epsilon_2}_x, a(u^{\epsilon_2}) - a(u))_{L_2}\right|. \qquad (14)$$

Since $a(u) \in W^1_2$ and $u^\epsilon(\cdot,t) \to u(\cdot,t)$ in L_2 as $\epsilon \to +0$ and by (13), the expressions $[a(u)h_x]'_x$ and $h_x u^\epsilon_x$ are bounded in L_2 uniformly with respect to $\epsilon \in (0,1]$, to $t \in [0,T]$ and to h in the unit sphere in H^2. In addition, clearly $a(u^\epsilon) \to a(u)$ in L_2 as $\epsilon \to +0$. Therefore, each of the three terms in the right-hand side of (14) goes to 0 as $\epsilon_i \to +0$ uniformly with respect to h in the unit sphere in H^2 and to $t \in [0,T]$.

So, we have proved that indeed, the sequence $\{u_t^\epsilon(\cdot,t)\}_{\epsilon\in(0,1]}$ is converging in the space $C([0,T];H^{-2})$ as $\epsilon \to +0$. In addition, one can prove by complete analogy that $u_t(\cdot,t) \in C([0,T];H^{-2})$ and that $u_t^\epsilon(\cdot,t) \to u_t(\cdot,t)$ in this space as $\epsilon \to +0$. Note also that one can prove completely as above that $u_{tt}^\epsilon(\cdot,t) \to u_{tt}(\cdot,t)$ in $C([0,T];H^{-2})$ as $\epsilon \to +0$ and that equation (1) written for our function u holds in the sense of the space $C([0,T];H^{-2})$.

Take an arbitrary $t \in [0,T]$. In view of (13), an arbitrary sequence $\{u_t^{\epsilon_n}(\cdot,t)\}_{n=1,2,3,\dots}$ with $\epsilon_n \to +0$ is relatively compact in the space L_2 taken with the topology of the weak convergence. Let w be an arbitrary limit point of such a sequence in this sense. But by the proved earlier, $u_t^{\epsilon_n}$ goes to u_t in $C([0,T];H^{-2})$. Thus, $w = u_t(\cdot,t)$ and thus $u_t(\cdot,t) \in L_2$ for any $t \in [0,T]$.

Let us prove that $u_t(\cdot,t) \in C_w([0,T];L_2)$. Let $\{t_n\}_{n=1,2,3,\dots} \subset [0,T]$ and $t \in [0,T]$ be arbitrary, such that $t_n \to t$ as $n \to \infty$. In view of (13) and the proved above, it suffices to prove only that $u_t(\cdot,t_n) \to u_t(\cdot,t)$ weakly in L_2 as $n \to \infty$. By (13), the sequence $\{u_t(\cdot,t_n)\}_{n=1,2,3,\dots}$ is relatively compact in the space L_2 taken with the topology of the weak convergence. Let w be an arbitrary limit point of this sequence in this sense. But then, since $u_t(\cdot,t_n) \to u_t(\cdot,t)$ in H^{-2} as $n \to \infty$, we have: $w = u_t(\cdot,t)$, so that indeed $u_t(\cdot,t) \in C_w([0,T];L_2)$.

One can prove by analogy with the arguments above that $u_t(\cdot,t) \in C_w([0,T];L_p)$ for any $p \in [2,\infty)$ and that, in view of (13),

$$\|u_t(\cdot,t)\|_{L_p} \leqslant C, \quad p \in [2,\infty),\ t \in [0,T],$$

where $C > 0$ is the constant in (13). Therefore, $\|u_t(\cdot,t)\|_{L_\infty} \leqslant C$ for any $t \in [0,T]$. In addition, since as is proved earlier, $u_t \in W_p^1(\mathbb{R} \times [0,T])$ for any $p \in [2,\infty)$, the norm $\|u_t(\cdot,t)\|_{L_\infty}$ is a measurable function of $t \in [0,T]$. Thus, indeed $u_t(\cdot,t) \in L_b([0,T];L_\infty)$.

So, we have proved that problem (1)–(2) has a solution $u(\cdot,t)$ that belongs to $C([0,T];L_2) \cap L_b([0,T];W_\infty^1) \cap C_w([0,T];W_2^1)$ and is such that $u_t(\cdot,t) \in C_w([0,T];L_2) \cap L_b([0,T];L_\infty)$. Now, let us prove the uniqueness of such a solution. Suppose that $u_1(x,t)$ and $u_2(x,t)$ are arbitrary two solutions of this problem each of which belongs to the function spaces above. Again, let us take $b \in [0,T]$, $\eta(x,t) = \int\limits_b^t [A(u_1(x,s)) - A(u_2(x,s))]ds$ for $t \in [0,b]$ and $\eta(x,t) \equiv 0$ for $t \geqslant b$, and introduce $k(x,t) = \dfrac{A(u_1(x,t)) - A(u_2(x,t))}{u_1(x,t) - u_2(x,t)}$ if $u_1(x,t) \neq u_2(x,t)$, $k(x,t) = a(u_1(x,t))$ otherwise. It is easily seen that $\eta(x,t) \in W_2^1(\mathbb{R}\times[0,T])$ and that $\eta(x,T) \equiv 0$. In addition, one can verified that $k(x,t)$ is a continuous function of $(x,t) \in \mathbb{R} \times [0,T]$, that

$$0 < a_1 \leqslant k(x,t) \leqslant a_2 < +\infty \quad \text{for any } (x,t) \in \mathbb{R} \times [0,T]$$

and that $k_t(x,t) \in L_\infty(\mathbb{R} \times [0,T])$.

Now, substitute our function $\eta(x,t)$ in identity (4). Then, repeating transformations in (12), we obtain for $w(x,t) = u_1(x,t) - u_2(x,t)$:

$$\int_{\mathbb{R}} k(x,b) w^2(x,b) dx - \int_0^b dt \int_{\mathbb{R}} k_t'(x,t) w^2(x,t) dx + \\ + \int_{\mathbb{R}} \left\{ \int_0^b dt [A(u_1(x,t)) - A(u_2(x,t))]_x' \right\}^2 dx = 0$$

for any $b \in [0,T]$. Hence. $u_1(x,t) \equiv u_2(x,t)$ by Gronwell's lemma. The uniqueness of a solution $u(x,t)$ of problem (1)–(2) that belongs to the function spaces indicated in the statement of Theorem 1 is proved for $t > 0$. For $t < 0$, all our proofs can be made by complete analogy. By a standard procedure, this weak solution can be uniquely continued on a maximal interval as Theorem 1 says. So, Theorem 1 is proved. Now, Theorem 2 follows from Theorem 0 and remark 3 established in Sec. 2. Our proofs are complete.

Remark 4. It follows, in fact, from our proof of Theorem 1 that our weak solution $u(x,t)$ of problem (1)–(2), the existence and uniqueness of which we have proved above, for any $p \in [2, +\infty)$ and sufficiently small $\delta > 0$ belongs in addition to $C_w([-T^- + \delta, T^+ - \delta]; W_p^1)$, that $u_t(\cdot,t) \in C_w([-T^- + \delta, T^+ - \delta]; L_p)$ and that $u_{tt} \in C([-T^- + \delta, T^+ - \delta]; H^{-2})$. In addition, it follows from our proof of Theorem 1 that after the substitution of our weak solution u in equation (1) for any sufficiently small $\delta > 0$ the equality in (1) holds at least in the sense of the space $C([-T^- + \delta, T^+ - \delta]; H^{-2})$.

Remark 5. The author hopes that methods developed in this article may be found useful in studies of problems of a more general kind, for example, problems of the type of the initial value problem for the equation

$$u_{tt} = (a(t,x,u)u_x)_x + f(t,x,u)$$

under suitable assumptions on the functions a and f. As for applications of our methods in the multidimensional case when $x \in \mathbb{R}^d$ with some integer $d \geqslant 2$, it seems that, to do this, one needs, in particular, to derive a suitable multidimensional analog of identity (8) for this case.

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