Stationary axially symmetric gravitation fields in Einstein theory

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1 Introduction

In many recent astrophysical applications of the theory of dense matter it is necessary to investigate the properties of rapidly rotating compact objects within general relativity theory. The reason of this development is the hope that changes in the internal structure of the dense matter, e.g. during phase transitions, could have observable consequences for the dynamics of the rotational behavior of these objects. Particular examples are the observations of glitches and postglitch relaxation in pulsars, which are discussed as signals for superfluidity in nuclear matter [1] and the suggestion that the braking index is remarkably enhanced when a quark matter core occurs in the center of a pulsar during its spin–down evolution [2]. Further constraints for the nuclear equation of state come from the observation of quasi–periodic brightness oscillations (QPO's) in low–mass–X–ray binaries, which entail mass and radius limits for rapidly rotating neutron stars [3].

The problem of rotation in the general relativity theory was and remains one of the central and complicated problems [4]. Besides the modern methods of numerical solutions of this problem the method of perturbation theory [5] is physically the most systematic approach for the solution of the problem for stationary gravitational fields and their sources.

From the practical point of view for definitions of the integral characteristics of the astrophysical objects it is important to analyze the asymptotical expansion of the metric tensor at large distances from the stars, to be able to compare the results with observational data. One can of course introduce the physical parameters of the configuration using the symmetry properties of the object and the gravitational field by expressing them in terms of conserved quantities.

Using the method of perturbation theory we are going to calculate the total mass, angular momentum and shape deformation from the iterative solution of the gravitational field equations in case of hydrodynamical, thermodynamical and chemical equilibrium for given total baryon number and angular velocity Ω of the object. The perturbation method allows to solve the problem for all possible angular velocities.

A number of papers have been devoted to the problem of rotating stellar configurations within the framework of Newton's theory. Exact analytical solutions outside the mass distribution are known in the case of Newton's theory. Interior solutions of the problem were reviewed in [6] – [12].

Kerr found the exterior solution of the rotation problem within the framework of the Einstein theory [13].

The general solution of Einstein's equations for small angular velocities was discussed in [14] and [15] in the Ω approximation. It was shown that, in this approximation, the diagonal components of the metric tensor outside the mass distribution retain the Schwarzschild form, but there is also the nondiagonal component

$$g_{03} = -\frac{2I}{r}\,\sin^2\theta\,,$$

where I is the total angular momentum. This approximation corresponds to the rotation of a sphere with allowance for Coriolis forces but without change of shape. The solution depends on two parameters, namely, the mass and the total angular momentum, and can be obtained from the Kerr solution by expanding the latter in terms of the total angular momentum, retaining only terms that are linear in I. When the next approximation in Ω (terms of the order of Ω^2) is taken into account, the configuration becomes nonspherical and centrifugal forces and quadrupole moments appear. To find the integral characteristics of rotating stars in this approximation we must have at our disposal both the interior and exterior solutions of Einstein's equations, and join them on the stellar surface.

2 Form of the four−dimensional interval

From the geometric standpoint, constant axially symmetric gravitational fields can be divided into two types, namely, static and stationary fields. Static fields are produced by nonmoving oblate bodies with axially symmetric mass distributions. This type of mass distribution cannot be present in the absence of rotation in a system consisting of gravitating gases and liquids. In fact, this would require a very special stress field inside the mass distribution. In the absence of rotation or special stress fields, the mass distribution must necessarily be spherically symmetric. Stationary axially symmetric fields are defined as those produced during the rotation of masses with time–independent angular velocities. The metrics for the two types of field are quite different. We shall be concerned exclusively with stationary fields, since they are the only fields, which are of interest in physics.

We shall consider the metric for a gravitational field produced by a stationary rotation of a mass distribution. It is clear that, in this case, the mass distribution and the gravitational field must be axially symmetric, i.e.

$$g_{ik} = g_{ik}\left(r, \theta, \Omega\right). \tag{1}$$

In the general case, Ω may depend on R and θ. We shall use the following notation $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$.

Suppose that the observer is located in a fixed reference frame. We shall assume that the rotation occurs in the clockwise direction (we are using a system of units in which $G = c = 1$):

$$\Omega = \frac{\mathrm{d}\varphi^*}{\mathrm{d}x^0}\,. \tag{2}$$

Let us apply the coordinate transformation $x^{\acute{0}} = -x^0$, $x^{\acute{1}} = x^1$, $x^{\acute{2}} = x^2$, $x^{\acute{3}} = -x^3$. This will not affect the sign of the angular velocity. Consequently, g_{ik}

will not undergo any change either. Since $\mathrm{d}^2 s$ must be invariant, it follows that

$$g_{01} = g_{13} = g_{23} = g_{20} = 0\,. \tag{3}$$

Therefore, in the most general case, the four–dimensional interval can be written in the form

$$\mathrm{d}^2 s = g_{00}\mathrm{d}^2 x^0 + g_{11}\mathrm{d}^2 x^1 + g_{22}\mathrm{d}^2 x^2 + g_{33}\mathrm{d}^2 x^3 + 2g_{12}\mathrm{d}x^1\mathrm{d}x^2 + 2g_{03}\mathrm{d}x^0\mathrm{d}x^3\,. \tag{4}$$

Further simplification can be achieved in this expression by suitable transformation of the coordinates r and θ. We shall demand that in the new coordinates

$$g'_{12} = 0\,, \quad g'_{33} = g'_{22}\sin^2\theta\,, \tag{5}$$

where it will be convenient to substitute

$$g'_{11} = \mathrm{e}^{\lambda} \quad g'_{22} = \mathrm{e}^{\mu}\,, \quad g'_{03} = \omega \mathrm{e}^{\mu}\sin^2\theta\,, \quad g'_{00} = \omega^2\mathrm{e}^{\mu}\sin^2\theta - \mathrm{e}^{\nu}\,. \tag{6}$$

In the new notation, the four–dimensional interval can be written in the form

$$\mathrm{d}^2 s = \mathrm{e}^{\lambda}\mathrm{d}^2 r + \mathrm{e}^{\mu}\big(\mathrm{d}^2\theta + \sin^2\theta\,\mathrm{d}^2\varphi\big) + 2\omega\mathrm{e}^{\mu}\sin^2\theta\,\mathrm{d}\varphi\,\mathrm{d}t + \big(\omega^2\mathrm{e}^{\mu}\sin^2\theta - \mathrm{e}^{\nu}\big)\mathrm{d}^2 t\,, \tag{7}$$

where λ, μ, ω and ν are functions of r, θ and Ω. Since the metric must be invariant under the transformation $t \to -t$, and the angular velocity changes sign under this transformation, it is clear that all the components of the metric tensor other than ω will be even functions of Ω. Moreover, ω must be an odd function of Ω.

It is important to note that, in contrast to the static axially symmetric case, the stationary situation is such that the component g_{03} of the metric tensor cannot be reduced to zero by any choice of the reference frame. In the static case, g_{ik} are functions of r and θ (there is no rotation and no dependence on Ω) and, therefore, the requirement that $\mathrm{d}s$ be invariant under the transformation $t \to -t$ means that g_{03} is zero.

3 Einstein equations for axial symmetry

The general form of the metric for an axial symmetric space–time manifold is

$$\mathrm{d}^2 s = \mathrm{e}^{\lambda}\mathrm{d}^2 r + \mathrm{e}^{\mu}\left(\mathrm{d}^2\theta + \sin^2\theta(\Omega+\omega)^2\mathrm{d}^2 t\right) - \mathrm{e}^{\nu}\mathrm{d}^2 t \tag{8}$$

written in a spherical symmetric coordinate system in order to obtain as a limiting case the Schwarzschild solution. This line element is time–translational and axial–rotational invariant; all metric functions are dependent on the coordinate distance from the coordinate center r and azimuthal angle θ between the radius vector and the axis of symmetry.

The physical characteristics of the rotating object depend on the centrifugal forces in the local inertial frame of the observer. In general relativity due to the

Lenz–Thirring law rotational effects are described by ϖ the difference of the angular velocity Ω and the frame dragging frequency $-\omega$

$$\varpi \equiv \Omega + \omega(r, \theta) \,. \tag{9}$$

The energy–momentum tensor of stellar matter can be approximated by the expression of the energy–momentum tensor of an ideal liquid

$$T_\mu^\nu = (P + \rho) u_\mu u^\nu - p \delta_\mu^\nu \,, \tag{10}$$

where u^μ is the 4–velocity of matter, P the pressure and ρ the energy density.

We assume that the star due to high viscosity (ignoring the super–fluid component of the matter) rotates stationary as a solid body with an angular velocity Ω that is independent of the spatial coordinates. The time scales for changes in the angular velocity, which we will consider in our applications, are well separated from the relaxation times at which hydrodynamical equilibrium is established, such that the assumption of a rigid rotator model is justified.

Therefore there are only two non-vanishing components of the velocity

$$\begin{aligned} u^\varphi &= \Omega u^t \\ u^t &= \frac{1}{\sqrt{\mathrm{e}^\nu - \mathrm{e}^\mu \varpi^2 \sin^2 \theta}} \,. \end{aligned} \tag{11}$$

Once the energy–momentum tensor (10) is fixed by the choice of the equation of state for stellar matter, the unknown metric functions ν, λ, μ, ω can be determined by the set Einstein field equations of which we use the following four combinations.

There are three Einstein equations for the determination of the diagonal elements of the metric tensor

$$\begin{aligned} G_r^r - G_t^t &= 8\pi G \left(T_r^r - T_t^t \right) , \\ G_\theta^\theta + G_\varphi^\varphi &= 8\pi G \left(T_\theta^\theta + T_\varphi^\varphi \right) , \\ G_\theta^r &= 0 \,, \end{aligned} \tag{12}$$

and one for the determination of the nondiagonal element

$$G_\varphi^t = 8\pi G T_\varphi^t \,. \tag{13}$$

Here G is the gravitational constant and the G_μ^ν is the Einstein tensor.

We use also an equation for the hydrodynamical equilibrium

$$H(r, \theta) \equiv \int \frac{\mathrm{d}P}{P + \rho} = \frac{1}{2} \ln \left[u^t(r, \theta) \right] + \mathrm{const} \,, \tag{14}$$

where the gravitational enthalpy H is introduced as a function of the energy and pressure distribution.

These Einstein equations allow us to find four unknown functions, determining the gravitational field in empty space:

$$\begin{aligned} G^r_r - G^t_t &= 0\,, \\ G^\theta_\theta + G^\varphi_\varphi &= 0\,, \\ G^r_\theta &= 0\,, \\ G^t_\varphi &= 0\,. \end{aligned} \tag{15}$$

4 Solution method

The problem of the rotation can be solved iteratively by using a perturbation expansion of the metric tensor in a Taylor series with respect to the angular velocity. As a small parameter for this expansion we use the ratio of the rotational energy to the gravitational one of a homogenous Newtonian star

$$\beta \equiv \frac{E_{\rm rot}}{E_{\rm grav}} = \left(\frac{\Omega}{\overline{\Omega}}\right)^2, \tag{16}$$

where $\overline{\Omega}^2 = 8\pi G\rho(0)$ and $\rho(0)$ is the mass density in the center of the configuration. This expansion gives sufficiently correct solutions already at $O(\Omega^2)$, since the expansion parameter is limited to values $\Omega/\overline{\Omega} \ll 1$ by the condition of mechanical stability of the rigid rotation. This can be easily seen by considering as an upper limit for attainable angular velocities the so-called Kepler one $\Omega_K = \sqrt{GM/R_e^3}$ with M being the total mass and R_e the equatorial radius. For homogenous Newtonian spherical stars $\Omega < \Omega_K = \overline{\Omega}/\sqrt{6}$.

The expansion of the metric tensor is given by

$$g_{ik}(r,\theta,\Omega) = \sum_{j=0}^{\infty} \left(\frac{\Omega}{\overline{\Omega}}\right)^j g_{ik}^{(j)}(r,\theta)\,. \tag{17}$$

According to the symmetries of the metric coefficients we have even orders $j = 0, 2, \ldots$ for the diagonal elements

$$\begin{aligned} \lambda(r,\theta,\Omega) &= \lambda^0(r) + \sum_{n=1}^{\infty} \beta^n \lambda^{(n)}(r,\theta)\,, \\ \nu(r,\theta,\Omega) &= \nu^0(r) + \sum_{n=1}^{\infty} \beta^n \nu^{(n)}(r,\theta)\,, \\ \mu(r,\theta,\Omega) &= \mu^0(r) + \sum_{n=1}^{\infty} \beta^n \mu^{(n)}(r,\theta)\,, \end{aligned} \tag{18}$$

and odd orders only for the frame dragging frequency

$$\omega(r,\theta,\Omega) = \sqrt{\beta}\sum_{n=0}^{\infty} \beta^n \omega^{(n)}(r,\theta)\,. \tag{19}$$

All functions with superscript (0) denote the solution of the static configuration and therefore they are only functions of r, the others are the corrections corresponding to the rotation.

This series expansion allows to transform the Einstein equations into a coupled set of equations for the coefficient functions, which can be solved by recursion. At zeroth order we recover the nonlinear problem of the static spherically symmetric star configuration (Tolman–Oppenheimer–Volkoff equations).

Obtained series we substitute for Einstein equations and choose those terms of expansion, which correspond to that degree of β, in approximation of which we wish to solve the problem. In order to find the Einstein equations in any approximation, it is necessary to have λ^n, μ^n, ν^n and ω^n functions for all previous approximations. Therefore the problem must be solved step-by-step, beginning from the zero approximation. In spite of that, we found a common method of solution for all even and all odd orders of approximation.

5 Zeroth order: Static spherically symmetric star models

The functions of the spherically symmetric solution can be found from Eqs. (12) and (14) in zeroth order of the Ω–expansion.

That is the solution of the following equations (Tolman–Oppenheimer–Volkoff)

$$\frac{\mathrm{d}P^0(r)}{\mathrm{d}r} = -G\left[P^0(r)+\rho^0(r)\right]\frac{m(r)+4\pi P^0(r)r^3}{r\big[r-2Gm(r)\big]}\,, \tag{20}$$

where $m(r)$ is the distribution of accumulated mass

$$m(r) = 4\pi\int_0^r \rho(r')r'^2\mathrm{d}r' \tag{21}$$

within sphere of radius r. For the gravitational potentials we have

$$\begin{aligned}\lambda^0(r) &= -\ln\left(1-\frac{2Gm(r)}{r}\right)\\ \nu^0(r) &= -\lambda^0(R_0) - 2G\int_r^{R_0}\frac{m(r')+4\pi P^0(r')r'^3}{r'\big(r'-2Gm(r')\big)}\,\mathrm{d}r'\,.\end{aligned} \tag{22}$$

R_0 is the spherical radius of the star, which is defined by $P^0(R_0)=0$. The set of Eqs. (20) and (21) fulfills the following conditions at the center of the configuration: $\rho^0(0)=\rho(0)$ and $m(0)=0$. The central mass density $\rho(0)$ is the parameter of the spherical configuration. The total mass of the spherically distributed matter in the self-consistent gravitational field is $M_0\big(\rho^0(0)\big)=m(R_0)$.

For empty space in zeroth order we have Schwarzschild metric, and Einstein equations are

$$\begin{aligned}\mathrm{e}^{-\lambda^0}\left(\frac{1}{r^2}-\frac{\lambda_1^0}{r}\right)-\frac{1}{r^2} &= 0\,,\\ \mathrm{e}^{-\lambda^0}\left(\frac{1}{r^2}+\frac{\nu_1^0}{r}\right)-\frac{1}{r^2} &= 0\,.\end{aligned} \tag{23}$$

The solutions of these equations are

$$\mathrm{e}^{-\lambda^0} = \mathrm{e}^{\nu^0} = 1 - \frac{2m}{r} \quad \text{and} \quad \mathrm{e}^{\mu^0} = r^2\,, \tag{24}$$

where $m \equiv m(R_0)$ is the mass of spherically symmetric configuration.

6 Odd $(\Omega^1, \Omega^3, \ldots)$ orders of approximation

From Einstein equations only $G_3^0 = 0$ contains odd degrees of angular velocity. If we substitute known functions $\lambda^0, \ldots, \lambda^n, \nu^0, \ldots, \nu^n, \mu^0, \ldots, \mu^n, \omega^0, \ldots, \omega^{n-1}$ from previous approximations, we will obtain an equation for ω^n (Ω^{2n+1} approximation). For all approximations left sides of this equation is the same. They differ from each other only by right sides:

$$\omega^n_{11} + \frac{4}{r}\,\omega^n_1 + \frac{1}{r(r-2m)}\,(\omega^n_{22} + 3\omega^n_2 \operatorname{ctg}\theta) = W^n(r,\theta)\,, \tag{25}$$

$$\begin{aligned} W^0(r,\theta) &= 0\,,\\ W^1(r,\theta) &= -\omega^0_1\left(2\mu^1_1 - \tfrac{1}{2}\,\lambda^1_1 - \tfrac{1}{2}\,\nu^1_1\right)\\ &\ \ \ldots \end{aligned}$$

($n = 0$ corresponds to Ω^1 approximation, and $n = 1$ — to Ω^3.)

The solution of Eq. (25) will be sought in the form

$$\omega^n(r,\theta) = \sum_{k=0}^{\infty} \omega^n_k(r) P^{(1)}_{k+1}(\cos\theta)\,, \quad P^{(1)}_k(\gamma) \equiv \frac{\mathrm{d}P_k(\gamma)}{\mathrm{d}\gamma}\,. \tag{26}$$

In that case, we obtain for ω^n_k

$$\omega^n_{(k)\,11} + \frac{4}{r}\,\omega^n_{(k)\,1} - \frac{k(k+3)}{r(r-2m)}\,\omega^n_{(k)} = W^n_k(r)\,, \quad k = 0, \ldots, 2n\,, \tag{27}$$

$$\begin{aligned} W^0_k(r) &= 0\\ W^1_k(r) &= -\sum_{\ell=0,2} \omega^0_1 \left[2\mu^1_{(\ell)\,1} - \tfrac{1}{2}\,\lambda^1_{(\ell)\,1} - \tfrac{1}{2}\,\nu^1_{(\ell)\,1}\right] \frac{\delta_{k,\ell} - \delta_{k,\ell-2}}{2\ell+1}\,,\\ &\ \ \ldots \end{aligned}$$

Solution is

$$\omega^n_k = \frac{a^n_k}{r^{k+3}} F\left(k, k+3, 2k+4; \frac{2m}{r}\right) + b^n_k r^k F\left(-k, -k-3, -2k-2; \frac{2m}{r}\right) + w^n_k(r)\,. \tag{28}$$

Here a_k^n and b_k^n are constants of integration. Their values are determined from the continuity of $\omega(r,\theta)$ and its first derivative on the surface of the configuration. $b_k^n = 0$, because g_{03} is the final quantity at infinity.

$$\underline{\omega_k^n = \frac{a_k^n}{r^{k+3}} F\left(k, k+3, 2k+4; \frac{2m}{r}\right) + w_k^n(r)\,,} \tag{29}$$

$$\begin{aligned}
w_0^0(r) &= 0\,,\\
w_0^1(r) &= \frac{3a_0^0 A_0^1}{2r^4} - \frac{3a_0^0 A_2^1}{16m^5 r}\left[5 - \frac{7m}{r} - \frac{40m^2}{3r^2} + \frac{2m^3}{r^3} + \right.\\
&\qquad \left. + \frac{r}{m}\left(\frac{5}{2} - \frac{6m}{r} + \frac{4m^3}{r^3}\right)\ln\left(1 - \frac{2m}{r}\right)\right] - \frac{(a_0^0)^3}{10mr^6}\left(1 - \frac{27m}{r}\right),\\
w_2^1(r) &= -\frac{a_0^0 A_2^1}{8m^4 r^2}\left[3 - \frac{2m}{r} - \frac{3m^2}{r^2} + \frac{r}{2m}\left(3 + \frac{m}{r}\right)\left(1 - \frac{2m}{r}\right)\ln\left(1 - \frac{2m}{r}\right)\right] - \\
&\quad - \frac{(a_0^0)^3}{60m^2 r^5}\left(5 + \frac{9m}{r}\right)\left(1 - \frac{2m}{r}\right),\\
&\ldots
\end{aligned}$$

7 Even $(\Omega^2, \Omega^4, \ldots)$ orders of approximation

Einstein equations

$$\begin{aligned}
G_1^1 - G_0^0 &= 0\,,\\
G_2^2 + G_3^3 &= 0\,,\\
G_\varphi^t &= 0
\end{aligned} \tag{30}$$

contain even degrees of angular velocity. Substituting in these equations known functions $\lambda^0, \ldots, \lambda^{n-1}$, $\nu^0, \ldots, \nu^{n-1}$, $\mu^0, \ldots, \mu^{n-1}$, $\omega^0, \ldots, \omega^{n-1}$ from previous approximations, we obtain equations for unknown functions λ^n, ν^n, μ^n (Ω^{2n} approximation). Left sides of these equations are the same for any n, they differ from each other only by right sides:

$$\left(1 - \frac{2m}{r}\right)\left[\mu_{11}^n + \frac{2\mu_1^n - \lambda_1^n - \nu_1^n}{r}\right] + \frac{\lambda_{22}^n - \nu_{22}^n + (\lambda_2^n - \nu_2^n)\operatorname{ctg}\theta}{2r^2} = C^n(r,\theta)\,, \tag{31}$$

$$\begin{aligned}
&\left(1 - \frac{2m}{r}\right)(\mu_{11}^n + \nu_{11}^n) + \frac{2(r-m)\mu_1^n - (r-m)\lambda_1^n + (r+m)\nu_1^n}{r^2} + \\
&\qquad + \frac{\lambda_{22}^n + \nu_{22}^n + (\lambda_2^n + \nu_2^n)\operatorname{ctg}\theta}{2r^2} = D^n(r,\theta)\,,
\end{aligned} \tag{32}$$

$$\left(1 - \frac{2m}{r}\right)(\mu_{12}^n + \nu_{12}^n) - \frac{(r-m)\lambda_2^n + (r-3m)\nu_2^n}{r^2} = E^n(r,\theta)\,. \tag{33}$$

$$C^1(r,\theta) = 0\,,$$

$$\begin{aligned}
C^2(r,\theta) &= -\frac{1}{2}\left(1-\frac{2m}{r}\right)\mu_1^1\left(\mu_1^1-\lambda_1^1-\nu_1^1\right)- \\
&-\frac{1}{2r^2}\left\{\frac{1}{2}\left((\lambda_2^1)^2-(\nu_2^1)^2\right)-\left(\mu^1-\lambda^1\right)\left[\lambda_{22}^1-\nu_{22}^1+\left(\lambda_2^1-\nu_2^1\right)\operatorname{ctg}\theta\right]\right\}, \\
D^1(r,\theta) &= \frac{9(a_0^0)^2}{r^6}\sin^2\theta\,, \\
D^2(r,\theta) &= -\frac{1}{2}\left(1-\frac{2m}{r}\right)\left[\mu_1^1\left(\mu_1^1-\lambda_1^1+\nu_1^1\right)-\nu_1^1\left(\lambda_1^1-\nu_1^1\right)\right]- \\
&-\frac{1}{2r^2}\left\{\frac{1}{2}\left(\lambda_2^1+\nu_2^1\right)^2-\left(\mu^1-\lambda^1\right)\left[\lambda_{22}^1+\nu_{22}^1+\left(\lambda_2^1+\nu_2^1\right)\operatorname{ctg}\theta\right]\right\}- \\
&-\frac{3a_0^0}{r^2}\left[2\omega_1^1-\frac{3a_0^0}{r^4}\left(\mu^1-\nu^1\right)\right]\sin^2\theta\,, \\
E^1(r,\theta) &= 0\,, \\
E^2(r,\theta) &= \frac{1}{2}\left(1-\frac{2m}{r}\right)\left[\lambda_2^1\left(\mu_1^1+\nu_1^1\right)+\nu_2^1\left(\mu_1^1-\nu_1^1\right)\right]-\frac{3a_0^0}{r^2}\,\omega_2^1\sin^2\theta\,. \\
&\ldots
\end{aligned}$$

We can integrate the Eq. (33) by θ. Then

$$\mu_1^n=-\nu_1^n+\frac{r-m}{r(r-2m)}\,\lambda^n+\frac{r-3m}{r(r-2m)}\,\nu^n+M^n(r,\theta)\,. \tag{34}$$

$$M^1(r,\theta)=0\,,\quad \ldots.$$

Substituting μ_1^n and μ_{11}^n into Eqs. (31) and (32) we obtain

$$\begin{aligned}
&\left(1-\frac{2m}{r}\right)\left[\nu_{11}^n-\frac{m}{r(r-2m)}\,\lambda_1^n+\frac{2r-3m}{r(r-2m)}\,\nu_1^n-\frac{r^2-4mr+2m^2}{r^2(r-2m)^2}\,\lambda^n-\right. \\
&\left.-\frac{r^2-4mr+6m^2}{r^2(r-2m)^2}\,\nu^n\right]-\frac{1}{2r^2}\left[\lambda_{22}^n-\nu_{22}^n+(\lambda_2^n-\nu_2^n)\operatorname{ctg}\theta\right]=H^n(r,\theta)\,,
\end{aligned} \tag{35}$$

$$\lambda^n+\nu^n+\tfrac{1}{2}\left[\lambda_{22}^n+\nu_{22}^n+(\lambda_2^n+\nu_2^n)\operatorname{ctg}\theta\right]=K^n(r,\theta)\,, \tag{36}$$

$$\begin{aligned}
H^1(r,\theta) &= 0\,, \\
K^1(r,\theta) &= \frac{9(a_0^0)^2}{r^4}\sin^2\theta\,, \\
&\ldots
\end{aligned}$$

All diagonal elements of the metric tensor could be represented as a series expansion in Legendre polynomials:

$$\begin{aligned}
\lambda^n(r,\theta) &= \sum_{\ell=0}^{\infty}\lambda_\ell^n(r)P_\ell(\cos\theta)\,, \\
\nu^n(r,\theta) &= \sum_{\ell=0}^{\infty}\nu_\ell^n(r)P_\ell(\cos\theta)\,.
\end{aligned} \tag{37}$$

Substitute these series into the Eqs. (35) and (36):

$$\left(1-\frac{2m}{r}\right)\left[\nu^n_{(\ell)11}-\frac{m}{r(r-2m)}\,\lambda^n_{(\ell)1}+\frac{2r-3m}{r(r-2m)}\,\nu^n_{(\ell)1}-\frac{r^2-4mr+2m^2}{r^2(r-2m)^2}\,\lambda^n_\ell-\right.$$
$$\left.-\frac{r^2-4mr+6m^2}{r^2(r-2m)^2}\,\nu^n_\ell\right]+\frac{\ell(\ell+1)}{2r^2}\,(\lambda^n_\ell-\nu^n_\ell)=Q^n_\ell(r)\,,$$

$$\lambda^n_\ell=-\nu^n_\ell+L^n_\ell(r)\,,\quad \ell=0,\ldots,2n\,, \tag{38}$$
$$Q^1_0(r)=0\,,\qquad Q^1_2(r)=0\,,$$
$$L^1_0(r)=\frac{6(a^0_0)^2}{r^4}\,,\qquad L^1_2(r)=\frac{3(a^0_0)^2}{r^4}\,,$$
$$\ldots.$$

Finally we obtain equation for ν^n_ℓ:

$$\left(1-\frac{2m}{r}\right)\nu^n_{(\ell)11}+\frac{2(r-m)}{r^2}\,\nu^n_{(\ell)1}-\left[\frac{4m^2}{r^3(r-2m)}+\frac{\ell(\ell+1)}{r^2}\right]\nu^n_\ell=N^n_\ell(r)\,, \tag{39}$$

$$N^1_0(r)\;=\;6(a^0_0)^2\frac{r^2-8mr+10m^2}{r^7(r-2m)}$$
$$N^1_2(r)\;=\;-6(a^0_0)^2\frac{r^2+mr-5m^2}{r^7(r-2m)}$$
$$\ldots$$

The solution of this equation is

$$\nu^n_\ell=\frac{A^n_\ell}{r^\ell(r-2m)}\,F\left(\ell+1,\ell-1,2\ell+2;\frac{2m}{r}\right)+$$
$$+\frac{B^n_\ell r^{\ell+1}}{r-2m}\,F\left(-\ell,-\ell-2,-2\ell;\frac{2m}{r}\right)+n^n_\ell(r)\,, \tag{40}$$

where A^n_ℓ and B^n_ℓ are constants of integration. The value of A^n_ℓ is determined from continuity of $\nu(r,\theta)$ and its first derivative on the surface of the configuration. $B^n_\ell=0$, because g_{00} is the final quantity at infinity.

$$\underline{\nu^n_\ell=\frac{A^n_\ell}{r^\ell(r-2m)}\,F\left(\ell+1,\ell-1,2\ell+2;\frac{2m}{r}\right)+n^n_\ell(r)\,,} \tag{41}$$

$$n^1_0(r)=\frac{(a^0_0)^2}{2r^4}\,\frac{r-4m}{r-2m}\,,\qquad n^1_2(r)=\frac{(a^0_0)^2}{2mr^4}\,\frac{r^2-mr-2m^2}{r-2m}\,,$$

$$n^2_0(r)=-(A^1_0)^2\frac{m}{r}\,\frac{r-m}{(r-2m)^3}\left(5-10\,\frac{m}{r}+8\,\frac{m^2}{r^2}\right)+$$

$$
\begin{aligned}
&+(A^1_2)^2\,\frac{5r^3}{128m^9}\Bigg\{\frac{r^2(r-m)}{(r-2m)^3}\bigg(189-1512\,\frac{m}{r}+4464\,\frac{m^2}{r^2}-5616\,\frac{m^3}{r^3}+\\
&\qquad+2172\,\frac{m^4}{r^4}+528\,\frac{m^5}{r^5}+16\,\frac{m^6}{r^6}+64\,\frac{m^7}{r^7}-320\,\frac{m^8}{r^8}\bigg)+\\
&+\frac{3r}{2m}\left(117-468\,\frac{m}{r}+504\,\frac{m^2}{r^2}-72\,\frac{m^3}{r^3}-20\,\frac{m^4}{r^4}\right)\ln\left(1-\frac{2m}{r}\right)+\\
&+\frac{81r^2}{2m^2}\left(1-\frac{m}{r}\right)\left(1-\frac{2m}{r}\right)^2\ln^2\left(1-\frac{2m}{r}\right)\Bigg\}-\\
&-A^1_0A^1_2\,\frac{5}{m}\,\frac{1}{(r-2m)^3}\left(1-3\frac{m}{r}+6\,\frac{m^2}{r^2}-4\,\frac{m^3}{r^3}\right)-\\
&-(a^0_0)^2A^1_0\,\frac{1}{4m^2}\,\frac{1}{(r-2m)^3}\bigg(40-375\,\frac{m}{r}+1074\,\frac{m^2}{r^2}-364\,\frac{m^3}{r^3}-\\
&\qquad-3576\,\frac{m^4}{r^4}+6384\,\frac{m^5}{r^5}-3360\,\frac{m^6}{r^6}\bigg)-\\
&-(a^0_0)^2A^1_2\,\frac{1}{16m^6r}\Bigg[\frac{r^3}{(r-2m)^3}\bigg(69-561\,\frac{m}{r}+2834\,\frac{m^2}{r^2}-14610\,\frac{m^3}{r^3}+\\
&+52380\,\frac{m^4}{r^4}-98688\,\frac{m^5}{r^5}+74592\,\frac{m^6}{r^6}+13152\,\frac{m^7}{r^7}-33600\,\frac{m^8}{r^8}\bigg)+\\
&+\frac{15r}{m}\left(1-6\,\frac{m}{r}+48\,\frac{m^2}{r^2}-166\,\frac{m^3}{r^3}+165\,\frac{m^4}{r^4}\right)\ln\left(1-\frac{2m}{r}\right)\Bigg]-\\
&-a^0_0a^1_0\,\frac{3}{m^2r^2}\left(1-\frac{8m}{r}+\frac{10m^2}{r^2}\right)-\\
&-a^0_0a^1_2\left[\frac{2m}{r}\left(3+3\,\frac{m}{r}+4\,\frac{m^2}{r^2}+6\,\frac{m^3}{r^3}\right)-3\ln\left(1-\frac{2m}{r}\right)\right]-\\
&-(a^0_0)^4\,\frac{1}{20m^3r^2}\,\frac{1}{(r-2m)^3}\bigg(189-623\,\frac{m}{r}-324\,\frac{m^2}{r^2}-16944\,\frac{m^3}{r^3}+\\
&\qquad+143520\,\frac{m^4}{r^4}-404052\,\frac{m^5}{r^5}+497880\,\frac{m^6}{r^6}-230400\,\frac{m^7}{r^7}\bigg),
\end{aligned}
$$

$$
\begin{aligned}
n^2_2(r)&=-(A^1_2)^2\,\frac{75r}{448m^7}\,\frac{r^2}{3r^2-6mr+2m^2}\times\\
&\times\Bigg\{\frac{(r-m)(3r^2-6mr-2m^2)}{r(r-2m)^2}\left(9-36\,\frac{m}{r}+41\,\frac{m^2}{r^2}-10\,\frac{m^3}{r^3}-2\,\frac{m^4}{r^4}\right)+\\
&\qquad+\frac{3r}{m}\left(12-48\,\frac{m}{r}+53\,\frac{m^2}{r^2}-10\,\frac{m^3}{r^3}-6\,\frac{m^4}{r^4}\right)\ln\left(1-\frac{2m}{r}\right)+
\end{aligned}
$$

$$+\frac{45r^2}{4m^2}\left(1-\frac{m}{r}\right)\left(1-\frac{2m}{r}\right)^2\ln^2\left(1-\frac{2m}{r}\right)\Bigg\}-$$

$$-A_0^1A_2^1\,\frac{15}{8m^3r}\left[\frac{(r-m)\left(3r^4-12mr^3+12m^2r^2-4m^4\right)}{r(r-2m)^2\left(3r^2-6mr+2m^2\right)}+\frac{r}{2m}\ln\left(1-\frac{2m}{r}\right)\right]+$$

$$+(a_0^0)^2A_0^1\frac{1}{2r^5}\,\frac{11r^2+33mr-96m^2}{3r^2-6mr+2m^2}+$$

$$+(a_0^0)^2A_2^1\frac{3}{56m^4r}\,\frac{1}{3r^2-6mr+2m^2}\left[\frac{r^2}{(r-2m)^2}\left(150-177\,\frac{m}{r}-1964\,\frac{m^2}{r^2}+\right.\right.$$

$$\left.+4340\,\frac{m^3}{r^3}-548\,\frac{m^4}{r^4}-2030\,\frac{m^5}{r^5}-1120\,\frac{m^6}{r^6}\right)+$$

$$\left.+\frac{5r}{2m}\left(33-246\,\frac{m^2}{r^2}+178\,\frac{m^3}{r^3}+150\,\frac{m^4}{r^4}\right)\ln\left(1-\frac{2m}{r}\right)\right]+$$

$$+a_0^0a_0^1\,\frac{6}{r^4}\,\frac{r^2+mr-5m^2}{3r^2-6mr+2m^2}+$$

$$+a_0^0a_2^1\,\frac{1}{7}\,\frac{r^2}{3r^2-6mr+2m^2}\left[\frac{2}{5}\left(540-1845\,\frac{m}{r}+1485\,\frac{m^2}{r^2}+108\,\frac{m^4}{r^4}+\right.\right.$$

$$\left.+148\,\frac{m^5}{r^5}+60\,\frac{m^6}{r^6}\right)+$$

$$\left.+\frac{3r}{m}\left(36-159\,\frac{m}{r}+210\frac{m^2}{r^2}-70\,\frac{m^3}{r^3}\right)\ln\left(1-\frac{2m}{r}\right)\right]+$$

$$+(a_0^0)^4\frac{3}{28m^2r^2}\,\frac{1}{3r^2-6mr+2m^2}\left(20-277\,\frac{m}{r}+540\,\frac{m^2}{r^2}+937\,\frac{m^3}{r^3}-2079\,\frac{m^4}{r^4}\right),$$

$$n_4^2(r)=-(A_2^1)^2\,\frac{15r^3}{448m^7}\,\frac{1}{5r^2-10mr+m^2}\times$$

$$\times\left[\frac{(r-m)\left(3r^2-6mr-2m^2\right)}{r(r-2m)^2}\left(51-204\,\frac{m}{r}+230\,\frac{m^2}{r^2}-52\,\frac{m^3}{r^3}+12\,\frac{m^4}{r^4}\right)+\right.$$

$$+\frac{3r}{2m}\left(129-516\,\frac{m}{r}+568\,\frac{m^2}{r^2}-104\,\frac{m^3}{r^3}-40\,\frac{m^4}{r^4}\right)\ln\left(1-\frac{2m}{r}\right)+$$

$$\left.+\frac{117r^2}{2m^2}\left(1-\frac{m}{r}\right)\left(1-\frac{2m}{r}\right)^2\ln^2\left(1-\frac{2m}{r}\right)\right]-$$

$$+(a_0^0)^2A_2^1\,\frac{3}{224m^4r}\,\frac{1}{5r^2-10mr+m^2}\left[351-489\,\frac{m}{r}-356\,\frac{m^2}{r^2}-210\,\frac{m^3}{r^3}-336\,\frac{m^4}{r^4}+\right.$$

$$\left.+\frac{5r}{2m}\left(81-168\,\frac{m}{r}-40\frac{m^2}{r^2}+64\,\frac{m^3}{r^3}+120\,\frac{m^4}{r^4}\right)\ln\left(1-\frac{2m}{r}\right)\right]-$$

$$-a_0^0 a_2^1 \, \frac{1}{7} \, \frac{r^2}{5r^2 - 10mr + m^2} \left[\frac{2}{5} \left(270 - 765 \, \frac{m}{r} + 375 \, \frac{m^2}{r^2} + 105 \, \frac{m^3}{r^3} + 89 \frac{m^4}{r^4} + \right. \right.$$
$$\left. \left. + 88 \, \frac{m^5}{r^5} + 30 \, \frac{m^6}{r^6} \right) + \frac{3r}{m} \left(18 - 69 \, \frac{m}{r} + 70 \, \frac{m^2}{r^2} - 7 \, \frac{m^3}{r^3} \right) \ln \left(1 - \frac{2m}{r} \right) \right] -$$
$$- \frac{(a_0^0)^4}{280 m^2 r^4} \, \frac{1}{5r^2 - 10mr + m^2} \left(300 + 689 \, \frac{m}{r} - 3359 \, \frac{m^2}{r^2} + 804 \, \frac{m^3}{r^3} + 1890 \, \frac{m^4}{r^4} \right) .$$

8 Constants

The obtained solution must coincide with the Papapetrou metric [16] at infinity:

$$\begin{aligned} \mathrm{d}^2 s = & - \left(1 - \frac{2M}{r} \right) \mathrm{d}^2 t + \left(1 + \frac{2M}{r} \right) \mathrm{d}^2 r + \\ & + r^2 \left(1 + \frac{2M - 2m}{r} \right) \left(\mathrm{d}^2 \theta + \sin^2 \theta \, \mathrm{d}^2 \varphi \right) - \frac{4 I_z}{r} \, \sin^2 \theta \, \mathrm{d}\varphi \, \mathrm{d}t \, . \end{aligned} \tag{42}$$

Here M is the mass of a rotating star and I_z is the z component of its moment of inertia. Analyzing the asymptotical expansion of our metric tensor and comparing it with the Papapetrou metric, we can find, that

$$\begin{aligned} M &= m - \frac{1}{2} \sum_{n=1}^{\infty} \beta^n A_0^n \, , \\ I_z &= -\frac{1}{2} \sqrt{\beta} \sum_{n=0}^{\infty} \beta^n a_0^n \, . \end{aligned} \tag{43}$$

M and I_z contain only zeroth members of A_ℓ^n and a_k^n. Other members are included in quadrupole moment and moments of higher order.

Our purpose is to find a method for calculating any order of approximation also for the internal solution, thus to find physical characteristics in result.

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